Random walks in cones: exponential growth

Lecture #1

Analytic and probabilistic tools for lattice path enumeration

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77th Séminaire Lotharingien de Combinatoire
September 12, 2016
Strobl, Austria

Context: enumeration of lattice walks

▷ *Nearest-neighbor walks* in the plane $\mathbb{Z}^2$; admissible steps

$$\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

▷ $\mathfrak{S}$-walks: walks in $\mathbb{Z}^2$ starting at $(0,0)$ and using *steps in* $\mathfrak{S}$

▷ $\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{n} (i,j)\}$: number of $\mathfrak{S}$-walks ending at (i,j) and consisting of exactly n steps, possibly *confined to* some subdomain of $\mathbb{Z}^2$ (for us: *the quarter plane* Q)

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▷ Example with

$$\mathfrak{S} = \{\swarrow, \leftarrow, \nearrow, \rightarrow\}$$

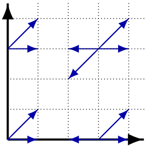
$$\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{0} (0,0)\} = 1$$

$$\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{2n+1} (0,0)\} = 0$$

$$\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{2} (0,0)\} = 2$$

$$\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{4} (0,0)\} = 11$$

$$\#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{6} (0,0)\} = 85$$



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Search: **seq:1,2,11,85**

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Sort: relevance | references | number | modified | created Format: long | short | data

A135404 Gessel sequence: the number of paths of length 2m in the plane, starting and ending at (0,1), with unit steps in the four directions (north, east, south, west) and staying in the region y>0, x>-y. +20 6

1, 2, 11, 85, 782, 8004, 88044, 1020162, 12294260, 152787976, 1946310467, 25302036071, 334560525538, 4488007049900, 60955295750460, 836838395382645, 11597595644244186, 162074575606984788, 2281839419729917410, 32340239369121304038, 461109219391987625316, 6610306991283738684600 (list; graph; refs; listen; history; text; internal format)

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$$Q_{\mathfrak{S}}(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} \#_Q^{\mathfrak{S}}\{(0,0) \xrightarrow{n} (i,j)\} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]]$$

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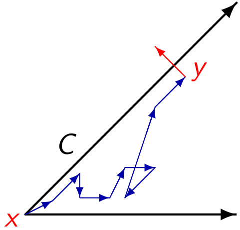
Questions: Given $\mathfrak{S}$, what can be said about $Q(x,y)$?
Structure? (algebraic/D-finite) *Explicit form*? *Asymptotics*?

$Q(0,0) \rightsquigarrow$ counts $\mathfrak{S}$-walks returning to the origin (excursions)
$Q(1,1) \rightsquigarrow$ counts $\mathfrak{S}$-walks with prescribed length

Structure of the series of lectures

Counting numbers in cones $C \subset \mathbb{Z}^d$

- *Excursions* from x to y:
  $\#_C^{\mathfrak{S}}\{x \xrightarrow{n} y\}$
  Probability: *local limit theorem*



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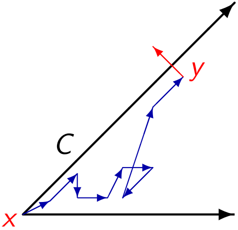

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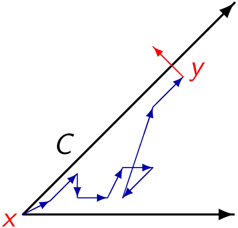

A rich asymptotic behavior

- $\#^{\mathfrak{S}}_{C}\{x \xrightarrow{n} C\} \sim \kappa \cdot V(x) \cdot \rho^n \cdot n^{-\alpha}$

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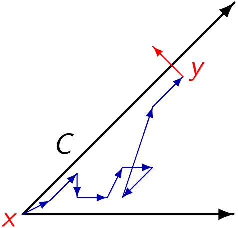

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ρ: exp. growth $\rightsquigarrow$ Mon.
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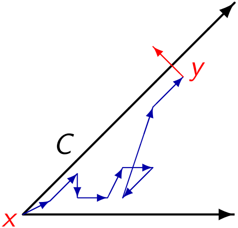

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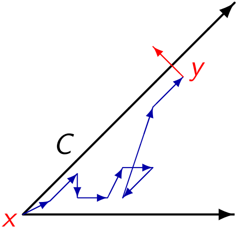

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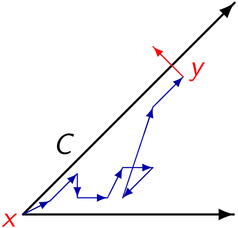

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Simple remark: V, ρ, α depend on C & $\mathfrak{S}$ — hence also on d

Existing results on the exponential growth (1/3)

1D: the half-line

▷ Known formulas

✎ Banderier & Flajolet '02; Banderier & Wallner '16

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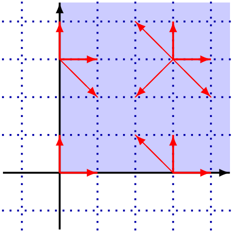
2D: the quarter plane



▷ Numerical conjectures: values of ρ for all quadrant small step models

✎ Bostan & Kauers '08

▷ Proof of the conjectures

✎ Bousquet-Mélou & Mishna '10; Fayolle & R. '12

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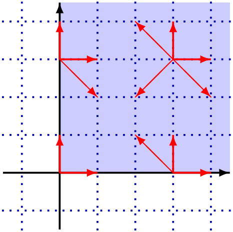

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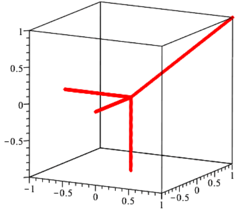
3D: the octant



- ▷ Asymptotic guessing of ρ

  ✎ Bacher, Kauers & Yatchak ’16

Existing results on the exponential growth (2/3)

Combinatorial approach for an upper bound of ρ_Q

- ▷ For any half-plane $Q \subset H$, $\boxed{\rho_Q \leqslant \rho_H}$ ✎ Johnson, Mishna & Yeats '13
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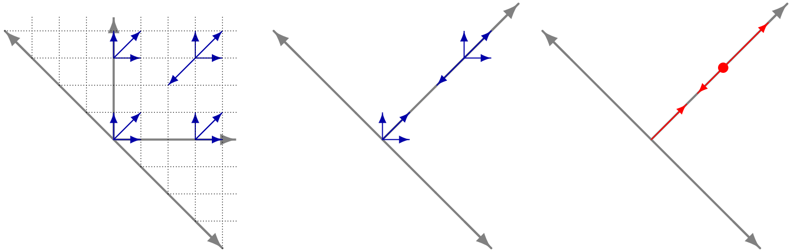

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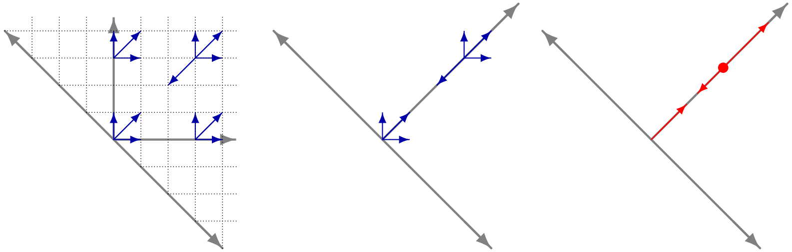

Key observations ✎ Johnson, Mishna & Yeats '13

- ▷ For small step quadrant walks, $\boxed{\rho_Q = \min_{H \supset Q} \rho_H}$
- ▷ There is a *best half-space* $\begin{cases} \text{not necessarily unique} \\ \text{not necessarily } \perp \text{ to } \textit{drift } \sum_{s \in \mathfrak{S}} s \end{cases}$

▷ Let $\mathfrak{S} \subset \mathbb{Z}^d$ such that $\langle \mathfrak{S} \rangle = \mathbb{Z}^d$ and C be any *convex cone*

Laplace transform

▷ *Laplace transform* (exp. gen. function) of $\mathfrak{S}$: $L^{\mathfrak{S}}(x) = \sum_{s \in \mathfrak{S}} e^{\langle x, s \rangle}$

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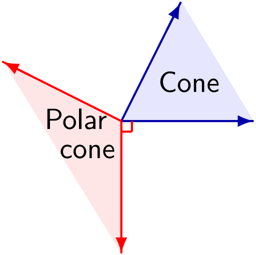
Cones, polar cones & dual cones



▷ Polar cone: $\{x : \langle x, y \rangle \leqslant 0, \forall y \in C\}$

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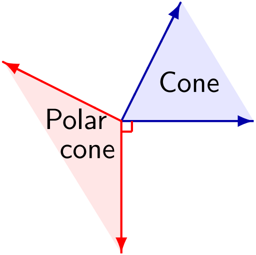

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▷ $\boxed{\rho_C^{\mathfrak{S}} = \min_{\text{dual cone}} L^{\mathfrak{S}}} = L^{\mathfrak{S}}(x_0)$, x_0 *unique* ✎ Garbit & R. '16

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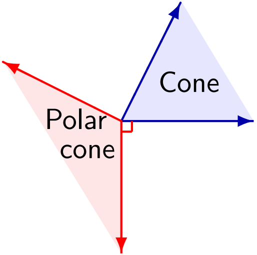

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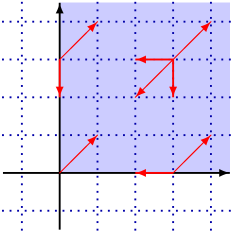
▷ If $\langle \mathfrak{S} \rangle \neq \mathbb{Z}^d$, ρ depends on the starting point ✎ Garbit '16

Miscellaneous examples

A concrete example in the quadrant



- *Minimize* on $\mathbb{R}_+^2$ *the function* $L^{\mathfrak{S}}(x,y) = e^{x+y} + e^{-x} + e^{-y} + e^{-x-y}$
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- ρ *algebraic number* of degree 4

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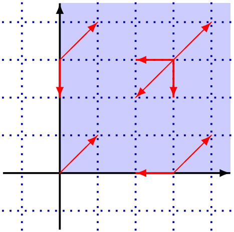

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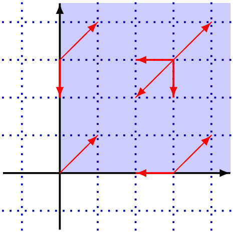

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Drift in the cone C and maximal exponential growth

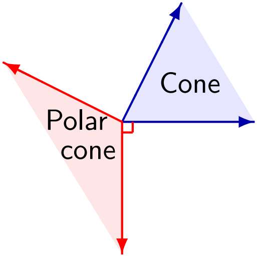
- If *drift* $\sum_{s\in\mathfrak{S}} s \in C$ (possibly 0), then $x_0 = 0$ and

$$\boxed{\rho_C^{\mathfrak{S}} = L^{\mathfrak{S}}(0) = \#\mathfrak{S}}$$

$\rightsquigarrow$ law of large numbers

The half-plane identity

Apparition of the polar & dual cones



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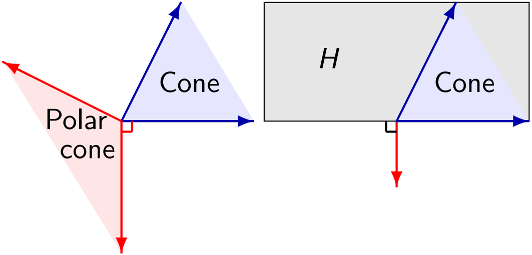

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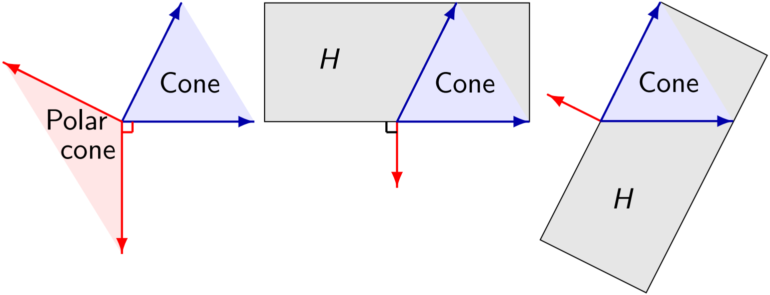

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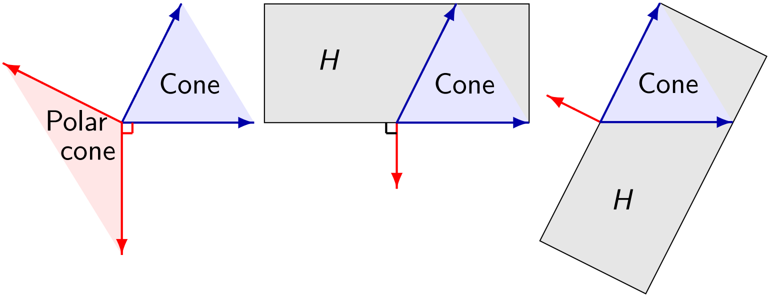

A three-line proof of the half-plane identity *(any convex cone)*

- ▷ $\boxed{\rho_C \leqslant \min_{H \supset C} \rho_H}$ ✎ Johnson, Mishna & Yeats '13
- ▷ $\boxed{\rho_C = \min_{x \in \text{dual cone}} L^{\mathfrak{S}}(x)}$ ✎ Garbit & R. '16
- ▷ For $H = x^{\perp}$, $\rho_H = L^{\mathfrak{S}}(x)$ ✎ Banderier & Flajolet '02

Exponential growth of the excursions

Obvious remark

- $\#_C^{\mathfrak{S}}\{x \xrightarrow{n} y\} \leqslant \#_C^{\mathfrak{S}}\{x \xrightarrow{n} C\}$

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▷ $\#_C^{\mathfrak{S}}\{x \xrightarrow{n} y\} \leqslant \#_C^{\mathfrak{S}}\{x \xrightarrow{n} C\} \Longrightarrow \boxed{\rho(\text{excursions}) \leqslant \rho}$

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- $\rho(\text{excursions}) = \min_{\mathbb{R}^d} L^{\mathfrak{S}} \leqslant \min_{\text{dual cone}} L^{\mathfrak{S}} = \rho$

  ✎ Iglehart '74; Garbit '08; Denisov & Wachtel '15

- Proof: Cramér's transform

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Reluctant case

Case $\boxed{\rho(\text{excursions}) = \rho}$ is called *reluctant*. Results:

- Random generation ✎ Lumbroso, Mishna & Ponty '16
- Exact asymptotics ✎ Duraj '14
- Intuitively: a typical walk is located not far way from the origin
- 1D: non-positive drift

Reformulation of the problem: probability theory

General hypotheses

- Finite *step set* $\mathfrak{S} \subset \mathbb{Z}^d$ $(d \geqslant 1)$
- *Convex cone* $C \subset \mathbb{R}^d$

Reformulation of the problem: probability theory

General hypotheses

- Finite *step set* $\mathfrak{S} \subset \mathbb{Z}^d$ $(d \geqslant 1)$
- *Convex cone* $C \subset \mathbb{R}^d$

RW with steps in $\mathfrak{S}$

Let $\{S(n)\}_{n \geqslant 0}$ be a *RW* whose increments have the *uniform law in* $\mathfrak{S}$, i.e.,

- $S(n) = x + X(1) + \cdots + X(n)$, where the $X(i)$ are i.i.d.
- $\mathbb{P}[X(i) = s] = \dfrac{1}{\#\mathfrak{S}}$

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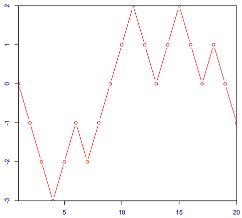
Main equation combinatorics/probability

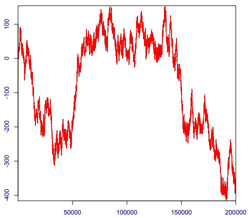
- $$\boxed{\mathbb{P}^x[\tau > n] = \frac{\#\{x \xrightarrow{n} C\}}{(\#\mathfrak{S})^n}}$$

Why introducing this probability?

Technical reasons

- Possibility of using *Brownian motion* and *Donsker theorem*





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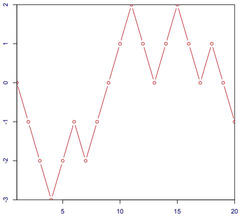

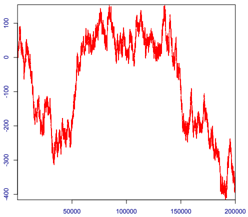

Motivations in other fields than combinatorics

- ▷ Links with *representation theory*
- ▷ *Conditioned RW in cones* (quantum RW, random matrices, non-colliding RW, etc.)
- ▷ More details to come tomorrow

Zero drift case $\sum_{s\in\mathfrak{S}} s = 0$

Non-exponential decay

✎ Garbit '07

If drift $\sum_{s\in\mathfrak{S}} s = 0$ ($\iff \mathbb{E}[X(i)] = 0$) then

$$\boxed{\rho = 1} \quad \iff \quad \lim_{n\to\infty} \mathbb{P}^{x}[\tau > n]^{1/n} = 1$$

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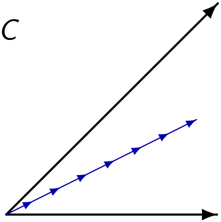

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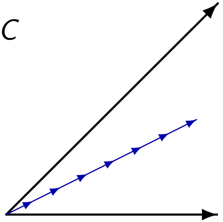

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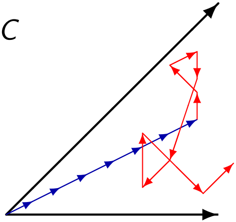

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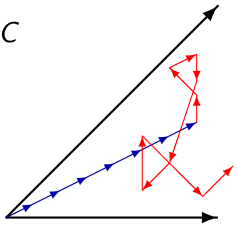

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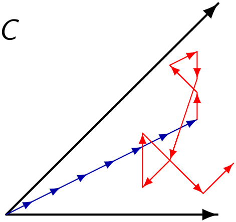

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Non-zero drift $\sum_{s \in \mathfrak{S}} s \neq 0$ (1/2)

Exponential change of measure (Girsanov or Cramér)

$$\mathbb{P}[X(i) = s] = \frac{1}{\#\mathfrak{S}} \qquad \longrightarrow \qquad \boxed{\mathbb{P}[X(i) = s] = \frac{e^{\langle s,x \rangle}}{L^{\mathfrak{S}}(x)}}$$

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$$\mathbb{P}^y[\tau>n]=\left(\frac{L^{\mathfrak{S}}(x)}{\#\mathfrak{S}}\right)^n e^{\langle x,y\rangle}\mathbb{E}^y[\tau>n,e^{-\langle x,S(n)\rangle}] \qquad (\forall x)$$

$$\leqslant\left(\frac{L^{\mathfrak{S}}(x)}{\#\mathfrak{S}}\right)^n e^{\langle x,y\rangle} \qquad (\forall x:\langle x,S(n)\rangle\geqslant 0)$$

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An easy upper bound

$$\limsup_{n\to\infty} \mathbb{P}^y[\tau > n]^{1/n} = \frac{\rho}{\#\mathfrak{S}} \leqslant \frac{\min_{\text{dual cone}} L^{\mathfrak{S}}}{\#\mathfrak{S}}$$

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Lower bound (reminder)

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A simple case

If *global minima* of $L^{\mathfrak{S}}$ on $\mathbb{R}^d$ reached on the dual cone (say at x_0):

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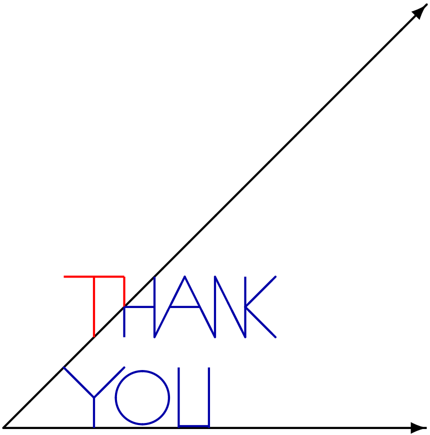
General case

Minimum on Q at $x_0 = (x_0^{(1)}, \dots, x_0^{(d)})$:

$$\frac{\partial L^{\mathfrak{S}}(x_0)}{\partial x_i} \begin{cases} \geqslant 0 & \forall i \\ = 0 & \forall i \text{ such that } x_0^{(i)} > 0 \end{cases}$$

A few open questions

- Find the *exact asymptotics* (not only the exponential growth)
  $\rightsquigarrow$ come tomorrow!
- Existence of the Laplace transform $\Longrightarrow$ *exponential moments*
  What about weighted step sets without exponential moments?
  (typically, L^2-moments)
- *Variations* of the models (e.g., lattice paths with catastrophes)
  ✎ Banderier & Wallner '16



cnrs

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