Random walks in cones: critical exponents

Lecture #2

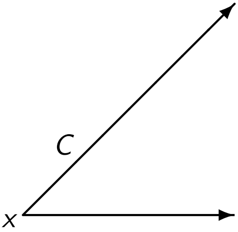
Analytic and probabilistic tools for lattice path enumeration

Kilian Raschel

77th Séminaire Lotharingien de Combinatoire
September 13, 2016
Strobl, Austria

Random processes (RW & BM) in cones

First exit time from a cone C



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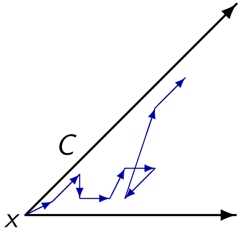

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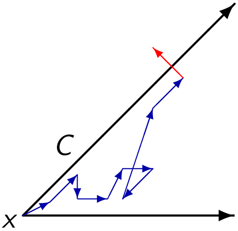

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- $\tau_C = \inf\{n \in \mathbf{N} \ : S(n) \notin C\}$ (S RW)
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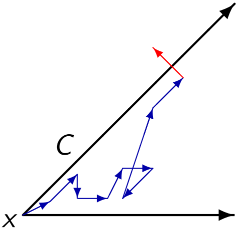

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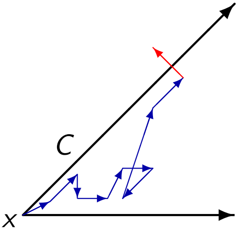

Persistence probabilities $\rightsquigarrow$ total number of walks

- $\mathbf{P}_x[\tau_C > n] \sim \kappa \cdot V(x) \cdot \rho^n \cdot n^{-\alpha}$

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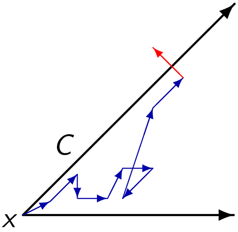

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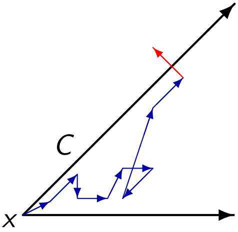

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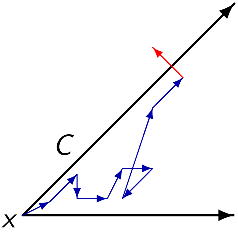

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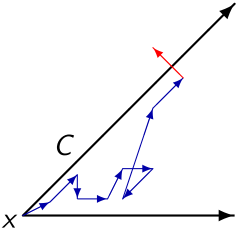

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- $\mathbf{P}_x[\tau_C > n, S(n) = y] \sim \kappa \cdot V(x,y) \cdot \rho^n \cdot n^{-\alpha}$

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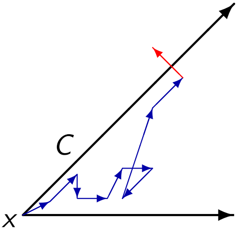

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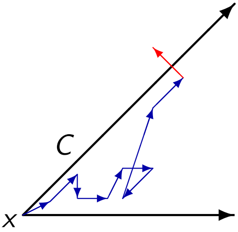

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Aim of the talk: understanding the critical exponents α

Definition of random walks & motivations

Random walk on $\mathbf{Z}^d$

- A *random walk* $\{S(n)\}_{n\geqslant 0}$ is

$$S(n) = x + X(1) + \cdots + X(n),$$

where the $X(i)$ are i.i.d.

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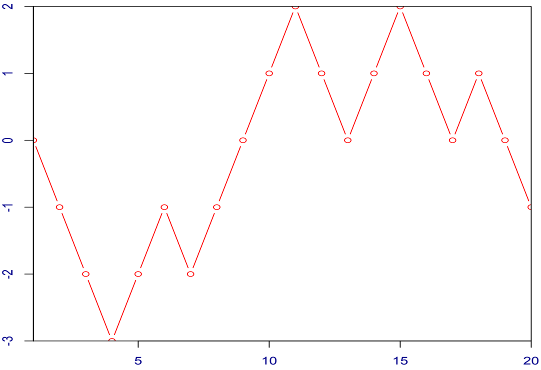

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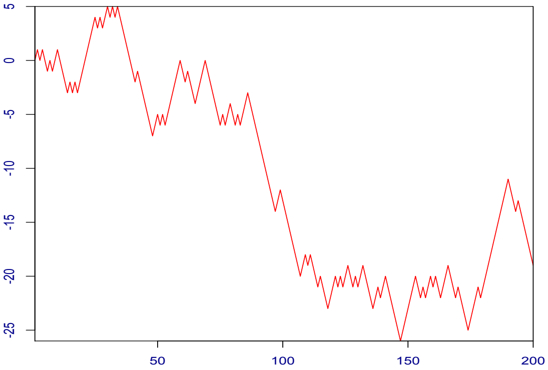

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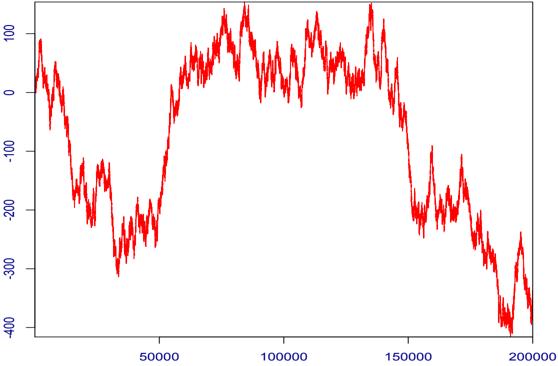

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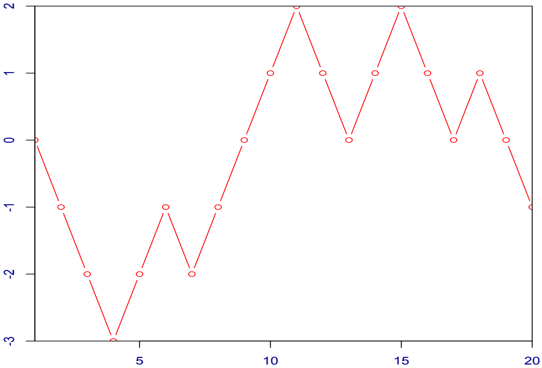
Motivations

- *Persistence probabilities* in statistical physics
- Constructing *processes conditioned* never to leave cones
- *Asymptotics* of numbers of walks
- *Transcendental nature* of functions counting walks in cones
$\rightsquigarrow$ Alin Bostan's course at AEC
- Important & combinatorial cones (quarter/half/slit plane, orthants, Weyl chambers, etc.)

Non-constrained walk with $\mathfrak{S} = \{-1, 1\}$



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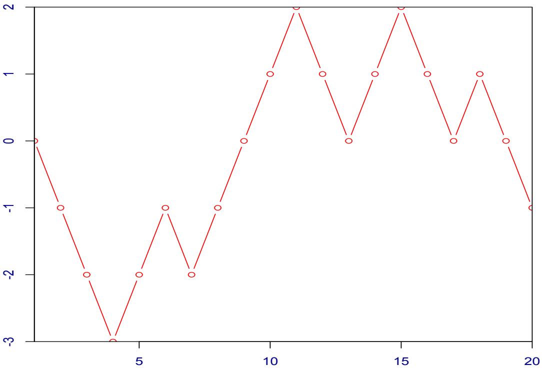

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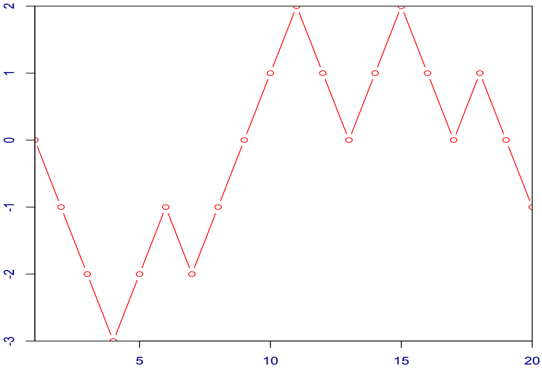

- $\#\{x \xrightarrow{n} \mathbf{Z}\} = 2^n$ Walk $\rightsquigarrow$ Exponent 0
- $\#\{x \xrightarrow{n} y\} = \binom{n}{\frac{n+(y-x)}{2}} \sim \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$ Bridge $\rightsquigarrow$ Exponent $\frac{1}{2}$

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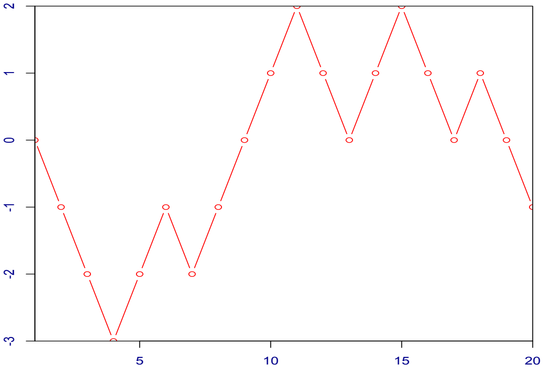

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- $\sum \frac{1}{\sqrt{n}} = \infty$: *recurrence* of the simple random walk in $\mathbf{Z}$

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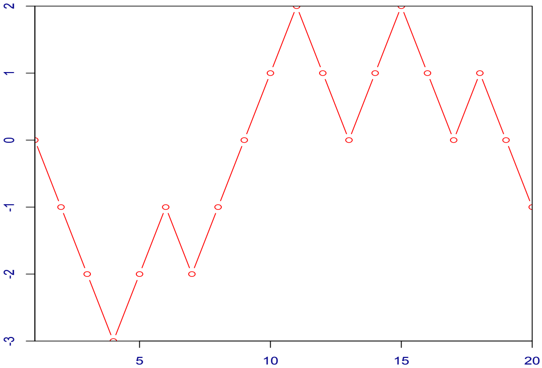

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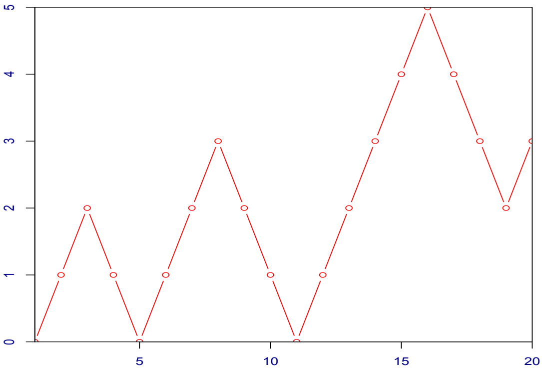
- Constant $\sqrt{\frac{2}{\pi}}$ *independent of x & y* in the asymptotics

Constrained walk with $\mathfrak{S} = \{-1, 1\}$ (Dyck paths)



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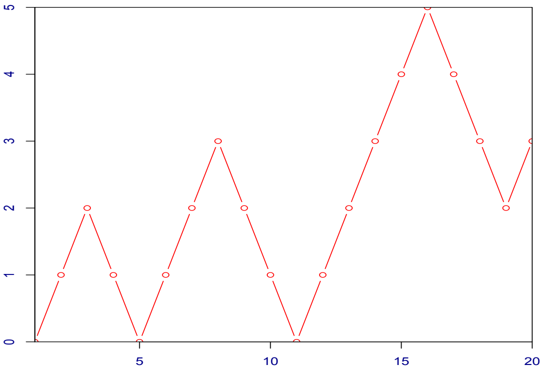

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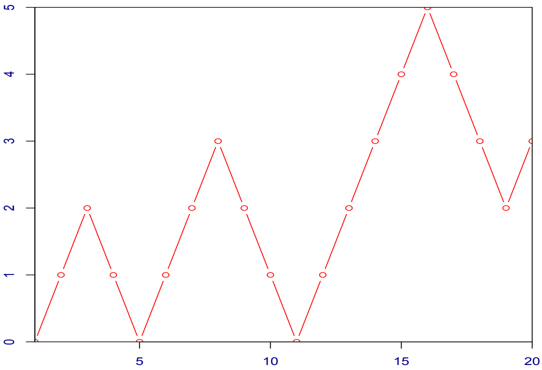

- ▷ $\#_{\mathbf{N}}\{x \xrightarrow{n} \mathbf{N}\} \sim \dfrac{2^n}{\sqrt{n}}$ — Meanders ⇝ Exponent $\dfrac{1}{2}$
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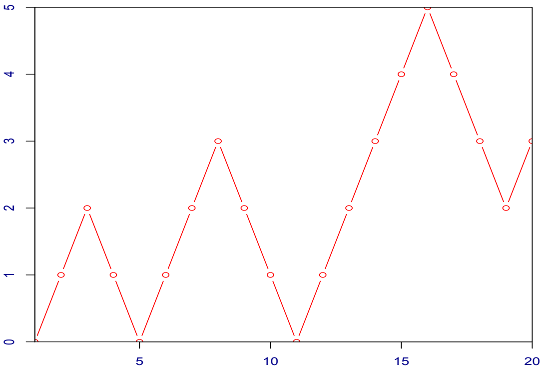

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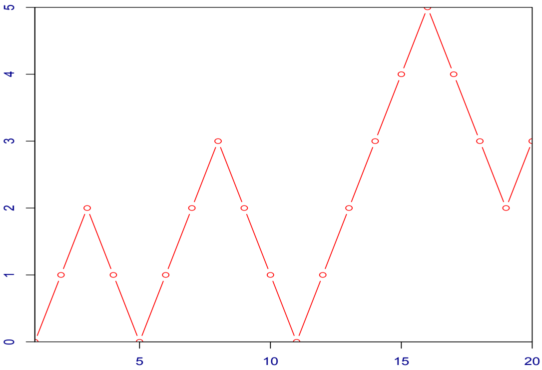

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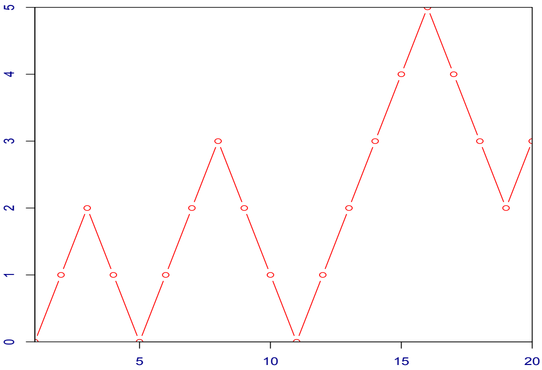

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- *Wiener-Hopf* techniques in probability theory
- See ✎ Bousquet-Mélou & Petkovšek '00; Banderier & Flajolet '02

Beyond the algebraic exponents 0, $\frac{1}{2}$ & $\frac{3}{2}$

Weighted models in dimension 1

Drift $\sum_{s\in\mathfrak{S}} s$ governs the exponents, which are still 0, $\frac{1}{2}$ & $\frac{3}{2}$

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The simple walk in two-dimensional wedges

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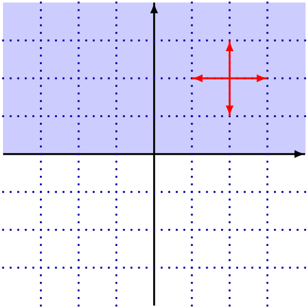

- Half-plane:
  one-dimensional case
- Dyck paths
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{1}{2}$
- Excursions:
  $\rightsquigarrow$ Exponent $2 = \frac{3}{2} + \frac{1}{2}$

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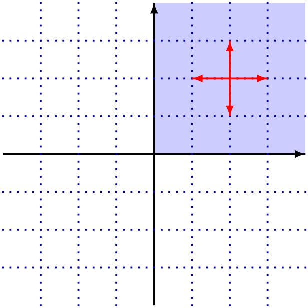

- Quarter plane: product of two one-dimensional cases
- Reflection principle
- Total number of walks:
  $\rightsquigarrow$ Exponent $1 = \frac{1}{2} + \frac{1}{2}$
- Excursions:
  $\rightsquigarrow$ Exponent 3

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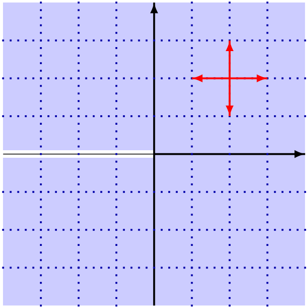

- Slit plane:
  - ✎ Bousquet-Mélou & Schaeffer '00
- Highly non-convex cone
- Total number of walks:
  - ⇝ Exponent $\frac{1}{4}$
- Excursions:
  - ⇝ Exponent $\frac{3}{2}$

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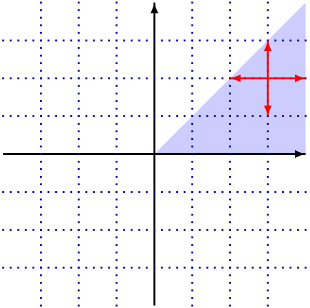

- 45°: ✎ Gouyou-Beauchamps '86
- See
  ✎ Bousquet-Mélou & Mishna '10
- Total number of walks:
  ⇝ Exponent 2
- Excursions:
  ⇝ Exponent 5

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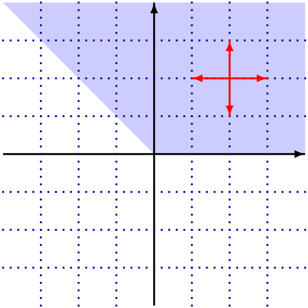

- 135°: Gessel
- See ✎ Kauers, Koutschan & Zeilberger '09, etc.
- Total number of walks:
  ⇝ Exponent $\frac{2}{3}$
- Excursions:
  ⇝ Exponent $\frac{7}{3}$

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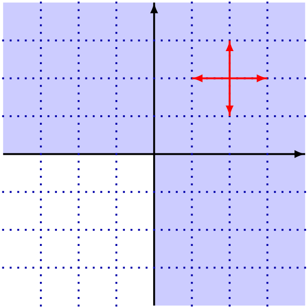


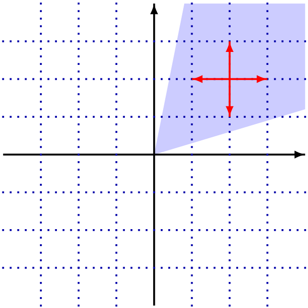
- Walks avoiding a quadrant
- See ✎ Bousquet-Mélou '15; Mustapha '15
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{1}{3}$
- Excursions:
  $\rightsquigarrow$ Exponent $\frac{5}{3}$

Beyond the algebraic exponents $0, \frac{1}{2}$ & $\frac{3}{2}$

Weighted models in dimension 1

Drift $\sum_{s\in\mathfrak{S}} s$ governs the exponents, which are still 0, $\frac{1}{2}$ & $\frac{3}{2}$

The simple walk in two-dimensional wedges



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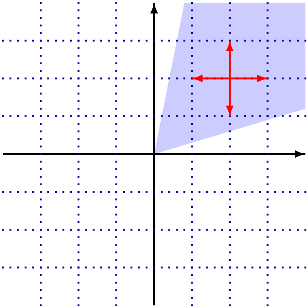

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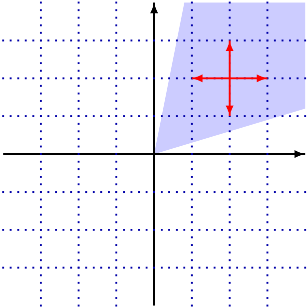

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Conclusion: 1D case not enough

- Dramatic change of behavior: every exponent is possible!
- *Non-D-finite* behaviors (first observed by Varopoulos '99)

Brownian motion on R

Law of large numbers

$$\frac{X(1) + \cdots + X(n)}{n^{1}} \xrightarrow{\text{a.s.}} \mathbf{E}[X(1)]$$

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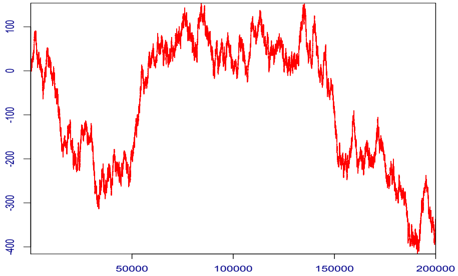

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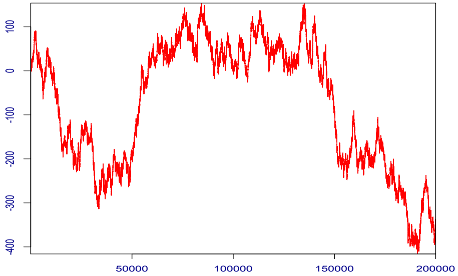

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$$\left\{\frac{S(\lfloor nt \rfloor)}{\sqrt{n}}\right\}_t \xrightarrow{\text{law}} \{B(t)\}_t$$

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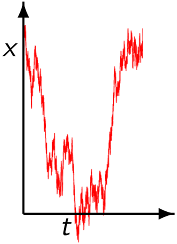
Remainder of this section: computing $\alpha\{\text{BM}\}$ (easier)

Two derivations of the BM persistence probability in R

Reflection principle



$$
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\mathbf{P}_x[T_{(0,\infty)} > t] &= \mathbf{P}_0[\min_{0 \leqslant u \leqslant t} B(u) > -x] \\
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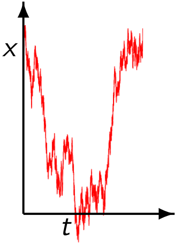

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Heat equation

Function $g(t;x) = \mathbf{P}_x[T_{(0,\infty)} > t]$ satisfies

$$\begin{cases}
\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) g(t;x) = 0, & \forall x \in (0,\infty),\ \forall t \in (0,\infty) \\
g(0;x) = 1, & \forall x \in (0,\infty) \\
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Dimension d: explicit expression for $\mathbf{P}_x[T_C > t]$

Heat equation

✎ Doob '55

For essentially *any domain* C in *any dimension* d, $\mathbf{P}_x[T_C > t]$ & $p^C(t; x, y)$ ($\mathbf{P}_x[T_C > t] = \int_C p^C(t; x, y) \mathrm{d}y$) satisfy *heat equations*

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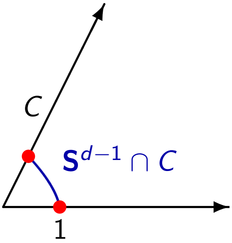
Dirichlet eigenvalues problem

✎ Chavel '84



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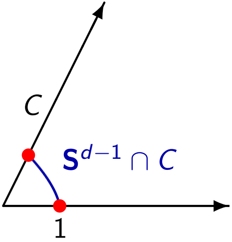

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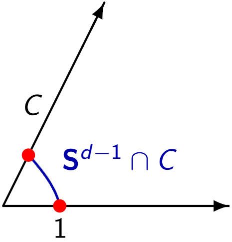

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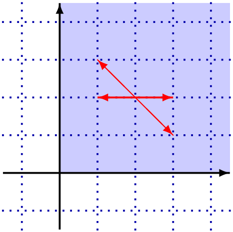
Exercise

Recover the exponent $\frac{\pi}{2\theta}$ of the persistence probability for a simple random walk in a two-dimensional wedge of opening angle θ

Example #1: Gouyou-Beauchamps model

In the quarter plane



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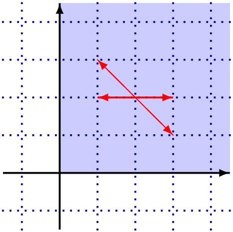

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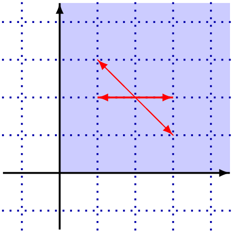

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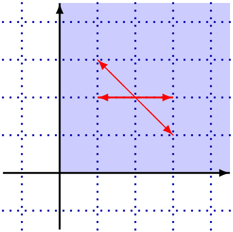

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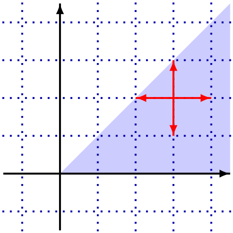

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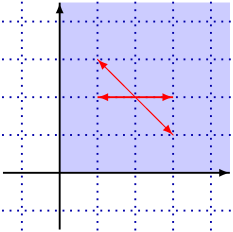

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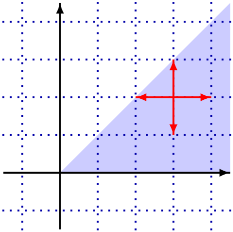

- Wedge of angle $\theta = \frac{\pi}{4}$
- Total number of walks:
  $\rightsquigarrow$ Exponent $\frac{\pi}{2\theta} = 2$
- Excursions:
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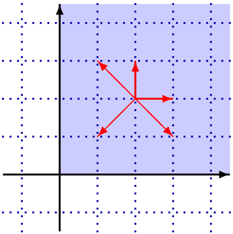
Example #2: quadrant walks

A scarecrow



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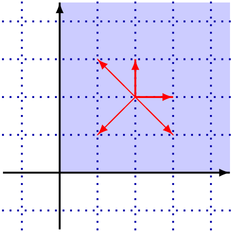

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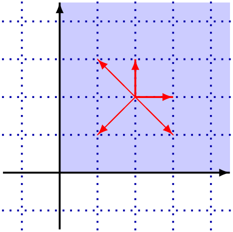

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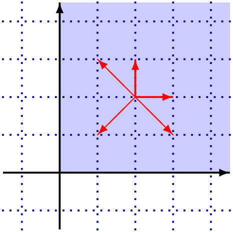

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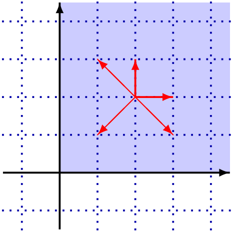

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✎ Bostan, R. & Salvy '14

- Systematic computation of $\alpha = \arccos\{\text{algebraic number}\}$

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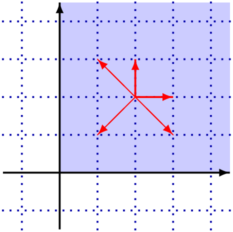

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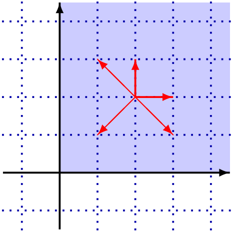
- ▷ Walks with small steps in $\mathbf{N}^2$:
  - ▷ $\alpha \in \mathbf{Q}$ iff
  - ▷ *generating function* of the excursions is D-finite iff
  - ▷ the *group of the model* is finite

Example #2: quadrant walks

A scarecrow



- ▷ $\mathbf{E} = (0,0)$ & $\mathbf{V} = \begin{pmatrix} 4 & -3 \\ -3 & 4 \end{pmatrix} \neq \mathrm{id}$
- ▷ $\theta = \arccos\left(-\frac{1}{4}\right) \Longrightarrow \alpha = \frac{\pi}{\theta} + 1 \notin \mathbf{Q}$
- ▷ $\sum_{n=0}^{\infty} \#_{\mathbf{N}^2}\{(0,0) \xrightarrow{n} (0,0)\} t^n$ non-D-finite

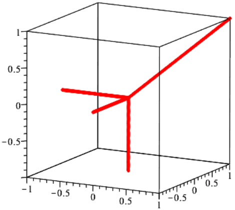
In dimension 2 (excursions only) ✎ Bostan, R. & Salvy '14

- ▷ Systematic computation of $\alpha = \arccos\{\text{algebraic number}\}$
- ▷ Walks with small steps in $\mathbf{N}^2$:
  - ▷ $\alpha \in \mathbf{Q}$ iff
  - ▷ *generating function* of the excursions is D-finite iff
  - ▷ the *group of the model* is finite
- ▷ If $\sum_{s \in \mathfrak{S}} s \neq 0$, first perform a *Cramér transform*

Three-dimensional models

Example: Kreweras 3D

Model with jumps:



Three-dimensional models

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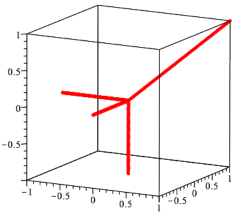

Exponent $\alpha = 2\sqrt{\lambda_1 + \frac{1}{4}} - \frac{1}{2}$

Three-dimensional models

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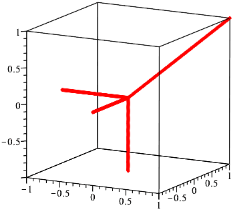

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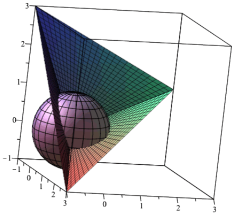

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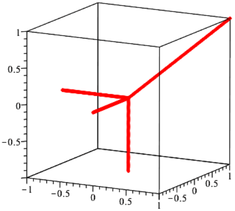

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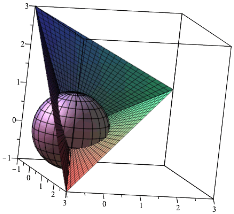

Value of λ_1? $\lambda_1 \in \mathbf{Q}$?

Three-dimensional models

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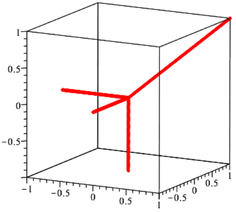

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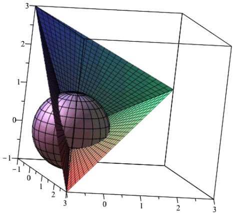


Value of λ_1? $\lambda_1 \in \mathbf{Q}$?

General theory (still to be done!)

- ▷ Classification & resolution of some finite group models

  ✎ Bostan, Bousquet-Mélou, Kauers & Melczer '16

- ▷ Asymptotic simulation ✎ Bacher, Kauers & Yatchak '16

  ⇝ Conjectured Kreweras exponent: 3.3257569

- ▷ Equivalence finite group iff D-finite generating functions?

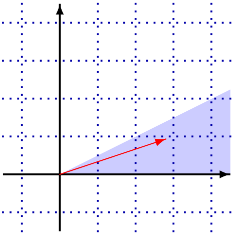
Non-universal exponents: six cases

Excursions: formula for α independent of the drift $\sum_{s\in\mathfrak{S}} s$

Non-universal exponents: six cases

Excursions: formula for α independent of the drift $\sum_{s\in\mathfrak{S}} s$

Case #1: interior drift



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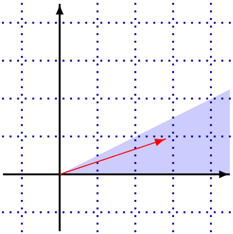

- Law of large numbers: $\mathbf{P}[\forall n, S(n) \in C] > 0$
- Exponent $\alpha = 0$

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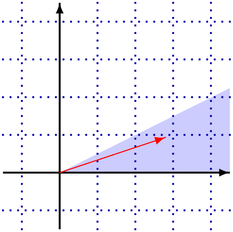

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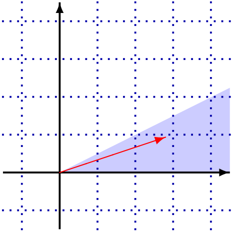

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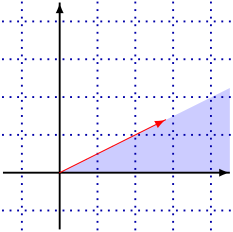

- Half-plane case
- Exponent $\alpha = \frac{1}{2}$

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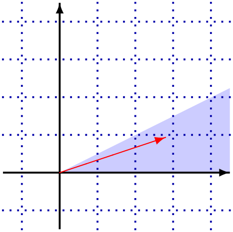

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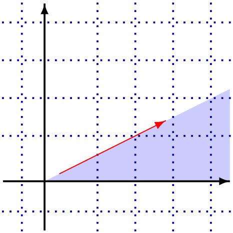

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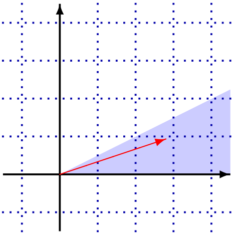

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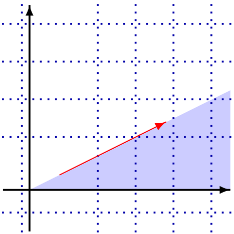

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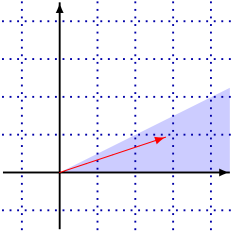

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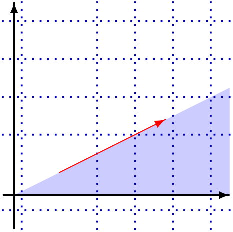

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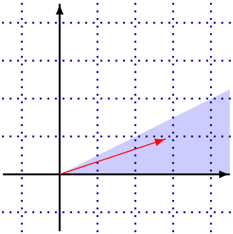

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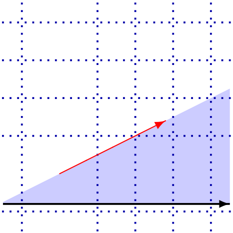

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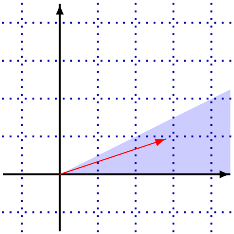

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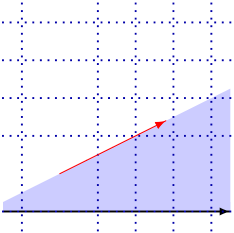

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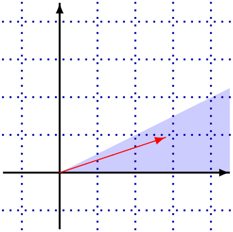

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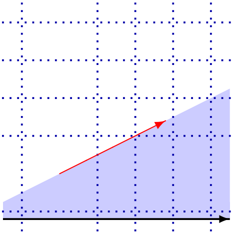

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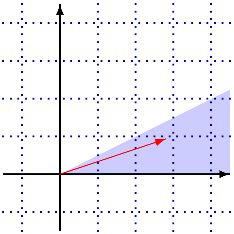

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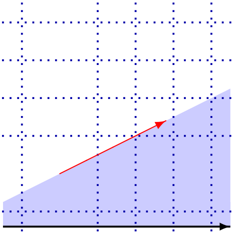

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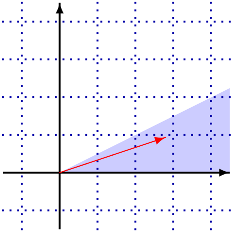

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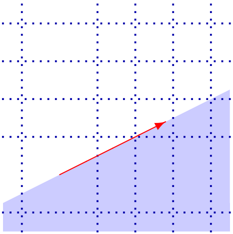

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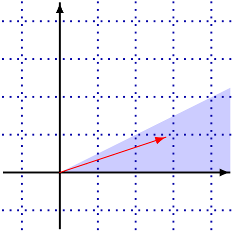

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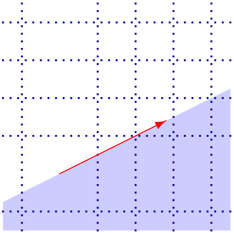
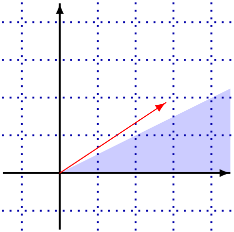


- Half-plane case
- Exponent $\alpha = \frac{1}{2}$
- *Cannot* be used as a filter to detect non-D-finiteness
- Exponent $\alpha = \frac{i}{2}$ for non-smooth boundary

Non-universal exponents: six cases

Case #3: directed drift



- Half-plane case
- Exponent $\alpha = \frac{3}{2}$
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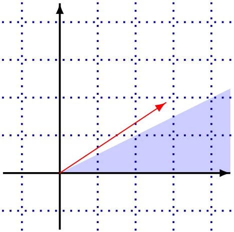
Non-universal exponents: six cases

Case #3: directed drift



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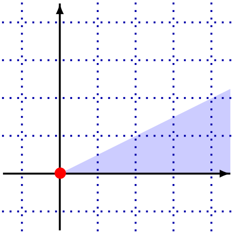
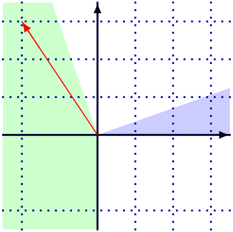
Case #4: zero drift



- See ✎ Varopoulos '99; Denisov & Wachtel '15
- Exponent $\alpha_1 = 2\sqrt{\lambda_1 + (\frac{d}{2} - 1)^2} - (\frac{d}{2} - 1)$
- *Can* be used as a filter to detect non-D-finiteness

Non-universal exponents: six cases

Case #5: polar interior drift



- See ✎ Duraj '14
- Exponent $2\alpha_1 + 1$
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Non-universal exponents: six cases

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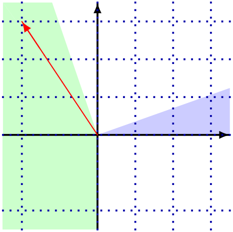

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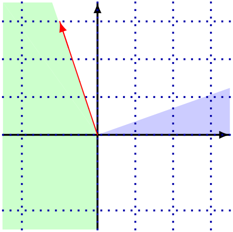
- Exponent $2\alpha_1 + 1$
- *Can* be used as a filter to detect non-D-finiteness

Case #6: polar boundary drift



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Non-universal exponents: six cases

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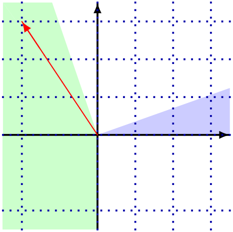

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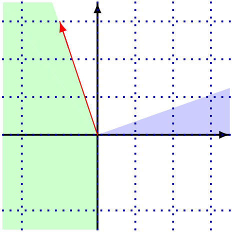

- Exponent $\alpha_1 + 1$
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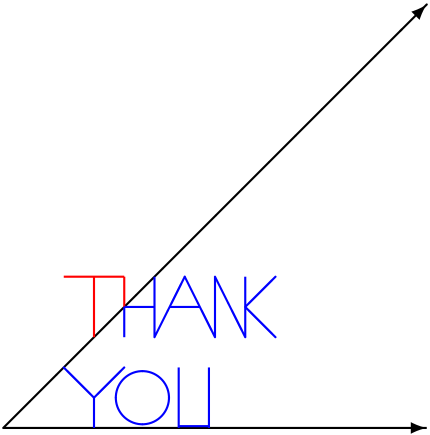
Six-exponents-result: joint with R. Garbit & S. Mustapha



cnrs

LMPT Laboratoire de Mathématiques et Physique Théorique

fdp Fédération **Denis Poisson**
Orléans - Tours