Pieri Rules and Oscillating Tableaux

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This talk is based on a paper arXiv:1606.02375.

Plan:

- Burrill's conjecture on standard tableaux and oscillating tableaux
- Representation theoretical proof of Burrill's conjecture
- Generalizations of Burrill's conjecture

Burrill's Conjecture

Standard Tableaux

Let λ be a partition with $|\lambda| = k$. A standard tableau of shape λ is a filling of the cells of the diagram of λ by positive integers $1, 2, \ldots, k$ satisfying

- each positive integer from 1 to k appears exactly once,
- the entries in each row and each column are increasing.

Standard tableaux of shape λ are in bijection with sequences of partitions

$$\emptyset = \lambda^{(0)},\ \lambda^{(1)},\ \ldots,\ \lambda^{(k)} = \lambda$$

satisfying

$$\lambda^{(i-1)} \subset \lambda^{(i)}, \quad |\lambda^{(i-1)}| + 1 = |\lambda^{(i)}| \quad (1 \le i \le k).$$

Example

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \longleftrightarrow \emptyset \subset \begin{array}{|c|}\hline 1 \\ \hline\end{array} \subset \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \subset \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \subset \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \subset \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

Oscillating Tableaux

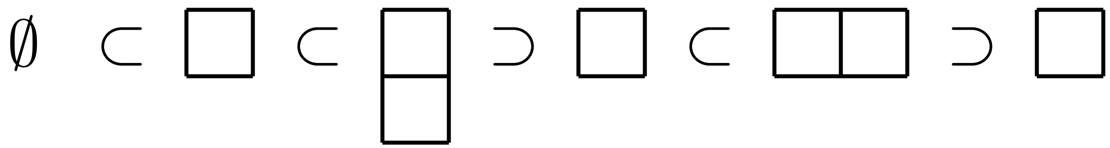
For a nonnegative integer k and a partition λ, an oscillating tableau of length k and shape λ is a sequence of partitions

$$\emptyset = \lambda^{(0)},\ \lambda^{(1)},\ \ldots,\ \lambda^{(k)} = \lambda$$

satisfying

$$\begin{aligned} & \lambda^{(i-1)} \subset \lambda^{(i)}, \quad |\lambda^{(i-1)}| + 1 = |\lambda^{(i)}|, \\ \text{OR} \quad & \lambda^{(i-1)} \supset \lambda^{(i)}, \quad |\lambda^{(i-1)}| - 1 = |\lambda^{(i)}|. \end{aligned}$$

Example The sequence



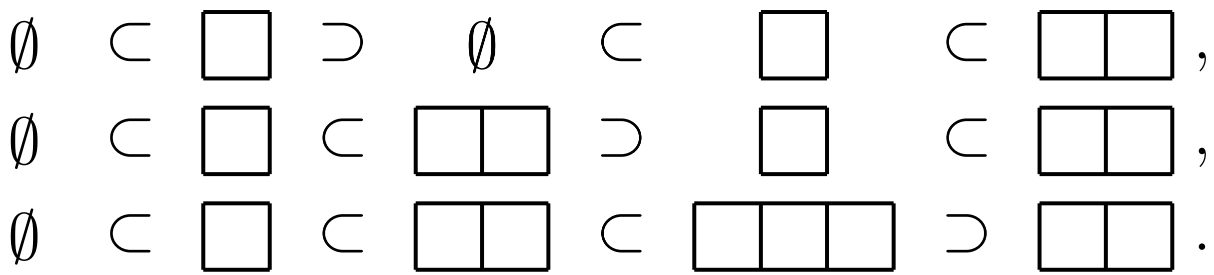
is an oscillating tableau of length 5 and shape (1).

Burrill's Conjecture

Theorem 1 (Krattenthaler, Burrill–Courtiel–Fusy–Melczer–Mishna)
Given nonnegative integers k, m and n, the following are equinumerous:

(a) Oscillating tableaux $\left(\lambda^{(i)}\right)_{i=0}^{k}$ of length k and shape (m) (the one-row partition of size m) such that $l(\lambda^{(i)}) \leq n$ for each i.

Example If $k = 4$, $m = 2$ and $n = 1$, then



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(b) Standard tableaux whose shape λ satisfies $|\lambda| = k$, $c(\lambda) = m$, and $l(\lambda) \leq 2n$. Here $c(\lambda)$ is the number of columns of odd length in the diagram of λ.

Example If $k = 4$, $m = 2$ and $n = 1$, then

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

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(b) Standard tableaux whose shape λ satisfies $|\lambda| = k$, $c(\lambda) = m$, and $l(\lambda) \leq 2n$. Here $c(\lambda)$ is the number of columns of odd length in the diagram of λ.

Krattenthaler and Burrill et al. gave bijective proofs to this theorem. In this talk, we give another proof and generalizations based on the representation theory of classical groups.

Representation Theoretical Proof of Burrill's Conjecture

Representation theory of $\mathbf{GL}_N(\mathbb{C})$

The irreducible polynomial representations of the general linear group $\mathbf{GL}_N$ are parametrized by partitions of length $\leq N$.

$$\begin{array}{ccc} \{\text{irred. polyn. repr. of } \mathbf{GL}_N\} / \cong & \longleftrightarrow & \{\text{partitions of length} \leq N\} \\ V_\lambda & \longleftrightarrow & \lambda \end{array}$$

For example,

$$V_\emptyset = \mathbb{C} \quad \text{(trivial repr.)}, \qquad V_\square = \mathbb{C}^N \quad \text{(natural repr.)},$$
$$V_{(r)} = S^r(V_\square), \qquad V_{(1^r)} = \bigwedge^r (V_\square).$$

Pieri Rule (simplest case) For a partition μ with $l(\mu) \leq N$, we have

$$V_\mu \otimes V_\square \cong \bigoplus_\lambda V_\lambda,$$

where λ runs over all partitions of length $\leq N$ such that

$$\lambda \supset \mu, \quad |\lambda| = |\mu| + 1.$$

Representation theory of $\mathbf{Sp}_{2n}(\mathbb{C})$

The irreducible representations of the symplectic group $\mathbf{Sp}_{2n} \subset \mathbf{GL}_{2n}$ are parametrized by partitions of length $\leq n$.

$$\begin{array}{ccc} \{\text{irred. repr. of } \mathbf{Sp}_{2n}\} \,/\cong & \longleftrightarrow & \{\text{partitions of length} \leq n\} \\ W_{\langle\lambda\rangle} & \longleftrightarrow & \lambda \end{array}$$

For example,

$$W_{\langle\emptyset\rangle} = \mathbb{C} \quad (\text{trivial repr.}), \qquad W_{\langle\Box\rangle} = \mathbb{C}^{2n} \quad (\text{natural repr.}),$$
$$W_{\langle(r)\rangle} = S^r(W_{\langle\Box\rangle}), \qquad W_{\langle(1^r)\rangle} = \bigwedge^r(W_{\langle\Box\rangle})/\bigwedge^{r-2}(W_{\langle\Box\rangle}).$$

Pieri Rule (simplest case) For a partition μ with $l(\mu) \leq n$, we have

$$W_{\langle\mu\rangle} \otimes W_{\langle\Box\rangle} \cong \bigoplus_{\lambda} W_{\langle\lambda\rangle},$$

where λ runs over all partitions of length $\leq n$ such that

$$\lambda \supset \mu,\ |\lambda| = |\mu| + 1, \quad \text{or} \quad \lambda \subset \mu,\ |\lambda| = |\mu| - 1.$$

Theorem 1 (Krattenthaler, Burrill–Courtiel–Fusy–Melczer–Mishna) Given nonnegative integers k, m and n, the following are equinumerous:

(a) Oscillating tableaux $\left(\lambda^{(i)}\right)_{i=0}^{k}$ of length k and shape (m) (the one-row partition of size m) such that $l(\lambda^{(i)}) \leq n$ for each i.

(b) Standard tableaux whose shape λ satisfies $|\lambda| = k$, $c(\lambda) = m$, and $l(\lambda) \leq 2n$. Here $c(\lambda)$ is the number of columns of odd length in the diagram of λ.

Representation-theoretical Proof of Theorem 1

We compute the multiplicity

$$\left[\mathrm{Res}^{\mathbf{GL}_{2n}}_{\mathbf{Sp}_{2n}} V_{\square}^{\otimes k} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}}$$

in two ways, where (m) is the one-row partition of size m.

Since $\mathrm{Res}^{\mathbf{GL}_{2n}}_{\mathbf{Sp}_{2n}} V_{\square} = W_{\langle \square \rangle}$, we have

$$\left[\mathrm{Res}^{\mathbf{GL}_{2n}}_{\mathbf{Sp}_{2n}} V_{\square}^{\otimes k} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}} = \left[W_{\square}^{\otimes k} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}}.$$

By iteratively applying Pieri rule for $\mathbf{Sp}_{2n}$, we have

$$\left[W_{\square}^{\otimes k} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}}$$
$$= \# \left\{ \left(\lambda^{(i)} \right)_{i=0}^{k} : \begin{array}{r} \text{oscillating tableau of length } k \text{ and shape } (m) \\ \text{such that } l(\lambda^{(i)}) \leq n \text{ for } 0 \leq i \leq k \end{array} \right\}.$$

Representation-theoretical Proof of Theorem 1 (continued)

On the other hand, by iteratively applying Pieri rule for $\mathbf{GL}_{2n}$, we have

$$V_{\square}^{\otimes k} \cong \bigoplus_{\lambda} V_{\lambda}^{\oplus f^{\lambda}},$$

where λ runs over all partition with $|\lambda| = k$ and $l(\lambda) \leq 2n$, and f^{λ} denotes the number of standard tableaux of shape λ. Hence we have

$$\left[\mathrm{Res}_{\mathbf{Sp}_{2n}}^{\mathbf{GL}_{2n}} V_{\square}^{\otimes k} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}} = \sum_{\lambda} f^{\lambda} \left[\mathrm{Res}_{\mathbf{Sp}_{2n}}^{\mathbf{GL}_{2n}} V_{\lambda} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}}.$$

Therefore the proof is completed by using

Proposition For a partition of length $\leq 2n$, we have

$$\left[\mathrm{Res}_{\mathbf{Sp}_{2n}}^{\mathbf{GL}_{2n}} V_{\lambda} : W_{\langle (m) \rangle} \right]_{\mathbf{Sp}_{2n}} = \begin{cases} 1 & \text{if } c(\lambda) = m, \\ 0 & \text{otherwise.} \end{cases}$$

Generalizations of Burrill's Conjecture

Generalization

Our representation-theoretical proof suggests the following generalizations:

- Replace the representation $V_{\square}^{\otimes k}$ by

$$S^{\alpha}(V_{\square}) = S^{\alpha_1}(V_{\square}) \otimes \cdots \otimes S^{\alpha_k}(V_{\square}),$$
$$\bigwedge^{\alpha}(V_{\square}) = \bigwedge^{\alpha_1}(V_{\square}) \otimes \cdots \otimes \bigwedge^{\alpha_k}(V_{\square}).$$

- Consider the representations of the orthogonal group $\mathbf{O}_N$ or $\mathbf{SO}_N$.

Classical Pieri rules describe the irreducible decompositions of

$$V_{\lambda} \otimes S^{k}(V_{\square}), \quad V_{\lambda} \otimes \bigwedge^{k}(V_{\square})$$

for $\mathbf{GL}_N$. We need similar decomposition formula for $\mathbf{Sp}_{2n}$ and $\mathbf{O}_N$, $\mathbf{SO}_N$.

Column-strict tableaux

A column-strict tableau (also called a semistandard tableau) of shape λ and weight $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a filling of the cells of the diagram of λ by positive integers $1, 2, 3, \ldots$ such that

- i appears α_i times,
- the entries in each row is weakly increasing,
- the entries in each column are strictly increasing, and

Column-strict tableaux of shape λ and weight α are in bijection with sequences of partitions

$$\emptyset = \lambda^{(0)},\ \lambda^{(1)},\ \ldots,\ \lambda^{(k)} = \lambda$$

such that

$$\lambda^{(i)}/\lambda^{(i-1)} \text{ is a horizonal } \alpha_i\text{-strip for each } i.$$

Example

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

$$\longleftrightarrow \quad \emptyset \subset \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \subset \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \subset \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array} \subset \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

Row-strict Tableaux

A row-strict tableau of shape λ and weight $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a filling of the cells of the diagram of λ by positive integers $1, 2, 3, \ldots$ satisfying

- i appears α_i times,
- the entries in each row is strictly increasing, and
- the entries in each column are weakly increasing.

Row-strict tableaux of shape λ and weight α are in bijection with sequences of partitions

$$\emptyset = \lambda^{(0)},\ \lambda^{(1)},\ \ldots,\ \lambda^{(k)} = \lambda$$

such that

$$\lambda^{(i)}/\lambda^{(i-1)} \text{ is a vertical } \alpha_i\text{-strip for each } i.$$

Pieri rules for $\mathbf{Sp}_{2n}$ (Sundaram)

Given partitions λ and μ with $l(\lambda)$, $l(\mu) \leq n$, we have

$$\left[W_{\langle\mu\rangle} \otimes S^r(W_{\langle\square\rangle}) : W_{\langle\lambda\rangle}\right]_{\mathbf{Sp}_{2n}} = \#\left\{ \xi : \begin{array}{l} \text{partition with } l(\xi) \leq n \text{ such that} \\ \bullet\ \mu \supset \xi \subset \lambda \\ \bullet\ \mu/\xi \text{ and } \lambda/\xi \text{ are horizontal strips} \\ \bullet\ |\mu/\xi| + |\lambda/\xi| = r \end{array} \right\},$$

$$\left[W_{\langle\mu\rangle} \otimes \Lambda^r(W_{\langle\square\rangle}) : W_{\langle\lambda\rangle}\right]_{\mathbf{Sp}_{2n}} = \#\left\{ \xi : \begin{array}{l} \text{partition with } l(\xi) \leq n \text{ such that} \\ \bullet\ \mu \subset \xi \supset \lambda \\ \bullet\ \xi/\mu \text{ and } \xi/\lambda \text{ are vertical strips} \\ \bullet\ |\xi/\mu| + |\xi/\lambda| = r \end{array} \right\}.$$

Generalizations

Theorem 2 (Krattenthaler) For a sequence $\alpha = (\alpha_1, \dots, \alpha_k)$ of nonnegative integers, and nonnegative integers m and n, the following are equinumerous:

(a) Down-up sequences

$$\emptyset = \lambda^{(0)} \supset \lambda^{(1)} \subset \lambda^{(2)} \supset \lambda^{(3)} \subset \lambda^{(4)} \supset \cdots \subset \lambda^{(2k-2)} \supset \lambda^{(2k-1)} \subset \lambda^{(2k)} = (m)$$

of partitions such that

- $l(\lambda^{(i)}) \leq n$,
- $\lambda^{(2i-2)}/\lambda^{(2i-1)}$ and $\lambda^{(2i)}/\lambda^{(2i-1)}$ are horizontal strips, and
- $|\lambda^{(2i-2)}/\lambda^{(2i-1)}| + |\lambda^{(2i)}/\lambda^{(2i-1)}| = \alpha_i$.

(b) Column-strict tableaux of weight α whose shape λ satisfies $c(\lambda) = m$ and $l(\lambda) \leq 2n$.

Generalizations

Theorem 3 For a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of nonnegative integers, and nonnegative integers m and n, the following are equinumerous:

(a) Up-down sequences

$$\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \supset \lambda^{(2)} \subset \lambda^{(3)} \supset \lambda^{(4)} \subset \cdots \supset \lambda^{(2k-2)} \subset \lambda^{(2k-1)} \supset \lambda^{(2k)} = (m)$$

of partitions such that

- $l(\lambda^{(i)}) \leq n$,
- $\lambda^{(2i-1)}/\lambda^{(2i-2)}$ and $\lambda^{(2i-1)}/\lambda^{(2i)}$ are both vertical strips, and
- $|\lambda^{(2i-1)}/\lambda^{(2i-2)}| + |\lambda^{(2i-1)}/\lambda^{(2i)}| = \alpha_i$.

(b) Row-strict tableaux of weight α whose shape λ satisfies $c(\lambda) = m$ and $l(\lambda) \leq 2n$.

Orthogonal Group Case

We can use the representation theory of the orthogonal group $\mathbf{O}_N(\mathbb{C})$ to prove

Theorem 4 Given nonnegative integers k, m and N, the following are equinumerous:

(a) Oscillating tableaux $\left(\lambda^{(i)}\right)_{i=0}^{k}$ of length k and shape (1^m) (the one-column partition of size m) such that $(\lambda^{(i)})'_1 + (\lambda^{(i)})'_2 \leq N$ for each i.

(b) Standard tableaux whose shape λ satisfies $|\lambda| = k$, $r(\lambda) = m$ and $l(\lambda) \leq N$. Here $r(\lambda)$ is the number of rows of odd length in the diagram of λ.

We have similar (but a bit complicated) Pieri rules for $\mathbf{O}_N$ and $\mathbf{SO}_N$ and equinumerous results analogous to Theorems 2 and 3.