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BINOMIAL SPECIES AND COMBINATORIAL EXPONENTIATION

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Abstract. We introduce a "binomial" species, $B(X,Y) = (1+X)^{\uparrow}Y = E(Y\,\mathrm{Lg}(1+X))$, where $E(X)$ is the species of finite sets and $\mathrm{Lg}(1+X)$ is the combinatorial logarithm. The expansion of B includes, by specialization of variables, the classical binomial expansion, binomial expansions for symmetric functions, and (q,t)-series. We also define and study a new exponentiation operation, $F^{\uparrow}G$, between species.

1. Introduction

The present paper is written under the framework of the theory of combinatorial species of structures founded in 1981 by André Joyal [Joy81]. We assume that the reader already possesses a minimal knowledge concerning "ordinary" species. See, for example, the book [BLL98] for basic concepts, results and early references about species. For completeness and to help the reader, we recall the notion of a multisort weighted species on variables $X, Y, \ldots$, called *sorts* and variables $u, v, \ldots$, called *weight counters*.

Informally speaking, a *multisort weighted species* is a class $F = F(X, Y, \ldots)$ of weighted structures built on arbitrary finite sets of elements of sorts $X, Y, \ldots$. Each structure is given a weight in the form of a power product $u^i v^j \cdots$ in the weight-counter variables. The class F must be closed under relabelling its structures along bijections between the sets of their underlying elements that preserve sorts and weights. A structure in the class F is called an F-structure, for short.[1]

In the special case where the weight of each F-structure is $1 = u^0 v^0 \cdots$ (the trivial monomial), F is called an *ordinary multisort species*.

Many operations between species have been defined in order to describe or recursively define various species. The main operations on species include *sum* $(+)$, *difference* $(-)$, *Cauchy* or *juxtaposition product* $(\cdot)$, *Hadamard* or *superposition product* $(\times)$, *division* $(/)$,

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[1] Algebraically speaking, a *species* is simply a functor of the form $F : \mathbf{S}\times\mathbf{S}\times\cdots \longrightarrow \mathbf{S_w}$, where $\mathbf{S}$ is the category of finite sets and bijections and $\mathbf{S_w}$ is a category of finite weighted sets and weight-preserving functions between them. In the Cartesian product $\mathbf{S}\times\mathbf{S}\times\cdots$, an object U in the first factor is interpreted as a finite set of elements of sort X, an object of the second factor is interpreted as a finite set of elements of sort Y, etc. Given $[U, V, \ldots]$ in $\mathbf{S}\times\mathbf{S}\times\cdots$, an element $s \in F[U, V, \ldots]$ is called an F-structure on the disjoint union $U \sqcup V \sqcup \cdots$. When an infinite family of sorts is given, the Cartesian product $\mathbf{S}\times\mathbf{S}\times\cdots$ is interpreted in the *weak* sense, i.e., the objects $[U, V, \ldots]$ are such that $U \sqcup V \sqcup \cdots$ is finite.

substitution[2] $(\circ)$, *partial derivations* $(\partial/\partial X, \partial/\partial Y, \dots)$.

Example 1.1. Figure 1 shows a G-structure belonging to the two-sort weighted species $G = G(X,Y)$ of simple graphs on finite sets made of black nodes (sort X) and white nodes (sort Y). The weight of a graph being given by

$$u^{\#\text{connected black components}} v^{\#\text{connected white components}} t^{\#\text{connected mixed components}}. \tag{1.1}$$

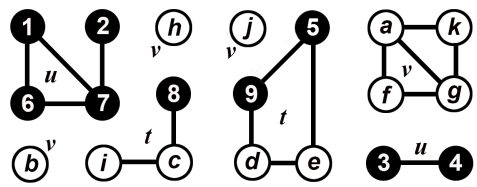

FIGURE 1. A $G(X,Y)$-structure on $\{1,\dots,9\} \sqcup \{a,\dots,k\}$ having weight $u^2v^4t^2$.

Example 1.2. Let $L = L(X)$ and $C = C(X)$ be the species of finite *linear orders* and (nonempty) *oriented cycles*. Figure 2 describes the fact that any oriented cycle made on a set of black nodes (X-structures) $\sqcup$ a set of white nodes each having weight u (uY-structures) can be naturally viewed as either an oriented cycle made of black nodes only ($C(X)$-structure) or an oriented cycle made of weighted white nodes each of which being followed by a linearly ordered set of black nodes ($C(uYL(X))$-structure).

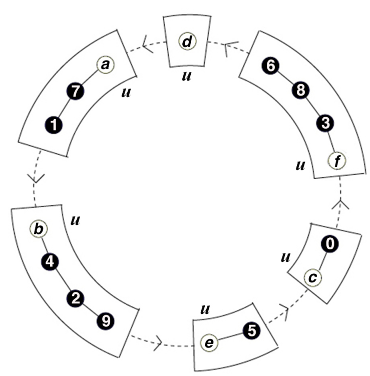

FIGURE 2. The combinatorial equation $C(X + uY) = C(X) + C(uYL(X))$.

Example 1.3. Let $E = E(X)$ be the species of finite sets[3] and $A = A(X)$ be the one-sort unweighted species of *arborescences* (= rooted trees). Figure 3 shows that the species $A(X)$ can be recursively defined by the combinatorial equation

$$A(X) = XE(A(X)) \tag{1.2}$$

[2]If $F = F(X)$ is 1-sort, $F \circ G$ is also written $F(G)$. If $F = F(X,Y,\dots)$, we usually write $F(G,R,\dots)$ instead of $F \circ (G,R,\dots)$. Also, $F \cdot G$ is generally written in the form FG.

[3]E is the first letter of the french word *ensembles* which means *sets*.

since any rooted tree can be naturally viewed as a root (i.e., an X-structure) followed by a set of rooted tree (i.e., an $E(A(X))$-structure). In this figure, the underlying set is $U = \{0, 1, \ldots, 9, a, b, \ldots, k\}$.

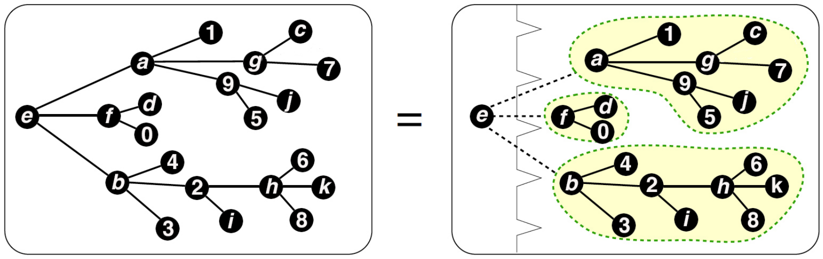

FIGURE 3. The combinatorial recursive definition $A(X) = XE(A(X))$.

Example 1.4. Given a species $F = F(X, Y, \ldots)$, the derivative species $\frac{\partial}{\partial X}F(X, Y, \ldots)$ is defined as follows: s is a $\frac{\partial}{\partial X}F$-structure on $U \sqcup V \sqcup \cdots$ if and only if s is an F-structure on $(U \sqcup \{\bullet\}) \sqcup V \sqcup \cdots$ where $\bullet$ is an *unlabelled* element of sort X outside U. The weight of s, as a $\frac{\partial}{\partial X}F$-structure, is that of s, as an F-structure[4]. Take, for example, the species $\Phi_u = \Phi_u(X, Y)$ whose structures are *functions* from finite sets of black elements (X-structures) to finite sets of white elements (Y-structures), the weight of a function being given by $u^{\#\text{non empty fibers of } f}$. Figure 4 shows that $\frac{\partial}{\partial X}\Phi_u(X, Y) = uE(X)Y\Phi_u(X, Y)$ (the underlying set of the $\frac{\partial}{\partial X}\Phi_u$-structure in this case is $\{a, b, \ldots, g\} \sqcup \{1, 2, \ldots, 5\}$, and the weight is u^3). One can check that the following combinatorial differential equality also holds: $\frac{\partial}{\partial Y}\Phi_u(X, Y) = \Phi_u(X, Y) + uE_+(X)\Phi_u(X, Y)$, where $E_+(X)$ is the species of nonempty finite sets of elements of sort X.

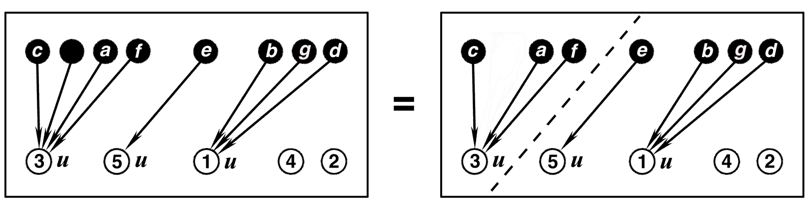

FIGURE 4. The combinatorial equation $\frac{\partial}{\partial X}\Phi_u(X, Y) = uE(X)Y\Phi_u(X, Y)$.

We include/recall in Subsections 1.1–1.3 of this introduction some more advanced material and special notational conventions about species that will be used later: *combinatorial power series versus species*, *underlying formal power series*, *substitution of power series into species*. Subsection 1.4 contains an overview of the main items of the remaining Sections 2–4 of the paper: the tools of *combinatorial logarithm* and *pseudo-singletons*, *binomial species*, *generalized binomial coefficients*, and a new operation of *exponentiation between species*. Appendix A recalls the substitution formulas for weighted species. Appendix B discusses formal summability for families of species.

[4]Similarly, $\frac{\partial}{\partial Y}F$ is defined by adding an extra unlabelled element $\circ$ of sort Y on the set V, etc.

1.1. **Encoding species by combinatorial power series.** Yeong-Nan Yeh has shown in [Yeh86] that weighted multisort species $F = F(X, Y, \dots)$ with weight counters $u, v, \dots$ can conveniently be encoded by *combinatorial power series*. These are series of the form

$$\sum_{n,k,\dots,H} f_{n,k,\dots,H} X^n Y^k \cdots /H, \tag{1.3}$$

where, for *any* tuple $(n, k, \dots)$ of integers, H runs through a system of representatives of the conjugacy classes of the Young subgroup $S_{n,k,\dots}$ of $S_{n+k+\cdots}$ in $S_{n+k+\cdots}$,[5] and

$$f_{n,k,\dots,H} = f_{n,k,\dots,H}(u, v, \dots) \in \mathbb{C}[[u, v, \dots]] \tag{1.4}$$

are formal power series with complex coefficients in the weight variables $u, v, \dots$.

In fact, the subgroups H are taken (up to conjugacy) as the stabilizers of the F-structures (up to isomorphism) built on the multi-sorted set $[n] \sqcup [k] \sqcup \cdots$ (disjoint union), where $[n] = \{1, 2, \dots, n\}$. The elements of the summands $[n], [k], \dots$ in this disjoint union are interpreted as singletons of sorts $X, Y, \dots$, respectively. Since automorphisms of structures must be *sort-preserving*, the elements $h \in H$ are sort-preserving permutations of $[n] \sqcup [k] \sqcup \cdots$, that is, elements of $S_{n,k,\dots}$. Since $S_{n,k,\dots} \cong S_n \times S_k \times \cdots$, the elements $h \in H$ will be written in the form $h = (h_1, h_2, \dots)$, where $h_1 \in S_n, h_2 \in S_k, \dots$.

An alternate form for the combinatorial power series (1.3) puts emphasis on the individual terms in its full expansion. It can be written as

$$\sum_{\mu,H} c_{\mu,H}\, \mu\, X^n Y^k \cdots /H, \tag{1.5}$$

where $\mu = u^i v^j \cdots$ runs through all power products of the variables $u, v, \dots$, and the coefficients $c_{\mu,H}$ are complex numbers depending on μ and H. Note that H determines $n, k, \dots$ in (1.5) and that $i + j + \cdots + n + k + \cdots < \infty$ in each term.

Under this setting, (1.3) (or (1.5)) is called by Yeh the *molecular expansion* of the species F into its *molecular* (i.e., irreducible) components $X^n Y^k \cdots /H$. The coefficients $f_{n,k,\dots,H} \in \mathbb{N}[[u, v, \dots]]$ in (1.3) are power series with nonnegative integer coefficients describing the family of weights assigned to F-structures (up to isomorphism) whose stabilizer is conjugate to H and the coefficients $c_{\mu,H}$ in (1.5) are nonnegative integers (see Example 1.5 below in the case of a 1-sort weighted species).

Two species $F(X, Y, \dots)$ and $G(X, Y, \dots)$ are naturally equivalent (as functors) if and only if they have the same expansion (1.3) or (1.5). In this case, we say that F and G are *combinatorially equal* and simply write $F = G$.

Allowing negative integral coefficients in $f_{n,k,\dots,H}$ in (1.3) (that is, $f_{n,k,\dots,H} \in \mathbb{Z}[[u, v, \dots]]$), or negative integral coefficients $c_{\mu,H}$ in (1.5) (that is, $c_{\mu,H} \in \mathbb{Z}$), we are led to the notion of weighted *virtual* species in the sense of Joyal [Joy85]. These are formal differences, $F - G$, between weighted species F and G.

[5] Two combinatorial monomials $X^n Y^k \cdots /H$ and $X^{n'} Y^{k'} \cdots /H'$ are considered as equal (or similar) if $n = n', k = k', \dots$ and H, H' are conjugate in $S_{n,k,\dots}$. Hence, similar terms are collected in (1.3) and (1.5).

The above main operations on species, $+$, $-$, $\cdot$, $\times$, $/$, $\circ$, $\partial/\partial X, \partial/\partial Y, \ldots$, have all been extended by Joyal [Joy85] and Yeh [Yeh86] to allow complex coefficients in $f_{n,k,\ldots,H}$ (that is, $f_{n,k,\ldots,H} \in \mathbb{C}[[u,v,\ldots]]$) in (1.3).[6]

Because of these facts, *any* series of the form (1.3) or (1.5), in any number of variables, $X, Y, \ldots$, will generally be called a species in the present text. The set of species will be denoted, for short, by

$$\mathbb{C}_{u,v,\ldots} \| X, Y, \ldots \|, \qquad \text{where } \mathbb{C}_{u,v,\ldots} = \mathbb{C}[[u,v,\ldots]]. \tag{1.6}$$

Since $X^nY^k\cdots/\{id_{n,k,\ldots}\} \cong X^nY^k\cdots$, monomials in the usual sense in $X, Y, \ldots$ are special cases of combinatorial monomials $X^nY^k\cdots/H$. We have the inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{C} \subset \mathbb{C}[[X]] \subset \mathbb{C}[[X,Y,\ldots]] \subset \mathbb{C}_{u,v,\ldots}[[X,Y,\ldots]] \subset \mathbb{C}_{u,v,\ldots} \| X, Y, \ldots \| . \tag{1.7}$$

This implies that non-negative integers, complex numbers and power series in the usual sense are all special cases of combinatorial (weighted) power series.

Recall that "ordinary" species are elements of $\mathbb{N} \| X, Y, \ldots \|$ and that "ordinary" weighted species are elements of $\mathbb{N}_{u,v,\ldots} \| X, Y, \ldots \|$.

Example 1.5. To illustrate the notion of molecular expansion in the case of ordinary weighted species on one sort, X, of singletons, consider, for example, the species $\mathcal{A}_w = \mathcal{A}_w(X)$ of *arborescences* weighted by

$$w(\text{rooted tree}) = u^{\#\text{internal nodes} \neq \text{root}} v^{\#\text{leaves}}. \tag{1.8}$$

Denote by $X^n = X^n/\{id_n\}$ the species of *linear orders* of length n (where $\{id_n\}$ denotes the trivial subgroup of S_n) and by $E_n = E_n(X) = X^n/S_n$ the species of *n-sets*[7] (that is, linear orders of length n up to an arbitrary permutation of their elements). Figure 5 (in which the labels of the underlying elements have been omitted for greater readability) shows that some of the first terms of the molecular expansion of the species $\mathcal{A}_w$ look as follows:

$$\begin{aligned} \mathcal{A}_w = \mathcal{A}_w(X) = X + vX^2 + v^2XE_2 + uvX^3 + v^3XE_3 + uv^2X^2E_2 + (uv^2 + u^2v)X^4 \\ + \cdots + (2uv^3 + u^2v^2)X^3E_2 + \cdots . \end{aligned} \tag{1.9}$$

Hence, $\mathcal{A}_w = \mathcal{A}_w(X) \in \mathbb{N}_{u,v} \| X \| \subset \mathbb{C}_{u,v} \| X \|$, in this case.

Example 1.6. Virtual species (i.e., elements of $\mathbb{Z}_{u,v,\ldots} \| X, Y, \ldots \|$) are combinatorially also very useful. For instance, substituting an ordinary (non virtual) species into a virtual species may well produce another ordinary (non virtual) species.[8] For example, consider

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[6]Although most operations are routinely extended from ordinary to virtual species, substitution ($\circ$) for virtual species is rather delicate to define. In fact, Yeh has shown that these extensions can be made using coefficients $c_{\mu,H} \in \mathbb{K}$, where $\mathbb{K}$ is any *binomial ring*; that is, a ring with torsion-free additive group, containing $k(k-1)\cdots(k-n+1)/n!$ for every $k \in \mathbb{K}$ and integer $n \geq 0$. In the present paper, we find it convenient to take $\mathbb{K} = \mathbb{C}$. Appendix A describes this substitution for species in $\mathbb{C}_{u,v,\ldots} \| X, Y, \ldots \|$.

[7]In particular, $E_0(X) = X^0/S_0 = 1$ is called the species of the *empty set*, and $E_1(X) = X^1/S_1 = X$ is the species of *singletons* (i.e., 1-element sets). The species of *all* finite sets is expanded as $E = E(X) = \sum_{n\geq 0} X^n/S_n$.

[8]The situation is similar to the Cardano method for solving real third degree polynomial equations: one uses complex numbers when the three roots are real.

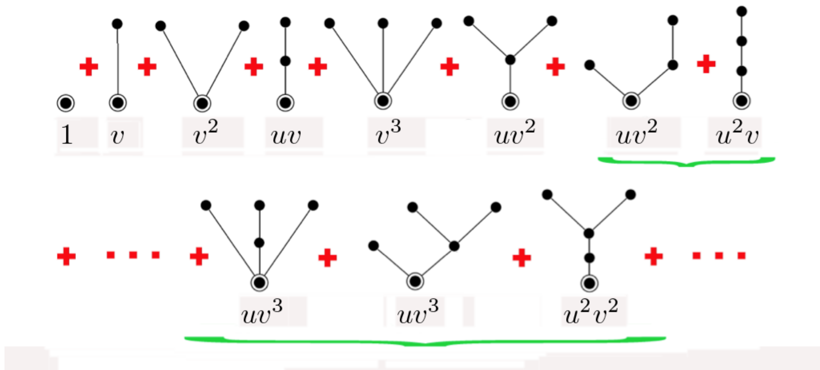


FIGURE 5. Some molecular components of the species $\mathcal{A}_w(X)$ of arborescences weighted by $w(\text{rooted tree}) = u^{\#\text{internal nodes} \neq \text{root}} v^{\#\text{leaves}}$.

the ordinary species $A(X)$ and $a(X)$ of rooted and unrooted (free) trees. Pierre Leroux has shown (see [BLL98, p. 280]) that

$$a(X) = \mathcal{V} \circ A(X), \tag{1.10}$$

where $\mathcal{V}(X)$ is the virtual species defined by

$$\mathcal{V}(X) = X + E_2(X) - X^2. \tag{1.11}$$

Combinatorial equation (1.10) is called the *dissymmetry theorem for trees*. Its importance stems from the fact that one can easily deduce from it all the combinatorial, enumerative and asymptotic properties of trees from those of rooted trees, despite the fact that trees have more complicated automorphisms than rooted trees.

Example 1.7. Virtual species are also used to define the multiplicative inverse of species by making use of geometric series. Here is how it works. Let $F = F(X, Y, \dots)$ be an ordinary weighted species satisfying[9] $F(\mathbf{0}) = 1$. Since F can then be rewritten in the form $F = 1 + F_+$ with $F_+(\mathbf{0}) = 0$, the expression $1/F$ (also denoted F^{-1}) is defined by the virtual species

$$1/F = 1 - F_+ + F_+^2 - F_+^3 + \cdots + (-1)^n F_+^n + \cdots. \tag{1.12}$$

More generally, Equation (1.12) is used to *define* $1/F$ for any $F \in \mathbb{C}_{u,v,\dots} \parallel X, Y, \dots \parallel$ satisfying $F(\mathbf{0}) = 1$.

The multiplicative inverse of the species $E(X) = \sum_{n\geq 0} X^n/S_n = 1 + E_+(X)$ of finite sets has a special status. It can be expressed in another form:

$$1/E(X) = E(-X). \tag{1.13}$$

This is a consequence of the standard combinatorial equalities

$$E(X + Y) = E(X)E(Y), \quad E(0) = 1, \tag{1.14}$$

[9]If $F = F(X, Y, \dots)$, then $F(\mathbf{0})$ is a shorthand notation for the result of the simultaneous substitutions $X = 0, Y = 0, \dots, u = 0, v = 0, \dots$ in F.

that reflect the facts that any finite set made of X-singletons and Y-singletons is naturally the same as a finite set made of X-singletons "followed" by a finite set made of Y-singletons and that an assembly of nothing is an empty set. The substitution $Y := -X$ in (1.14) gives $1 = E(0) = E(X - X) = E(X)E(-X)$, from which we deduce (1.13).

1.2. **Other formal power series associated to species.** Apart from the basic combinatorial power series expansion

$$F(X, Y, \dots) = \sum_{n,k,\dots,H} f_{n,k,\dots,H} X^n Y^k \cdots / H \in \mathbb{C}_{u,v,\dots} \| X, Y, \dots \|, \tag{1.15}$$

various other underlying formal power series are associated to species. The main one is the *cycle index series*

$$Z_F = Z_F(x_1, x_2, x_3, \dots ; y_1, y_2, y_3, \dots ; \dots), \tag{1.16}$$

where $x_1, x_2, x_3, \dots ; y_1, y_2, y_3, \dots ; \dots$ are countable families of extra formal variables that are associated to $X, Y, \dots$, respectively. These variables are distinct from the weight variables $u, v, \dots$. We recall the definition of the cycle index series in the case of an *ordinary* weighted species:

$$Z_F = \sum_{n,k,\dots} \frac{1}{n!\, k! \cdots} \sum_{\sigma \in S_n,\, \tau \in S_k,\, \dots} |F[\sigma, \tau, \dots]|\, x_1^{c_1(\sigma)} x_2^{c_2(\sigma)} x_3^{c_3(\sigma)} \cdots y_1^{c_1(\tau)} y_2^{c_2(\tau)} y_3^{c_3(\tau)} \cdots, \tag{1.17}$$

in which $|F[\sigma, \tau, \dots]| \in \mathbb{N}[[u, v, \dots]]$ denotes the total weight[10] of the F-structures on $[n] \sqcup [k] \sqcup \cdots$ for which $(\sigma, \tau, \dots)$ is an automorphism, and $c_i(\sigma)$ denotes the number of cycles of length i in the permutation σ.

When the ordinary weighted species F is written as a combinatorial power series (1.3), it is not difficult to show, using Pólya theory, that

$$Z_F = \sum_{n,k,\dots,H} f_{n,k,\dots,H} P_H(x_1, x_2, x_3, \dots ; y_1, y_2, y_3, \dots ; \dots), \tag{1.18}$$

where P_H is the classical *Pólya cycle indicator polynomial* of the group H acting on the multi-sorted set $[n] \sqcup [k] \sqcup \cdots$:

$$P_H = \frac{1}{|H|} \sum_{(h_1, h_2, \dots) \in H} x_1^{c_1(h_1)} x_2^{c_2(h_1)} x_3^{c_3(h_1)} \cdots y_1^{c_1(h_2)} y_2^{c_2(h_2)} y_3^{c_3(h_2)} \cdots, \tag{1.19}$$

Because of this fact, it is natural to *define* Z_F by (1.18) for any combinatorial power series $F \in \mathbb{C}_{u,v,\dots} \| X, Y, \dots \|$. Hence, Z_F is a (generally infinite) $\mathbb{C}[[u, v, \dots]]$-linear combination of monomials in the variables $x_1, x_2, x_3, \dots ; y_1, y_2, y_3, \dots$. In other words,

$$Z_F \in \mathbb{C}[[u, v, \dots ; x_1, x_2, x_3, \dots ; y_1, y_2, y_3, \dots ; \dots]]. \tag{1.20}$$

Many operations, including $Z_F + Z_G$, $Z_F \cdot Z_G$, $Z_F \times Z_G$, $Z_F(Z_G, Z_R, \dots)$, $\frac{\partial}{\partial x_1} Z_F$, between cycle index series, have been defined in such a way that the map $F \mapsto Z_F$ turns out to be

[10]or total number, if the weight of each structure is 1.

compatible with the corresponding combinatorial operations on species:

$$Z_{F+G} = Z_F + Z_G,\ Z_{F\cdot G} = Z_F \cdot Z_G, Z_{F\times G} = Z_F \times Z_G, \tag{1.21}$$

$$Z_{F(G,R,\dots)} = Z_F(Z_G, Z_R, \dots),\ Z_{\frac{\partial}{\partial X}F} = \frac{\partial}{\partial x_1} Z_F, \quad \text{etc.} \tag{1.22}$$

The first equality in (1.22) is satisfied if $0 = G(\mathbf{0}) = R(\mathbf{0}) = \cdots$ or if $F = F(X, Y, \dots)$ is of finite total degree in $X, Y, \dots$. The notation $Z_F(Z_G, Z_R, \dots)$ refers to *plethystic substitution* of cycle-index series which is defined as follows.

Definition 1.1. The *plethystic substitution*, $f(g, r, \dots)$, of cycle index series

$$g, r, \dots \in \mathbb{C}[[u, v, \dots; x_1, x_2, x_3, \dots; y_1, y_2, y_3, \dots; \dots]] \tag{1.23}$$

in the cycle index series $f = f(u, v, \dots; x_1, x_2, x_3, \dots; y_1, y_2, y_3, \dots; \dots)$ is the cycle index series h given by the formula

$$h = f(u, v, \dots, g_1, g_2, g_3, \dots; r_1, r_2, r_3, \dots; \dots), \tag{1.24}$$

where, for each integer $k \geq 1$, the following notation is used:

$$f_k = f(u^k, v^k, \dots; x_k, x_{2k}, x_{3k}, \dots; y_k, y_{2k}, y_{3k}, \dots; \dots). \tag{1.25}$$

Note. In (1.25), each weight variable is raised to the power k and the lower index of each variable $x, y, \dots$ is multiplied by k. Summability conditions[11] must be satisfied for existence of series (1.24).

For *ordinary* weighted species, $F = F(X, Y, \dots)$, the other classical "counting" series, namely the *exponential generating series*, $F(x, y, \dots)$, and the *type generating series*, $\widetilde{F}(x, y, \dots)$[12], respectively, are classically defined by

$$F(x, y, \dots) = \sum_{n,k,\dots} |F[n, k, \dots]| \, \frac{x^n}{n!} \frac{y^k}{k!} \cdots, \tag{1.26}$$

$$\widetilde{F}(x, y, \dots) = \sum_{n,k,\dots} |\widetilde{F}[n, k, \dots]| \, x^n y^k \cdots, \tag{1.27}$$

where $|F[n, k, \dots]|$ is the total weight of the F-structures on $[n] \sqcup [k] \sqcup \cdots$ and $|\widetilde{F}[n, k, \dots]|$ is the total weight[13] of the unlabelled ones (that is isomorphism types of such structures). These two series are extended to *any* combinatorial power series $F \in \mathbb{C}_{u,v,\dots} \| X, Y, \dots \|$ by the obvious formulas

$$F(x, y, \dots) = Z_F(x, 0, 0, \dots; y, 0, 0, \dots; \dots), \tag{1.28}$$

$$\widetilde{F}(x, y, \dots) = Z_F(x, x^2, x^3, \dots; y, y^2, y^3, \dots; \dots). \tag{1.29}$$

[11] See Appendix B for a general discussion of summability.

[12] Again, $x, y, \dots$ are auxiliary formal variables, *distinct from the weight variables* $u, v, \dots$, that are associated to $X, Y, \dots$

[13] By definition, the weight of an unlabelled structure is the weight of one of its labelled representatives.

This implies, of course, that for general species given in the form (1.15), we have

$$F(x,y,\dots) = \sum_{n,k,\dots} \left(\sum_{H \le S_{n,k,\dots}} \frac{c_{n,k,\dots,H}}{|H|} \right) x^n y^k \cdots , \tag{1.30}$$

$$\widetilde{F}(x,y,\dots) = \sum_{n,k,\dots} \left(\sum_{H \le S_{n,k,\dots}} c_{n,k,\dots,H} \right) x^n y^k \cdots , \tag{1.31}$$

where $H \le S_{n,k,\dots}$ means that H runs through a system of representatives of the conjugacy classes of the group $S_{n,k,\dots}$.

Table 1 describes the underlying series of some common species/series that will play a role in the sequel. These are those of *singletons*, X, *finite sets*, $E(X)$, *analytic exponential*, $\exp(X)$, *cyclic permutations*, $C(X)$, *analytic logarithm*[14], $\log(1+X)$, *permutations*, $S(X)$, *linear orders*, $L(X) = 1 + X + X^2 + \cdots$, and *2-sort functions*, $\Phi(X,Y)$[15].

$F(X,Y,\dots)$	$Z_F(x_1,x_2,\dots;y_1,y_2,\dots;\dots)$	$F(x,y,\dots)$	$\widetilde{F}(x,y,\dots)$
X	x_1	x	x
$E(X) = \sum_{n\ge 0} X^n/S_n$	$\exp\left(\sum_{i\ge 1} \frac{x_i}{i}\right)$	$\exp(x)$	$\frac{1}{1-x}$
$\exp(X) = \sum_{n\ge 0} \frac{1}{n!} X^n$	$\exp(x_1)$	$\exp(x)$	$\exp(x)$
$C(X) = \sum_{n\ge 1} X^n/C_n$	$\sum_{i\ge 1} \frac{\phi(i)}{i} \log\left(\frac{1}{1-x_i}\right)$	$\log\left(\frac{1}{1-x}\right)$	$\frac{x}{1-x}$
$\log(1+X) = \sum_{n\ge 1} \frac{(-1)^{n-1}}{n} X^n$	$\log(1+x_1)$	$\log(1+x)$	$\log(1+x)$
$S(X) = E(C(X))$	$\prod_{i\ge 1} \frac{1}{1-x_i}$	$\frac{1}{1-x}$	$\prod_{i\ge 1} \frac{1}{1-x^i}$
$L(X) = \sum_{n\ge 0} X^n$	$\frac{1}{1-x_1}$	$\frac{1}{1-x}$	$\frac{1}{1-x}$
$\Phi(X,Y) = E(E(X)Y)$	$\exp\left(\sum_{i\ge 1} \frac{y_i}{i} \exp\left(\sum_{j\ge 1} \frac{x_{ij}}{j}\right)\right)$	$\exp(\exp(x)y)$	$\prod_{i\ge 0} \frac{1}{1-x^i y}$

Table 1. Underlying series of some common species/series.

The species $E(X)$ of finite sets is often called the *combinatorial exponential*. It is important to note that

$$E(X) \ne \exp(X). \tag{1.32}$$

Example 1.8. A typical example of a cycle index series associated to a *weighted* 2-sort species involves the species $\mathcal{A}_w(X,Y)$ of rooted trees with internal nodes (including the root) of sort X and leaves of sort Y weighted by

$$w(\text{rooted tree}) = u^{\#\text{internal nodes}}. \tag{1.33}$$

[14]The corresponding series for the *combinatorial logarithm*, $\mathrm{Lg}(1+X)$, will be given in Section 2.

[15]In Table 1, C_n is the standard cyclic subgroup of S_n, ϕ denotes the Euler totient function, and a Φ-structure on $[n] \sqcup [k]$ is a function $f : [n] \to [k]$.

Then since each $\mathcal{A}_w$-structure is canonically a node, weighted by u, followed by a (possibly empty) set of leaves or $\mathcal{A}_w$-structures, the following combinatorial equation holds:

$$\mathcal{A}_w(X,Y) = uXE(Y + \mathcal{A}_w(X,Y)). \tag{1.34}$$

From this equation, the cycle index series $a = a(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots) = Z_{\mathcal{A}_w}$ can be recursively computed to any degree in a computer algebra system using the formula

$$a = ux_1 \exp\left(\sum_{k\geq 1} \frac{y_k + a_k}{k}\right). \tag{1.35}$$

1.3. **Substituting power series into species.** Since $\mathbb{C} \subset \mathbb{C}_{u,v,\ldots} \subset \mathbb{C}_{u,v,\ldots} \| X, Y, \ldots \|$, every complex number $c \in \mathbb{C}$ or power series $\alpha \in \mathbb{C}_{u,v,\ldots}$ can be considered as species. In particular if $F = F(X, Y, \ldots)$ is a species, $a, b, \cdots \in \mathbb{C}$ and $\alpha, \beta, \cdots \in \mathbb{C}_{u,v,\ldots}$, then

$$F \circ (a, b, \ldots) = F(a, b, \ldots) \quad \text{and} \quad F \circ (\alpha, \beta, \ldots) = F(\alpha, \beta, \ldots) \tag{1.36}$$

should be species. In fact, we will see that

$$F(a, b, \ldots) \quad \text{and} \quad F(\alpha, \beta, \ldots) \in \mathbb{C}_{u,v,\ldots} \tag{1.37}$$

assuming summability conditions. Here is how it works in the case of 1-sort species $F(X)$.

First of all, recall that $1 = X^0/S_0$ is the species of the empty set (there is only one 1-structure and it "lives" on the empty set). Hence, $F(1)$ is the species whose structures are F-assemblies of 1-structures. That is, F-assemblies of empty sets. Such structures are precisely the *unlabelled* F-structures, each of which is living on the empty set. By Pólya theory, we then must have

$$F(1) = Z_F(1, 1, 1, \ldots) = \text{total weight of all unlabelled } F\text{-structures}, \tag{1.38}$$

which is an element of $\mathbb{C}_{u,v,\ldots}$ assuming summability of the right-hand side of (1.38).

More generally, for any $k \in \mathbb{N}$, $k = k \cdot 1$ is the species whose structures are empty sets "coloured" by a colour $i \in [k]$. There are exactly k such structures:

$$\{\}_1, \{\}_2, \ldots, \{\}_i, \ldots, \{\}_k, \tag{1.39}$$

each of which lives on the empty underlying set. This time, an $F(k)$-structure is a k-coloured unlabelled F-structure living on the empty set. By Pólya theory, we must have

$$F(k) = Z_F(k, k, k, \ldots) = \text{total weight of all unlabelled } k\text{-coloured } F\text{-structures}. \tag{1.40}$$

Now take the series $\alpha = u + v + \cdots \in \mathbb{C}_{u,v,\ldots}$ (i.e., the formal sum of all variables $u, v, \ldots$). Then, since $u + v + \cdots = u \cdot 1 + v \cdot 1 + \cdots$, a $(u+v+\cdots)$-structure is an empty set weighted by u, or by v, etc. Hence, an $F(u+v+\cdots)$-structure is an F-assembly of indistinguishable dots each of which being weighted by u, or by v, etc. Accordingly, invoking again Pólya theory, this means that

$$F(u + v + \cdots) = Z_F(u + v + \cdots, u^2 + v^2 + \cdots, u^3 + v^3 + \cdots, \ldots). \tag{1.41}$$

Similarly, using the same kind of combinatorial arguments, for $m, n, \dots \in \mathbb{N}$, we have

$$\begin{aligned} F(mu + nv + \cdots) &= F(\underbrace{u + u + \cdots}_{m} + \underbrace{v + v + \cdots}_{n} + \cdots) \\ &= Z_F(mu + nv + \cdots, mu^2 + nv^2 + \cdots, mu^3 + nv^3 + \cdots, \dots). \end{aligned} \tag{1.42}$$

More generally, we can replace $u, v, \dots$ in (1.42) by power products $\mu, \nu, \dots \in \mathbb{C}_{u,v,\dots}$ and obtain

$$F(m\mu + n\nu + \cdots) = Z_F(m\mu + n\nu + \cdots, m\mu^2 + n\nu^2 + \cdots, m\mu^3 + n\nu^3 + \cdots, \dots). \tag{1.43}$$

Finally, since the coefficient $c_{i,j,\dots} = c_{i,j,\dots}(m, n, \dots)$ of each individual monomial $c_{i,j,\dots} u^i v^j \cdots$ in the expansion of (1.43) is a polynomial in $m, n, \dots$ with coefficients in $\mathbb{Q}$, we can replace $m, n, \dots$ by any complex numbers $a, b, \dots$ and we have the following facts.

Lemma 1.1. *Let $F(X)$ be a 1-sort species and $\alpha = \alpha(u, v, \dots) \in \mathbb{C}_{u,v,\dots}$ be a formal power series in the weight variables $u, v, \dots$. Then, assuming summability,*

$$F(\alpha) = Z_F(\alpha_1, \alpha_2, \alpha_3, \dots), \tag{1.44}$$

where, for $k = 1, 2, \dots,$

$$\alpha_k = \alpha_k(u, v, \dots) = \alpha(u^k, v^k, \dots). \tag{1.45}$$

More generally, let $F(X, Y, \dots)$ be a species on many sorts $X, Y, \dots$ of singletons and $\alpha, \beta, \dots \in \mathbb{C}_{u,v,\dots}$. Then, assuming summability,

$$F(\alpha, \beta, \dots) = Z_F(\alpha_1, \alpha_2, \alpha_3, \dots; \beta_1, \beta_2, \beta_3, \dots; \dots). \tag{1.46}$$

Notational convention. Recall that auxiliary variables are lower case indeterminates $x, y, z, \dots$ associated to sorts $X, Y, Z, \dots$ that are distinct from weight variables and complex numbers. We will find it convenient, from a notational point of view, to extend (1.46) to series $\alpha, \beta, \dots \in \mathbb{C}_{x,y,\dots;u,v,\dots}$ so as to include possible substitutions of the auxiliary variables $x, y, \dots$. This is done by considering the auxiliary variables as "plethystically vanishing" by making use of the following notational convention:

Given $\alpha = \alpha(x, y, \dots; u, v, \dots) \in \mathbb{C}_{x,y,\dots;u,v,\dots}$, define α_k, $k = 1, 2, 3, \dots,$ by

$$\alpha_1 = \alpha_1(x, y, \dots; u, v, \dots) = \alpha(x, y, \dots; u, v, \dots), \tag{1.47}$$

$$\alpha_k = \alpha_k(x, y, \dots; u, v, \dots) = \alpha(0, 0, \dots; u^k, v^k, \dots), \quad \text{if } k > 1. \tag{1.48}$$

For each $k \geq 1$, the transformation $\alpha \mapsto \alpha_k$ is a $\mathbb{C}$-algebra endomorphism

$$(*)_k : \mathbb{C}_{x,y,\dots;u,v,\dots} \longrightarrow \mathbb{C}_{x,y,\dots;u,v,\dots}. \tag{1.49}$$

Making use of this convention, one can consider expressions such as

$$F(x, \beta, \dots), \quad F\left(\frac{u + 2v}{1 - v}, 3u^2 y, \dots\right), \tag{1.50}$$

as compact encodings for the (generally more complicated) series

$$Z_F(x,0,0,\dots;\beta_1,\beta_2,\beta_3,\dots;\dots), \tag{1.51}$$

$$Z_F\left(\frac{u+2v}{1-v},\frac{u^2+2v^2}{1-v^2},\frac{u^3+2v^3}{1-v^3},\dots;3u^2y,0,0,\dots;\dots\right). \tag{1.52}$$

As a consequence of (1.44) and (1.46) this notational convention is compatible with *sums*, *Cauchy products* and *substitution*:

$$(F+G+\cdots)(\alpha,\beta,\dots)=F(\alpha,\beta,\dots)+G(\alpha,\beta,\dots)+\cdots, \tag{1.53}$$

$$(F\cdot G\cdots)(\alpha,\beta,\dots)=F(\alpha,\beta,\dots)\cdot G(\alpha,\beta,\dots)\cdots, \tag{1.54}$$

$$F\circ(G,R,\dots)(\alpha,\beta,\dots)=F(G(\alpha,\beta,\dots),R(\alpha,\beta,\dots),\dots), \tag{1.55}$$

under the condition that both sides in (1.53)–(1.55) are formally summable in the sense of Definition B.1 in Appendix B. For substitution, the standard conditions are:

(i) the constant terms $G(\mathbf{0}),R(\mathbf{0}),\dots$ in the species $G,R,\dots$ are all 0,

or

(ii) the total degree of $F(X,Y,\dots)$ in $X,Y,\dots$ is finite.

Formulas (1.53)–(1.55) can be used to generate various power series identities from combinatorial identities between species.

Partial substitutions of series into species can also be considered. Expressions such as

$$F(\alpha,Y,\dots),\quad F\left(\frac{X}{1-q},\beta,\dots\right), \tag{1.56}$$

where $F\in\mathbb{C}_{u,v,\dots}\|X,Y,\dots\|$, $\alpha,\beta\in\mathbb{C}_{x,y,\dots;u,v,\dots}$, and q is a weight variable, will then be perfectively legitimate and freely used in the present paper.

Example 1.9. The notational convention presented in (1.46)–(1.48) provides a *uniform* compact notation for all basic "enumerative" series that are associated to species. For example, let $F=F(X)$ be a 1-sort species, x an auxiliary variable, and u,q be weight variables. Then $F(x)$, $F(u)$ and $F_q(u)=F(u/(1-q))$ compactly denote the three series[16]

$$F(x)=Z_F(x,0,0,\dots)=\sum_{n\geq 0}|F[n]|\frac{x^n}{n!},\quad F(u)=Z_F(u,u^2,u^3,\dots)=\sum_{n\geq 0}|\widetilde{F}[n]|u^n, \tag{1.57}$$

$$F_q(u)=F(u/(1-q))=Z_F\left(\frac{u}{1-q},\frac{u^2}{1-q^2},\dots\right)=\sum_{n\geq 0}|F_q[n]|\frac{u^n}{(q;q)_n}, \tag{1.58}$$

where

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k). \tag{1.59}$$

[16]The last series in (1.57) is denoted by $\widetilde{F}(u)$ in the classical theory of species (see (1.27) above). Since u is a weight variable, the tilde on F is now unnecessary due to our notational convention.

The last equality in (1.58) is due to Hélène Décoste [Déc93], and the coefficients $|F_q[n]|$ can be considered as "q-inventories" (or q-enumerations) of F-structures[17]. More precisely, Décoste [Déc93] showed that $|F_q[n]|$ is a polynomial in q, of degree at most $n(n-1)/2$, with nonnegative integer coefficients, which simultaneously "q-counts" (or "q-weights") both labelled and unlabelled F-structures in the following sense:

$$\lim_{q\to 1} |F_q[n]| = |F[n]|, \qquad \lim_{q\to 0} |F_q[n]| = |\widetilde{F}[n]|. \tag{1.60}$$

Example 1.10. The symbol F_q in (1.58) can be thought of as a "q-analogue" of the species F:

$$F_q = F_q(X) = F\left(\frac{1}{1-q}X\right) = F(X + qX + q^2X + q^3X + \cdots). \tag{1.61}$$

This means that F_q-structures are F-structures in which each underlying singleton is weighted by q^k, where k is an arbitrary integer ≥ 0.

A (q,t)-series can also be associated to any 2-sort species $F(X,Y)$ as follows:

$$F_{q,t}(u,v) = F\left(\frac{1}{1-q}u, \frac{1}{1-t}v\right) = \sum_{n,k\geq 0} |F_{q,t}[n,k]| \frac{u^n}{(q;q)_n} \cdot \frac{v^k}{(t;t)_k}, \tag{1.62}$$

where $F_{q,t} = F_{q,t}(X,Y) = F(\frac{1}{1-q}X, \frac{1}{1-t}Y)$ is the (q,t)-analogue of the species $F(X,Y)$.

Example 1.11. Take the species of finite sets $F = E = E(X)$, $\alpha = a\mu + b\nu + \cdots \in \mathbb{C}_{u,v,\ldots}$, where $a, b, \cdots \in \mathbb{C}$ and $\mu, \nu, \ldots$ are power products in the weight variables $u, v, \ldots$ Then, by (1.44) and Table 1,

$$E(\alpha) = E(a\mu + b\nu + \cdots) = \frac{1}{(1-\mu)^a(1-\nu)^b\cdots}, \tag{1.63}$$

and, in particular, taking $\mu = u, \nu = v, \ldots$ we get

$$E(\alpha) = E(au + bv + \cdots) = \frac{1}{(1-u)^a(1-v)^b\cdots}. \tag{1.64}$$

Also, if q is a weight variable $\neq u$ and $\alpha = u + uq + uq^2 + \cdots = u/(1-q)$, we have

$$E_q(u) = E(u/(1-q)) = \prod_{k\geq 0} \frac{1}{1-uq^k} = \sum_{n\geq 0} \frac{u^n}{(q;q)_n}, \tag{1.65}$$

which is one form of the classical q-analogue of the exponential series.

Furthermore, making also use of auxiliary variables $x, y, z, \ldots$, we have, for example,

$$E(ax + buv^3 + cz) = \exp(ax)\frac{1}{(1-uv^3)^b}\exp(cz), \quad a, b, c \in \mathbb{C}. \tag{1.66}$$

Example 1.12. Let $C = C(X)$ be the species of cyclic permutations and consider the weighted species $\mathrm{Oct}(X,Y) = C(X + t(Y^2 + Y^3 + \cdots))$ of 2-sort octopuses[18] weighted

[17] If $F = \mathcal{P}_k = E_k E$, is the species of k-subsets of sets, then $|F_q[n]| = |\mathcal{P}_q[n]| = \binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$, the usual q-analogue of the binomial coefficient $\binom{n}{k}$.

[18] Such a structure is an oriented cycle made of non-trivial tentacles (i.e., linearly ordered sets made of at least 2 points of sort Y) and "non-tentacle points" of sort X. A better name for such a structure would be *polypus*.

according to their number of tentacles: $t^{\#\text{tentacles}}$, t being a weight variable. Let x, y be auxiliary variables and u, v be weight variables distinct from t. From Table 1, it is easy to see that

$$\mathrm{Oct}(x,y) = -\log\left(1 - x - \frac{ty^2}{1-y}\right) = \sum_{i,j,k} a_{i,j,k}\frac{x^i}{i!}\frac{y^j}{j!}t^k, \tag{1.67}$$

$$\mathrm{Oct}(u,y) = -\log\left(1 - u - \frac{ty^2}{1-y}\right) - \sum_{i>1}\frac{\phi(i)}{i}\log\left(1-u^i\right) = \sum_{i,j,k} b_{i,j,k}u^i\frac{y^j}{j!}t^k, \tag{1.68}$$

$$\mathrm{Oct}(x,v) = -\log\left(1 - x - \frac{tv^2}{1-v}\right) - \sum_{i>1}\frac{\phi(i)}{i}\log\left(1-\frac{t^iv^{2i}}{1-v^i}\right) = \sum_{i,j,k} c_{i,j,k}\frac{x^i}{i!}v^jt^k, \tag{1.69}$$

$$\mathrm{Oct}(u,v) = -\sum_{i\geq 1}\frac{\phi(i)}{i}\log\left(1-u^i-\frac{t^iv^{2i}}{1-v^i}\right) = \sum_{i,j,k} d_{i,j,k}u^iv^jt^k, \tag{1.70}$$

where

$$a_{i,j,k} = \#k\text{-tentacle octopuses with } i \text{ non-tentacle points, } j \text{ tentacle points,} \tag{1.71}$$

$$b_{i,j,k} = \#k\text{-tentacle octopuses with } i \text{ unlabelled non-tentacle points, } j \text{ tentacle points,} \tag{1.72}$$

$$c_{i,j,k} = \#k\text{-tentacle octopuses with } i \text{ non-tentacle points, } j \text{ unlabelled tentacle points,} \tag{1.73}$$

$$d_{i,j,k} = \#k\text{-tentacle octopuses with } i \text{ unlabelled non-tentacle points, } j \text{ unlabelled tentacle points.} \tag{1.74}$$

Note. As said before, summability conditions must be satisfied when series are substituted into species. For example, take $F = S(X)$, the 1-sort species of permutations, and $\alpha = 1$ in (1.44). Then $\alpha_k = 1_k = 1$ for every $k \geq 1$, and, since

$$Z_S = \prod_{n\geq 1}\frac{1}{(1-x_n)} = \prod_{n\geq 1}\sum_{k\geq 0}x_n^k, \tag{1.75}$$

we have $S(1) = Z_S(1,1,1,\dots) = \infty$. That is, $S(1)$ is not summable. However, $S(x)$ and $S(u)$, where x is an auxiliary variable and u is a weight variable, are the familiar summable series

$$S(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad S(u) = \prod_{n\geq 1}\frac{1}{(1-u^n)} = \sum_{k\geq 0}p(k)u^k, \tag{1.76}$$

where $p(k)$ is the number of integer partitions of k.

1.4. **Overview of the remaining sections of the paper.** The first goal of the present paper is to extend the classical 2-variable "analytic" Newton binomial expansion

$$(1+X)^\wedge Y = (1+X)^Y = \exp(Y\log(1+X)) = \sum_{n\geq 0}\binom{Y}{n}X^n \in \mathbb{Q}[[X,Y]] \tag{1.77}$$

to the context of 2-sort combinatorial species. In order to do so, we proceed by analogy by simply replacing the analytic exponential and logarithmic series $\exp(X)$ and $\log(1+X)$ appearing in (1.77) by the species $E(X)$, of sets, and a virtual species $\mathrm{Lg}(1+X)$, due to Joyal [Joy86], called the *combinatorial logarithm*.[19] More precisely, the analogy consists in replacing (1.77) by

$$(1+X)^{\uparrow}Y \underset{\mathrm{def}}{=} E(Y\,\mathrm{Lg}(1+X)) = \sum_{n\geq 0} \binom{X,Y}{n} \in \mathbb{Z}\,\|\,X,Y\,\|, \tag{1.78}$$

in which the expressions $\binom{X,Y}{n}$, $n = 0,1,2,\ldots,$ denote the 2-sort species obtained by collecting all terms of degree n in X in the molecular expansion of $E(Y\,\mathrm{Lg}(1+X))$. We call them *generalized binomial coefficients* and $(1+X)^{\uparrow}Y$ is called the *binomial species*.[20] The binomial species is denoted by

$$B(X,Y) = (1+X)^{\uparrow}Y. \tag{1.79}$$

Our second goal is to apply the binomial species to generate various identities and to define a new combinatorial operation of exponentiation $F \uparrow G$ between species. Specifically, the remaining sections are are arranged as follows.

– In Section 2, we recall some basic facts about the two main tools used in the present paper: the *combinatorial logarithm*, denoted by $\mathrm{Lg}(1+X)$, and the species of *pseudo-singletons*, denoted by $\widehat{X}$.

The *combinatorial logarithm* is a virtual species defined as the *inverse*, under combinatorial substitution ($\circ$), of the species of non empty finite sets.

The species of *pseudo-singletons*, $\widehat{X} \in \mathbb{Q}\|\,X\,\|$, was introduced by the present author in [Lab90] as the *analytic logarithm* of the species $E(X)$ of finite sets. The species $\widehat{X}$ of pseudo-singletons is similar to the species X of singletons and will be used to make a connection between the two kinds of logarithms $\mathrm{Lg}(1+X)$ and $\log(1+X)$.

– In Section 3, definitions and basic properties of the binomial species $B(X,Y)$ and generalized binomial coefficients $\binom{X,Y}{n}$ are presented together with their underlying cycle index and counting series. Various formulas and identities are obtained through specialization of variables and plethystic notation. These identities include the classical binomial expansion of Newton and corresponding binomial expansions in the context of symmetric functions and (q,t)-series. A computational method for the expansion of the species $B(X,Y)$ and $\binom{X,Y}{n}$ to arbitrary large degrees in X is also presented.

– In Section 4, we use the binomial species to introduce a new operation of *combinatorial exponentiation* between species, $F^{\uparrow}G$, and study its properties with respect to other classical combinatorial operations and underlying series. Specific examples and applications of the combinatorial exponentiation are also presented.

[19] In fact, Joyal used the notation $\log(1+X)$ for the combinatorial logarithm, but we prefer to use $\mathrm{Lg}(1+X)$ in order to distinguish it from the analytic logarithm.

[20] We intentionally use the "uparrow notation" $(1+X)^{\uparrow}Y$ instead of $(1+X)^{\wedge}Y$ to make a distinction between the *species* (1.78) and the *series* (1.77). Of course, $(1+X)^{\uparrow}Y \neq (1+X)^{\wedge}Y$ in $\mathbb{Z}\,\|\,X,Y\,\|$.

2. Basic facts about the combinatorial logarithm and pseudo-singletons

2.1. **Definitions and underlying cycle index series.** By analogy with the fact that $\log(1+X)$ is the analytic substitutional inverse of

$$\exp_+(X) = \exp(X) - 1 = \sum_{n\geq 1} \frac{1}{n!} X^n, \tag{2.1}$$

the *combinatorial logarithm* is defined as follows.

Definition 2.1 ([Joy86]). The *combinatorial logarithm*, denoted by $\mathrm{Lg}(1+X)$, is the virtual species which is the combinatorial substitutional inverse of the species

$$E_+(X) = E(X) - 1 = \sum_{n\geq 1} X^n/S_n \tag{2.2}$$

of nonempty finite sets. In other words,

$$\mathrm{Lg}(1+X) \underset{\mathrm{def}}{=} E_+^{<-1>}(X), \tag{2.3}$$

where $F^{<-1>}(X)$ denotes the substitutional inverse of $F(X)$ in $\mathbb{C}\|X\|$. This inverse exists and is unique by the "implicit species theorem" (see [Joy86]).

We will often use the notation

$$\Omega(X) = \mathrm{Lg}(1+X), \tag{2.4}$$

for the combinatorial logarithm. Then the following combinatorial equations hold:

$$E_+ \circ \Omega = \Omega \circ E_+ = X, \quad E \circ \Omega = (1+E_+) \circ \Omega = 1 + X. \tag{2.5}$$

Furthermore, by analogy with the fact that the species X of singletons can be thought of as the *combinatorial* logarithm of the species E of sets (since $E = E(X)$), the species $\widehat{X}$ of *pseudo-singletons* is defined as its *analytic* logarithm in the following way.

Definition 2.2 ([Lab90]). Consider the classical power series expansion of the analytic logarithm $\log(1+X) = \sum_{n\geq 1}(-1)^{n-1}X^n/n \in \mathbb{Q}[[X]]$. The species $\widehat{X}$ of *pseudo-singletons* is defined by the summable series

$$\widehat{X} \underset{\mathrm{def}}{=} \log(E) = \log(1+E_+) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} E_+^n \in \mathbb{Q}\|X\| \ . \tag{2.6}$$

Hence, the species of sets is the analytic exponential of that of pseudo-singletons,

$$E(X) = \exp(\widehat{X}), \tag{2.7}$$

while $E(X) \neq \exp(X)$, see (1.32). The connection between the combinatorial logarithm $\mathrm{Lg}(1+X)$ and the analytic logarithm $\log(1+X) = \sum_{n\geq 1}(-1)^{n-1}X^n/n$ is easily made using the species $\widehat{X}$ of pseudo-singletons. In fact, if we define $\widehat{F}$ by $\widehat{X} \circ F$, the following basic combinatorial equation holds:

$$\mathrm{Lg}\widehat{(1+X)} = \log(1+X), \tag{2.8}$$

as a consequence of the equalities

$$\begin{aligned}\widehat{\mathrm{Lg}(1+X)} = \widehat{X}\circ\Omega = \log(1+E_+)\circ\Omega = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}E_+^k\circ\Omega = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}X^k \\ = \log(1+X),\end{aligned}\tag{2.9}$$

since $E_+^k\circ\Omega = (E_+\circ\Omega)^k = X^k$ by (2.5).

For purposes of comparison, the underlying series of X, $\log(1+X)$, and of their combinatorial counterparts $\widehat{X}$, $\mathrm{Lg}(1+X)$, are given in Table 2, in which $\mu(k)$ denotes the Möbius function of k.

$F(X)$	$Z_F(x_1,x_2,x_3,\dots)$	$F(x)$	$\widetilde{F}(x)$
X	x_1	x	x
$\log(1+X)$	$\log(1+x_1)$	$\log(1+x)$	$\log(1+x)$
$\widehat{X}$	$\sum_{i\geq 1}\frac{x_i}{i}$	x	$\log(\frac{1}{1-x})$
$\mathrm{Lg}(1+X)$	$\sum_{k\geq 1}\frac{\mu(k)}{k}\log(1+x_k)$	$\log(1+x)$	$x-x^2$

TABLE 2. Underlying series of X, $\log(1+X)$, $\widehat{X}$, $\mathrm{Lg}(1+X)$.

In Table 2, the fact that $Z_X = x_1$ is immediate, and the expression for $Z_{\log(1+X)}$ follows from $Z_{X^n} = Z_X^n = x_1^n$ by linearity. The expression for $Z_{\widehat{X}}$ follows from the computation

$$Z_{\widehat{X}} = Z_{\log}(Z_E) = \log(Z_E) = \log(\exp\textstyle\sum_{i\geq 1}x_i/i) = \sum_{i\geq 1}x_i/i.\tag{2.10}$$

Moreover, the expression for $Z_{\mathrm{Lg}(1+X)}$ is a consequence of (2.8) and Möbius inversion. To see this, define $\omega = Z_{\mathrm{Lg}(1+X)}$ and $\ell = Z_{\log(1+X)} = \log(1+x_1)$. Then, taking the cycle index of both sides of (2.8), we obtain

$$\sum_{i\geq 1}\frac{\omega_i}{i} = \ell = \ell_1, \quad \text{and hence} \quad \sum_{n|k}\frac{\omega_k}{k} = \frac{\ell_n}{n}.\tag{2.11}$$

Thus,

$$\omega = \omega_1 = \sum_{k\geq 1}\frac{\omega_k}{k}\sum_{n|k}\mu(n) = \sum_{n\geq 1}\mu(n)\sum_{n|k}\frac{\omega_k}{k} = \sum_{n\geq 1}\frac{\mu(n)}{n}\ell_n = \sum_{n\geq 1}\frac{\mu(n)}{n}\log(1+x_n).\tag{2.12}$$

Finally, the last entry in Table 2, $x-x^2$, can be established as follows:

$$\widetilde{\mathrm{Lg}}(1+x) = \sum_{k\geq 1}\frac{\mu(k)}{k}\log(1+x^k) = \sum_{n\geq 1}\sum_{d|n}(-1)^{d-1}\mu(n/d)\,\frac{x^n}{n} = x-x^2,\tag{2.13}$$

since the Dirichlet convolution $c(n) = (-1)^{n-1}*\mu(n)$ of the two arithmetic multiplicative functions $(-1)^{n-1}$ and $\mu(n)$ satisfies $c(1)=1$, $c(2)=-2$, and $c(n)=0$ for $n>2$.

Note. One can avoid the above Dirichlet convolution in establishing (2.13) by making use of the following lemma which involves Lambert series.

Lemma 2.1. *The Möbius function $\mu(n)$ satisfies the identities*

$$\text{(a)} \ \sum_{n\geq 1} \mu(n)\frac{x^n}{1-x^n} = x, \qquad \text{(b)} \ \sum_{n\geq 1} \mu(n)\frac{x^n}{1+x^n} = x - 2x^2. \tag{2.14}$$

Proof. Identity (2.14a) is an immediate consequence of the fact that $\sum_{d|k} \mu(d) = 1$ if $k = 1$ and 0 if $k > 1$. Identity (2.14b) follows from identity (2.14a) via the computation

$$\begin{aligned} x - 2x^2 &= \sum_{n\geq 1} \mu(n)\frac{x^n}{1-x^n} - \sum_{n\geq 1} \mu(n)\frac{2x^{2n}}{1-x^{2n}} \\ &= \sum_{n\geq 1} \mu(n)\frac{x^n}{1-x^n}\left(1 - \frac{2x^n}{1+x^n}\right) = \sum_{n\geq 1} \mu(n)\frac{x^n}{1+x^n}. \end{aligned}$$

□

Corollary 2.2. *The Möbius function $\mu(n)$ satisfies the identities*

$$\text{(a)} \ -\sum_{n\geq 1} \frac{\mu(n)}{n}\log(1-x^n) = x, \qquad \text{(b)} \ \sum_{n\geq 1} \frac{\mu(n)}{n}\log(1+x^n) = x - x^2. \tag{2.15}$$

Proof. These identities follow by integration since application of the operator $x\frac{d}{dx}$ to (2.15a) and (2.15b) gives (2.14a) and (2.14b). □

The behaviour of the combinatorial species E and Lg relative to sum, product and derivation is similar to that of the analytic exp and log.

Lemma 2.3. *Let X and Y be two sorts of singletons. Then we have*

$$E(X+Y) = E(X)E(Y), \qquad \exp(X+Y) = \exp(X)\exp(Y), \tag{2.16}$$

$$\frac{d}{dX}E(X) = E(X), \qquad \frac{d}{dX}\exp(X) = \exp(X), \tag{2.17}$$

$$\frac{d}{dX}\operatorname{Lg}(1+X) = \frac{1}{1+X}, \qquad \frac{d}{dX}\log(1+X) = \frac{1}{1+X}, \tag{2.18}$$

$$\operatorname{Lg}((1+X)(1+Y)) = \operatorname{Lg}(1+X) + \operatorname{Lg}(1+Y), \tag{2.19}$$

$$\log((1+X)(1+Y)) = \log(1+X) + \log(1+Y), \tag{2.20}$$

where $\operatorname{Lg}((1+X)(1+Y))$ *is interpreted as*

$$\operatorname{Lg}(1+(X+Y+XY)) = \operatorname{Lg}(1+X)\circ(X+Y+XY). \tag{2.21}$$

Proof. We prove only the formulas relative to E and Lg. Formula (2.16 left) was discussed in the introduction. Formula (2.17 left) is classical and is a consequence of the fact that, for any finite set U, the set $U \sqcup \{\bullet\}$ with an extra outside unlabelled element $\bullet$ can be canonically identified with the set U itself. Formula (2.18 left) follows from the combinatorial chain-rule: application of $\frac{d}{dX}$ to both sides of $E(\operatorname{Lg}(1+X)) = X$ gives

$E'(\mathrm{Lg}(1{+}X))\cdot\mathrm{Lg}'(1{+}X) = 1$. Hence, $E(\mathrm{Lg}(1{+}X))\cdot\mathrm{Lg}'(1{+}X) = (1{+}X)\cdot\mathrm{Lg}'(1{+}X) = 1$. The proof of Equation (2.19) runs as follows: let

$$A = \mathrm{Lg}((1+X)(1+Y)), \quad B = \mathrm{Lg}(1+X) + \mathrm{Lg}(1+Y).$$

Then by (2.5), (2.16), and (2.21),

$$E(A) = E \circ \mathrm{Lg}(1+X) \circ (X+Y+XY) = (1+X) \circ (X+Y+XY) = (1+X)(1+Y),$$
$$E(B) = E(\mathrm{Lg}(1+X) + \mathrm{Lg}(1+Y)) = E(\mathrm{Lg}(1+X))E(\mathrm{Lg}(1+Y)) = (1+X)(1+Y).$$

Hence, $E(A) = E(B)$, and $A = B$ by the uniqueness of inverse species. □

Note. By (2.18), $\mathrm{Lg}(1{+}X)$ and $\log(1{+}X)$ have the same derivative but their difference is far from being a constant. Such a phenomenon is quite frequent in the theory of species.

2.2. **Molecular expansions and identities involving $\mathrm{Lg}(1{+}X)$ and $\widehat{X}$.** Joyal [Joy86] obtained an expansion of $\mathrm{Lg}(1+X)$ involving (positive and negative) species of strictly increasing sequences in the lattices of equivalence relations on finite sets. Ira M. Gessel and Ji Li [GL11, Li12] obtained expressions for $\mathrm{Lg}(1+X)$ in terms of special classes of graphs and cographs. The explicit expansion of the combinatorial logarithm as a countable $\mathbb{Z}$-linear combination of irreducible species (molecular expansion) has been obtained recently by the author in [Lab13]. Its first terms, up to degree 6, are given by

$$\mathrm{Lg}(1+X) = \mathrm{Lg}(1+X)_+ - \mathrm{Lg}(1+X)_-, \tag{2.22}$$

where

$$\begin{aligned}\mathrm{Lg}(1+X)_+ = X &+ XE_2 + XE_3 + E_2\circ E_2 + X^3E_2 + XE_4 + E_2E_3 + X^3E_3\\ &+ 2\,X^2E_2^2 + XE_5 + E_2E_4 + E_3\circ E_2 + E_2\circ E_3 + \cdots,\end{aligned} \tag{2.23}$$

$$\begin{aligned}\mathrm{Lg}(1+X)_- = E_2 &+ E_3 + X^2E_2 + E_4 + X^2E_3 + XE_2^2 + E_5 + X^4E_2 + X^2E_4\\ &+ 2\,XE_2E_3 + E_2\cdot(E_2\circ E_2) + E_6 + E_2\circ(XE_2) + \cdots,\end{aligned} \tag{2.24}$$

in which, for example, $E_3 \circ E_2 = X^6/(S_3 \wr S_2)$, where $\circ$ denotes substitution of species and $\wr$ denotes the wreath product of groups.

One of the main classical applications of the combinatorial logarithm is to see that *any* expansion of it provides a kind of intricate "inclusion-exclusion" principle by which one can express the species F^{conn} of connected F-structures in terms of the species F itself. More precisely, if $F = 1 + F_+$ is a species satisfying $F(\mathbf{0}) = 1$ and made of "connected structures", then

$$F^{\mathrm{conn}} = \mathrm{Lg}(F) = \mathrm{Lg}(1+F_+) = \mathrm{Lg}(1+X) \circ F_+, \tag{2.25}$$

and (2.23)–(2.25) can be used to express F^{conn} in terms of F. One of the most interesting facts about (2.25) is that it can be used to *define* a (in general, virtual) species F^{conn} under the sole condition $F(\mathbf{0}) = 1$ even in the case when F is *not* of the form $F = E(G)$. That is, even when F-structures are not sets of "connected" structures. In particular,

$$\mathrm{Lg}(1+X) = (1+X)^{\mathrm{conn}}. \tag{2.26}$$

By (2.25) and Table 2, the corresponding underlying series for the species F^{conn} of connected F-structures are given by the formulas

$$F^{\mathrm{conn}}(x) = \log(F(x)), \quad Z_{F^{\mathrm{conn}}}(x_1, x_2, x_3, \dots) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log Z_F(x_k, x_{2k}, x_{3k}, \dots). \tag{2.27}$$

Example 2.1. The virtual species, $-\operatorname{Lg}(1-X)$, turns out to be closely related to *Lyndon words* and *free Lie algebras* and has been called $\operatorname{Lie}(X)$ by Joyal [Joy86]. Recall that a Lyndon word [Lyn54] is an aperiodic word on a totally ordered alphabet which is lexicographically minimal among all its circular shifts. Lyndon words can be used to build a basis for free Lie algebras. The following combinatorial equations hold:

$$\operatorname{Lie}(X) = -\operatorname{Lg}(1-X) = \operatorname{Lg}\left(\frac{1}{1-X}\right) = \operatorname{Lg}(L(X)), \tag{2.28}$$

where $L(X) = 1 + X + X^2 + \cdots$ is the species of linear orders. Hence, we can write

$$\operatorname{Lie} = L^{\mathrm{conn}}, \tag{2.29}$$

so that Lie can be thought of as the virtual species of "connected" linear orders and

$$Z_{\operatorname{Lie}}(x_1, x_2, x_3, \dots) = \sum_{k\geq 1} \frac{\mu(k)}{k} \log \frac{1}{1-x_k}, \quad \operatorname{Lie}(x) = \log\frac{1}{1-x}, \quad \widetilde{\operatorname{Lie}}(x) = x, \tag{2.30}$$

by Corollary 2.2, since $\widetilde{\operatorname{Lie}}(x) = -\sum_{k\geq 1} \frac{\mu(k)}{k} \log \frac{1}{1-x^k}$. Equations (2.28)–(2.30) are of fundamental importance. They will be used later in our treatment of the classical cyclotomic identity and a symmetric extension of it, due to Volker Strehl (see Example 3.1 below).

On the other hand, the molecular expansion of the species $\widehat{X}$ of *pseudo-singletons* is much simpler than that of the combinatorial logarithm. It can be obtained by simply expanding (2.6) as a sum of monomials in $E_1 = X, E_2, E_3, \dots$. Its first few terms, up to degree 6, are given by

$$\widehat{X} = \widehat{X}_+ - \widehat{X}_-, \tag{2.31}$$

where

$$\widehat{X}_+ = X + E_2 + \frac{1}{3}X^3 + E_3 + X^2E_2 + E_4 + \frac{1}{5}X^5 + X^2E_3 + XE_2^2 + E_5 \\ + X^4E_2 + X^2E_4 + 2XE_2E_3 + \frac{1}{3}E_2^3 + E_6 + \cdots, \tag{2.32}$$

$$\widehat{X}_- = \frac{1}{2}X^2 + XE_2 + \frac{1}{4}X^4 + XE_3 + \frac{1}{2}E_2^2 + X^3E_2 + XE_4 + E_2E_3 \\ + \frac{1}{6}X^6 + X^3E_3 + \frac{3}{2}X^2E_2^2 + XE_5 + E_2E_4 + \frac{1}{2}E_3^2 + \cdots. \tag{2.33}$$

In order to collect the homogeneous components of degree $m = 1, 2, \dots$ in the expansion (2.6) for $\widehat{X}$ we proceed as follows. Consider the species $E_m(tX)$ of finite m-sets of singletons in which each singleton has weight t. Since the weights behave multiplicatively, the weight of each m-set is t^m. This means that the following expansion holds in $\mathbb{N}_t||X||$:

$$E(tX) = 1 + E_+(tX) = 1 + tE_1(X) + t^2E_2(X) + \cdots + t^mE_m(X) + \cdots . \tag{2.34}$$

Now, denote by $P_m(X)/m \in \mathbb{Q}||X||$ the coefficient of t^m in the expansion of the analytic logarithm of (2.34) in ascending powers of t,

$$\log(E(tX)) = \log(1 + E_+(tX)) = \sum_{m\geq 1} \frac{t^m P_m(X)}{m}. \tag{2.35}$$

Equivalently, we can write

$$E(tX) = \exp\left(\sum_{m\geq 1} \frac{t^m P_m(X)}{m}\right). \tag{2.36}$$

Application of the differential operator $t\frac{d}{dt}$ to both sides of (2.34), taking (2.36) into account, produces the combinatorial equality

$$\sum_{m\geq 1} mt^m E_m(X) = \left(\sum_{i\geq 1} t^i P_i(X)\right) \cdot \left(\sum_{j\geq 0} t^j E_j(X)\right). \tag{2.37}$$

Comparing the coefficient of t^m on both sides, we deduce the following recursive scheme for the computation of $P_m(X)$:

$$P_1 = X, \quad mE_m = P_m + E_1P_{m-1} + E_2P_{m-2} + \cdots + E_{m-1}P_1, \quad m > 1, \tag{2.38}$$

which exhibits a similarity with the Newton-type relations between homogeneous and power sum symmetric functions h_m and p_m.[21] Note that this implies that $P_m = P_m(X) \in \mathbb{Z}||X||$ are *virtual* species of degree m, and that $P_m(1) = E_m(1) = 1$. Also, letting $t = 1$ in (2.35) and (2.36), we get the two fundamental equations

$$\widehat{X} = \sum_{m\geq 1} \frac{1}{m} P_m(X), \tag{2.39}$$

$$E(X) = \exp\left(\sum_{m\geq 1} \frac{1}{m} P_m(X)\right). \tag{2.40}$$

An important and very useful property of the virtual species P_m is that they are *plethystic linear* of order m in the following sense.

Proposition 2.4 ([Lab08]). *For every power series $\alpha, \beta, \cdots \in \mathbb{C}_{u,v,\ldots}$ and sorts of singletons $X, Y, \ldots$, we have*

$$P_m \circ (\alpha X + \beta Y + \cdots) = \alpha_m P_m(X) + \beta_m P_m(Y) + \cdots , \tag{2.41}$$

where the notational convention (1.47)–(1.48) *for $\alpha_m, \beta_m, \ldots$ is used. In particular:*

[21]The Newton relation reads $mh_m = p_m + h_1p_{m-1} + \cdots + h_{m-1}p_1$. In fact, the species $E_m(X)$ and $P_m(X)$ can be seen as "combinatorial liftings" to $\mathbb{Z}||X||$ of the series h_m and p_m (see (2.42)). Note that $Z_{P_m} = x_m$, but $P_m(P_n(X)) \neq P_{mn}(X)$ in general, while $(p_m)_n = p_{mn}$.

(a) *Taking* $\alpha = u + v + \dots$, $0 = \beta = \gamma = \cdots$, $X := 1$, *we have*

$$P_m(u + v + \cdots) = u^m + v^m + \cdots = p_m(u, v, \dots), \tag{2.42}$$

which is the usual m*-th power sum symmetric function in the variables* $u, v, \dots$.

(b) *Taking* $\alpha = a$, $\beta = b$, $\dots$, *where* $a, b, \dots$ *are complex numbers, we have*

$$P_m \circ (aX + bY + \cdots) = aP_m(X) + bP_m(Y) + \cdots, \tag{2.43}$$

which means that P_m *is* $\mathbb{C}$*-linear in the usual sense.*

(c) *Moreover,* $\widehat{X}$ *is also* $\mathbb{C}$*-linear:*

$$(aX + bY + \cdots)\widehat{\ } = a\widehat{X} + b\widehat{Y} + \cdots. \tag{2.44}$$

Proof. By (1.14) we have $E(kX) = E(X)^k$ for any $k \in \mathbb{N}$. Hence, replacing t in (2.36) by any power product μ in the variables $u, v, \dots$, we obtain

$$E(k\mu X) = E(\mu X)^k = \exp\left(k \sum_{m\geq 1} \frac{\mu^m P_m(X)}{m}\right) = \exp\left(\sum_{m\geq 1} \frac{k\mu^m P_m(X)}{m}\right).$$

More generally, using again (1.14), it follows that, for $\alpha = k\mu + \ell\nu + \cdots \in \mathbb{N}_{u,v,\dots}$, we have

$$E(\alpha X) = E((k\mu + \ell\nu + \cdots)X) = E(k\mu X)E(\ell\nu X)\cdots \tag{2.45}$$

$$= \exp\left(\sum_{m\geq 1} \frac{(k\mu^m + \ell\nu^m + \cdots)P_m(X)}{m}\right) = \exp\left(\sum_{m\geq 1} \frac{\alpha_m P_m(X)}{m}\right). \tag{2.46}$$

Since the coefficient $c_{i,j,\dots} = c_{i,j,\dots}(k, \ell, \dots)$ of each individual term

$$\omega X^n/H = \left(\sum_{i,j,\dots} c_{i,j,\dots} u^i v^j \cdots\right) X^n/H$$

appearing in the full molecular expansions of (2.45)–(2.46) is a polynomial[22] in $k, \ell, \dots$ with coefficients in $\mathbb{Q}$, we can replace $k, \ell, \dots$ by any complex numbers $a, b, \dots$ and (2.45)–(2.46) hold for every $\alpha \in \mathbb{C}_{u,v,\dots}$. Finally, (2.41) follows from the fact that

$$E(\alpha X + \beta Y + \cdots) = E(\alpha X)E(\beta Y)\cdots. \qquad \square$$

Note. Using notational convention (1.47)–(1.48), we can *define* $P_m \circ (\alpha X + \beta Y + \cdots)$ by (2.41) for any $\alpha, \beta, \dots \in \mathbb{C}_{x,y,\dots;u,v,\dots}$, where $x, y, \dots$ are auxiliary variables.

The following computational scheme for the expansion of the combinatorial logarithm, $\mathrm{Lg}(1 + X)$, will be used in the following sections of this paper. It is a consequence of the plethystic linearity of the virtual species $P_m(X)$.

[22] This is essentially due to the fact

$$E(k\mu X) = E(\mu X)^k = (1 + E_+(\mu X))^k = \sum_{i\geq 0} \frac{k(k-1)\cdots(k-i+1)}{i!} E_+(\mu X)^i.$$

Proposition 2.5 ([Lab08]). *Let*

$$\mathrm{Lg}(1+X) = \Omega(X) = \Omega_1(X) + \Omega_2(X) + \cdots + \Omega_n(X) + \cdots, \tag{2.47}$$

where $\Omega_n(X)$ is of degree n in X. Then $\Omega_1 = X$, and, for $n > 1$, we have

$$\Omega_n(X) = \frac{(-1)^{n-1}}{n} X^n - \sum_{1<d|n} \frac{1}{d} P_d \circ \Omega_{n/d}(X). \tag{2.48}$$

Proof. We give a more direct proof than that given in [Lab08]. By (2.8), (2.39), and (2.47), we can write

$$\widehat{\mathrm{Lg}(1+X)} = \left(\sum_{d\geq 1} \frac{1}{d} P_d\right) \circ \left(\sum_{k\geq 1} \Omega_k\right) = \log(1+X). \tag{2.49}$$

Now, since $P_d \circ \Omega_k$ is of degree kd in X, collecting terms of degree n in X on both sides of (2.49), we get

$$\sum_{d|n} \frac{1}{d} P_d \circ \Omega_{n/d}(X) = \frac{(-1)^{n-1}}{n} X^n,$$

from which (2.48) immediately follows since $P_1(X) = X$. □

The reader is referred to [Lab08] and [Lab13] for more information about pseudo-singletons and the combinatorial logarithm together with applications to the computation of the molecular expansion of certain classes of species (including rooted trees, for example).

3. The binomial species, basic properties, and associated expansions

3.1. Definition of the binomial species and generalized binomial coefficients.

The classical Newton binomial expansion of the 2-variable series

$$(1+X)^\wedge Y = (1+X)^Y = \exp(Y \log(1+X)) \tag{3.1}$$

can be stated as

$$(1+X)^Y = \sum_{n\geq 0} \binom{Y}{n} X^n, \quad \text{where } \binom{Y}{n} = \frac{Y(Y-1)(Y-2)\cdots(Y-n+1)}{n!}. \tag{3.2}$$

Hence, $(1+X)^Y \in \mathbb{Q}[[X,Y]]$. By analogy, we define the 2-sort binomial species $B(X,Y)$ and associate to it "generalized binomial coefficients" as follows.

Definition 3.1. Let X and Y be two sorts of singletons. The *binomial species*, $B(X,Y) = (1+X)^{\uparrow}Y$, is a (virtual) species defined by the combinatorial equation[23]

$$(1+X)^{\uparrow}Y = E(Y\,\mathrm{Lg}(1+X)), \tag{3.3}$$

[23] Recall that the notation $(1+X)^{\uparrow}Y$ is used instead of $(1+X)^\wedge Y = \exp(Y\log(1+X)) \in \mathbb{Q}[[X,Y]]$ to put the emphasis on the fact that $B(X,Y)$ is a species, *not* a power series in the variables X and Y.

where $E = E(X)$ is the species of finite sets and $\mathrm{Lg}(1+X)$ is the combinatorial logarithm. The *generalized binomial coefficients*, $\binom{X,Y}{n}$, $n = 0, 1, 2, \ldots$, are the species defined by

$$(1+X)^{\uparrow}Y = \sum_{n\geq 0} \binom{X,Y}{n}, \tag{3.4}$$

where $\binom{X,Y}{n}$ is the sum of the terms of degree n in the variable X in the molecular expansion of $B(X,Y)$.

Making use of first terms of the expansion of the combinatorial logarithm in (2.23)–(2.24) and the basic properties of the species E of finite sets, the first few generalized binomial coefficients are

$$\binom{X,Y}{0} = 1, \qquad \binom{X,Y}{1} = XY, \qquad \binom{X,Y}{2} = -YE_2(X) + E_2(XY), \tag{3.5}$$

$$\binom{X,Y}{3} = -YE_3(X) + XYE_2(X) - XY^2E_2(X) + E_3(XY), \tag{3.6}$$

$$\begin{aligned}\binom{X,Y}{4} = &-YE_4(X) + YE_2 \circ E_2(X) + XYE_3(X) - X^2YE_2(X) - XY^2E_3(X)\\ &+ X^2Y^2E_2(X) + Y^2(E_2(X))^2 - E_2(YE_2(X)) - YE_2(X)E_2(XY) + E_4(XY).\end{aligned} \tag{3.7}$$

Note. For each n, $\binom{X,Y}{n}$ is a finite sum, and, since the basic combinatorial operations on species are compatible with the corresponding analytic operations on formal power series, we see, by Tables 1 and 2, that the *underlying* exponential generating series $B(x,y)$ of $B(X,Y)$ and $\binom{x,y}{n}$ of $\binom{X,Y}{n}$ satisfy

$$B(x,y) = (1+x)^y, \qquad \binom{x,y}{n} = \binom{y}{n}x^n. \tag{3.8}$$

In fact, $B(X,Y)$ and $\binom{X,Y}{n}$ are (much) more refined mathematical objects than their analytic counterparts (3.8). For example, by regrouping similar terms in the computation of the *underlying* exponential power series of (3.6), we get

$$\begin{aligned}\binom{x,y}{3} &= \binom{X,Y}{3}_{X=x,Y=y}\\ &= (-yE_3(x) + xyE_2(x)) - xy^2E_2(x) + E_3(xy)\\ &= \underbrace{(-y\frac{x^3}{3!} + xy\frac{x^2}{2!})} - xy^2\frac{x^2}{2!} + \frac{(xy)^3}{3!}\\ &= \qquad y\frac{x^3}{3} \qquad - y^2\frac{x^3}{2} + y^3\frac{x^3}{6} = \frac{1}{3!}y(y-1)(y-2)x^3 = \binom{y}{3}x^3.\end{aligned}$$

Such nice factorizations do not occur in general for $\binom{X,Y}{n}$. For example, X or Y cannot be factored out in $\binom{X,Y}{n}$ for each value of $n \geq 2$.

3.2. **Basic properties of $B(X,Y)$ and $\binom{X,Y}{n}$.** Although structurally more complicated than their analytic counterparts, the binomial species and generalized binomial coefficients share with them some basic identities.

Proposition 3.1. *The binomial species $B(X,Y) = (1+X)^{\uparrow}Y$ satisfies the equations*

$$(1+X)^{\uparrow}(Y+Z) = (1+X)^{\uparrow}Y \cdot (1+X)^{\uparrow}Z, \tag{3.9}$$

$$((1+X)(1+Y))^{\uparrow}Z = (1+X)^{\uparrow}Z \cdot (1+Y)^{\uparrow}Z, \tag{3.10}$$

$$(1+X)^{\uparrow}(Y\cdot Z) = ((1+X)^{\uparrow}Y)^{\uparrow}Z, \tag{3.11}$$

$$\frac{\partial}{\partial X}(1+X)^{\uparrow}Y = (1+X)^{\uparrow}Y \cdot \frac{Y}{1+X}, \tag{3.12}$$

$$\frac{\partial}{\partial Y}(1+X)^{\uparrow}Y = (1+X)^{\uparrow}Y \cdot \mathrm{Lg}(1+X), \tag{3.13}$$

where X, Y, Z denote three sorts of singletons.

Proof. Formulas (3.9)–(3.10) follow from (1.14). Indeed, let $\Omega = \Omega(X) = \mathrm{Lg}(1+X)$. Then

$$\begin{aligned}
(1+X)^{\uparrow}(Y+Z) &= E((Y+Z)\Omega(X)) = E(Y\Omega(X) + Z\Omega(X)) \\
&= E(Y\Omega(X))E(Y\Omega(X)) = (1+X)^{\uparrow}Y \cdot (1+X)^{\uparrow}Z, \\
((1+X)(1+Y))^{\uparrow}Z &= (1+(X+Y+XY))^{\uparrow}Z = E(Z\,\mathrm{Lg}((1+X+Y+XY)) \\
&= E(Z\Omega(X) + Z\Omega(Y)) = E(Z\Omega(X))E(Z\Omega(Y)) \\
&= (1+X)^{\uparrow}Z \cdot (1+Y)^{\uparrow}Z.
\end{aligned}$$

The proof of (3.11) is more involved:

$$\begin{aligned}
((1+X)^{\uparrow}Y)^{\uparrow}Z &= (1+X)^{\uparrow}Z|_{X:=E_+(Y\Omega(X))} = E(Z\Omega(X))|_{X:=E_+(Y\Omega(X))} \\
&= E(Z\cdot\Omega\circ E_+(Y\Omega(X))) = E(Z\cdot Y\Omega(X)) = (1+X)^{\uparrow}(Y\cdot Z),
\end{aligned}$$

since $\Omega\circ E_+(X) = X$. The differential formulas (3.12)–(3.13) are consequences of the combinatorial chain-rule and (2.17)–(2.18):

$$\begin{aligned}
\frac{\partial}{\partial X}(1+X)^{\uparrow}Y &= \frac{\partial}{\partial X}E(Y\,\mathrm{Lg}(1+X)) = E(Y\,\mathrm{Lg}(1+X))\frac{\partial}{\partial X}Y\,\mathrm{Lg}(1+X) \\
&= (1+X)^{\uparrow}Y\cdot\frac{Y}{1+X}, \\
\frac{\partial}{\partial Y}(1+X)^{\uparrow}Y &= \frac{\partial}{\partial Y}E(Y\,\mathrm{Lg}(1+X)) = E(Y\,\mathrm{Lg}(1+X))\frac{\partial}{\partial Y}Y\,\mathrm{Lg}(1+X) \\
&= (1+X)^{\uparrow}Y\cdot\mathrm{Lg}(1+X).
\end{aligned}$$

□

Corollary 3.2. *The binomial coefficients $\binom{X,Y}{n}$ satisfy the "Vandermonde-like" identities*

$$\binom{X,Y+Z}{n} = \sum_{i+j=n}\binom{X,Y}{i}\binom{X,Z}{j}, \qquad n = 0,1,2,\ldots. \tag{3.14}$$

In particular, substitution of $Z := 1$ gives

$$\binom{X, Y+1}{n} = \binom{X, Y}{n} + \binom{X, Y}{n-1} X, \qquad n = 1, 2, \ldots. \tag{3.15}$$

Proof. For (3.14), simply collect terms of degree n in X on both sides of (3.9), for each $n \geq 0$. Formula (3.15) follows from (3.14) using the fact that

$$(1+X)^1 = E(1 \cdot \mathrm{Lg}(1+X)) = 1 + X = \binom{X, 1}{0} + \binom{X, 1}{1} + 0,$$

so that $\binom{X,1}{0} = 1$, $\binom{X,1}{1} = X$, and $\binom{X,1}{j} = 0$ for $j \geq 2$. $\square$

Note. Taking underlying exponential power series of identities (3.14) and (3.15), we are led to the classical identities

$$\binom{y+z}{n} = \sum_{i+j=n} \binom{y}{i}\binom{z}{j}, \quad \binom{y+1}{n} = \binom{y}{n} + \binom{y}{n-1}, \tag{3.16}$$

since $\binom{x,y}{n} = \binom{y}{n} x^n$ by (3.8).

Corollary 3.3. *The combinatorial partial derivative $\frac{\partial}{\partial X}\binom{X,Y}{n}$ is given by*

$$\frac{\partial}{\partial X}\binom{X, Y}{n} = Y \sum_{i=0}^{n-1} (-1)^i X^i \binom{X, Y}{n-1-i}. \tag{3.17}$$

Proof. Since the operator $\frac{\partial}{\partial X}$ decreases the degree in X of its operand by 1, the partial derivative $\frac{\partial}{\partial X}\binom{X,Y}{n}$ is the sum of terms of X- degree $n-1$ in $\frac{\partial}{\partial X}(1+X)^{\uparrow}Y$. Explicitly, by (3.12) we have

$$\frac{\partial}{\partial X}\binom{X, Y}{n} = \text{sum of terms of } X\text{- degree } n-1 \text{ in } Y \sum_{i \geq 0} (-1)^i X^i \sum_{j \geq 0} \binom{X, Y}{j}. \qquad \square$$

Corollary 3.4. *The binomial species $B(X, Y)$ satisfies*

$$B(-X, -Y) = B(X/(1-X), Y) \tag{3.18}$$

$$= B(X, Y) B(X^2, Y) B(X^4, Y) B(X^8, Y) \cdots. \tag{3.19}$$

Proof. First, by (1.12), we have

$$\frac{1}{1-X} = 1 + X + X^2 + X^3 + \cdots = 1 + X \cdot (1 + X + X^2 + \cdots) = 1 + \frac{X}{1-X}.$$

Hence, by (1.13) and (3.9) we get

$$\begin{aligned} B(-X, -Y) &= E(-Y \,\mathrm{Lg}(1-X)) = 1/E(Y \,\mathrm{Lg}(1-X)) = 1/(1-X)^{\uparrow}Y \\ &= \left(\frac{1}{1-X}\right)^{\uparrow} Y = \left(1 + \frac{X}{1-X}\right)^{\uparrow} Y = B(X/(1-X), Y), \end{aligned}$$

which establishes (3.18). On the other hand, (3.19) follows by a passage to the limit,

$$\begin{aligned}\left(\frac{1}{1-X}\right)^{\uparrow} Y &= ((1+X)(1+X^2)(1+X^4)(1+X^8)\cdots)^{\uparrow} Y \\ &= (1+X)^{\uparrow}Y \cdot (1+X^2)^{\uparrow}Y \cdot (1+X^4)^{\uparrow}Y \cdot (1+X^8)^{\uparrow}Y \cdots \\ &= B(X,Y)B(X^2,Y)B(X^4,Y)B(X^8,Y)\cdots ,\end{aligned}$$

making use of (3.10). □

Corollary 3.5. *The cycle index series $Z_B = Z_B(x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots)$ of the binomial species $B(X,Y)$ is given by*

$$Z_B = Z_{(1+X)^{\uparrow}Y} = \prod_{k\geq 1}(1+x_k)^{\frac{1}{k}\sum_{d|k}\mu(d)y_{k/d}}. \tag{3.20}$$

Proof. By Tables 1 and 2, and plethystic substitution, we have

$$\begin{aligned}Z_B &= Z_{E(Y\,\mathrm{Lg}(1+X))} = Z_E \circ (Z_Y \cdot Z_{\mathrm{Lg}(1+X)}) = \exp\sum_{i\geq 1}\tfrac{1}{i}(Z_Y\cdot Z_{\mathrm{Lg}(1+X)})_i \\ &= \exp\sum_{i\geq 1}\tfrac{1}{i}y_i \cdot \sum_{j\geq 1}\tfrac{\mu(j)}{j}\log(1+x_{ij}) = \exp\sum_{k\geq 1}\frac{1}{k}\left(\sum_{ij=k}\mu(j)y_i\right)\log(1+x_k) \\ &= \prod_{k\geq 1}(1+x_k)^{\frac{1}{k}\sum_{ij=k}\mu(j)y_i}.\end{aligned}$$

□

3.3. **Formulas obtained by specializing variables in $B(X,Y)$.** A variety of more or less “exotic” formulas, identities and q-identities will now be obtained by substituting power series for X and Y in the binomial species $B(X,Y)$ and by making use of Proposition 3.1, Corollaries 3.2–3.4 together with Tables 1 and 2.

Making use of the notational convention (1.46)–(1.48), we state and prove first the following general proposition in which

$$(1+\alpha)^{\uparrow}\beta \text{ means } (1+X)^{\uparrow}Y|_{X=\alpha,Y=\beta}, \text{ and } \binom{\alpha,\beta}{n} \text{ means } \left.\binom{X,Y}{n}\right|_{X=\alpha,Y=\beta}, \tag{3.21}$$

while $(1+\alpha)^{\beta}$ means the usual “analytic/algebraic exponentiation” of power series:

$$\sum_{n\geq 0}\frac{\beta(\beta-1)(\beta-2)\cdots(\beta-n+1)}{n!}\alpha^n = \sum_{n,k\geq 0}\frac{s(n,k)}{n!}\alpha^n\beta^k, \tag{3.22}$$

in which the $s(n,k)$'s are the (signed) Stirling numbers of the first kind.

Proposition 3.6 (Substitution of series in B). *Let $B(X,Y) = (1+X)^{\uparrow}Y = \sum_{n\geq 0}\binom{X,Y}{n}$ be the binomial species, $u, v, q, t, \ldots$ be weight variables, and $x, y, \ldots$ be auxiliary variables. Then, for power series $\alpha, \beta \in \mathbb{C}_{x,y,\ldots;u,v,q,t,\ldots}$, we have, assuming summability,*

$$B(\alpha,\beta) = (1+\alpha)^{\uparrow}\beta = \prod_{k\geq 1}(1+\alpha_k)^{\frac{1}{k}\sum_{d|k}\mu(d)\beta_{k/d}} = \sum_{n\geq 0}\binom{\alpha,\beta}{n}. \tag{3.23}$$

The binomial coefficients $\binom{\alpha,\beta}{n}, n \geq 0$, *are series satisfying the recursive scheme*

$$\binom{\alpha,\beta}{0} = 1, \quad \binom{\alpha,\beta}{n} = \frac{1}{n}\left[\theta(1)\binom{\alpha,\beta}{n-1} + \cdots + \theta(\ell)\binom{\alpha,\beta}{n-\ell} + \cdots\right], \; n > 0, \quad (3.24)$$

in which the $\theta(\ell)$*'s are explicitly given by*

$$\theta(1) = \alpha_1\beta_1, \quad \theta(2) = \alpha_2\beta_2 - \alpha_1^2\beta_1 - \alpha_2\beta_1, \quad \theta(3) = \alpha_3\beta_3 + \alpha_1^3\beta_1 - \alpha_3\beta_1, \quad (3.25)$$

$$\theta(\ell) = \sum_{ijk=\ell} (-1)^{i-1}\mu(j)\alpha_{jk}^i\beta_k, \quad \ell \geq 1. \quad (3.26)$$

Proof. Formula (3.23) is a direct consequence of formula (3.20) for the cycle index series of $B(X,Y)$ together with the substitution formula (1.46), taking into account the notational convention (1.47)–(1.48). The recursive scheme described by (3.24)–(3.26) is more delicate and can be established by introducing first an extra weight variable, s, in (3.23) as follows:

$$B(s\alpha,\beta) = (1+s\alpha)^{\uparrow}\beta = \prod_{k\geq 1}(1+s^k\alpha_k)^{\frac{1}{k}\sum_{d|k}\mu(d)\beta_{k/d}} = \sum_{n\geq 0}\binom{\alpha,\beta}{n}s^n, \quad (3.27)$$

where the last equality is due to the fact that $\binom{X,Y}{n}$ is of degree n in X. Then application of the differential operator $s\frac{d}{ds}$ to (3.27) gives

$$\sum_{n\geq 1} n\binom{\alpha,\beta}{n}s^n = s\frac{d}{ds}\prod_{k\geq 1}(1+s^k\alpha_k)^{\gamma(k)} = \sum_{m\geq 1} m\gamma(m)\frac{\alpha_m s^m}{1+\alpha_m s^m}\cdot\prod_{k\geq 1}(1+s^k\alpha_k)^{\gamma(k)}$$

$$= \sum_{m\geq 1} m\gamma(m)\frac{\alpha_m s^m}{1+\alpha_m s^m}\cdot\sum_{\nu\geq 0}\binom{\alpha,\beta}{\nu}s^\nu, \quad (3.28)$$

where $\gamma(m) = \frac{1}{m}\sum_{d|m}\mu(d)\beta_{m/d}$. Expanding (3.28) according to powers of s, we obtain

$$\sum_{n\geq 1} n\binom{\alpha,\beta}{n}s^n = \sum_{i,j,k,\nu\geq 1}(-1)^{i-1}\mu(j)\alpha_{jk}^i\beta_k\binom{\alpha,\beta}{\nu}s^{ijk+\nu}. \quad (3.29)$$

We conclude by equating the coefficient of s^n in both sides of (3.29) using the fact that $ijk+\nu = n$ if and only if $\nu = n-\ell$ and $ijk = \ell$. ☐

Note. While summability is needed in the infinite product (3.23), no summability conditions are needed in (3.24) since each $\binom{X,Y}{n}$ is of finite degree in X and Y. Hence the binomial coefficients $\binom{\alpha,\beta}{n}$ are always well-defined series. For example, let v be a weight variable, then taking $\alpha = 1$ and $\beta = v$ in (3.23), we have

$$B(1,v) = (1+1)^{\uparrow}v = \prod_{k\geq 1} 2^{\frac{1}{k}\sum_{d|k}\mu(d)v^{k/d}} = 2^{\sum_{n\geq 1}(\sum_{i\geq 1}\mu(i)/i)v^n/n}, \quad (3.30)$$

and the coefficient of v^n on the right-hand side of (3.30) is not a finite sum but a (conditionally convergent) infinite series $\frac{1}{n}\sum_{i\geq 1}\mu(i)/i$. Hence $B(1,v)$ is not well-defined. However, the corresponding binomial coefficients $\binom{1,v}{n}$ are well-defined and can be computed recursively via

$$\binom{1,v}{0} = 1, \quad \binom{1,v}{n} = \frac{1}{n}\left[\theta(1)\binom{1,v}{n-1} + \cdots + \theta(\ell)\binom{1,v}{n-\ell} + \cdots\right], \ n > 0, \tag{3.31}$$

in which the $\theta(\ell)$'s are given by

$$\theta(\ell) = \begin{cases} v^\ell, & \ell \text{ odd} \geq 1, \\ v^\ell - 2v^{\ell/2}, & \ell \text{ even} \geq 2. \end{cases} \tag{3.32}$$

Example 3.1. The cyclotomic identity reads

$$\frac{1}{1-au} = \prod_{n\geq 1}\left(\frac{1}{1-u^n}\right)^{M(a,n)}, \tag{3.33}$$

where

$$M(a,n) = \frac{1}{n}\sum_{d|n}\mu(n/d)a^d, \quad n = 1, 2, \ldots \tag{3.34}$$

are the (*aperiodic*) *necklace polynomials*. These polynomials are also called the *Lyndon polynomials* since $M(a,n)$ is the number of Lyndon words of length n over an a-letter ordered alphabet, for $a \in \mathbb{N}$ (see Example 2.1 above).

Now let a be a complex variable and u be a weight variable. The cyclotomic identity is a special case of (3.23) of Proposition 3.6, which is seen by taking $\alpha = -au$ and $\beta = -1$. Indeed, we have

$$\begin{aligned}\frac{1}{1-au} &= B(-au,-1) = \exp\sum_{i,j\geq 1}\frac{\mu(i)}{ij}\log\frac{1}{1-au^{ij}} = \exp\sum_{i,j,k\geq 1}\frac{\mu(i)}{ijk}a^k u^{ijk} \\ &= \exp\left(\sum_{n\geq 1}\left(\sum_{j\geq 1}\frac{u^{nj}}{j}\right)\left(\frac{1}{n}\sum_{ik=n}\mu(i)a^k\right)\right) = \prod_{n\geq 1}\left(\frac{1}{1-u^n}\right)^{M(a,n)}.\end{aligned}$$

Stated differently, this is a consequence of the combinatorial identities (see (2.28))

$$\frac{1}{1-aX} = L(aX) = B(-aX,-1) = E\circ \mathrm{Lg}\left(\frac{1}{1-aX}\right) = E(\mathrm{Lie}(aX)). \tag{3.35}$$

To see this, take a totally ordered alphabet A of a letters (for $a \in \mathbb{N}$). Then upon evaluation of (3.35) at $X = u = u\cdot 1$, each singleton becomes unlabelled and assigned a weight u, and $1/(1-au) = E(\mathrm{Lie}(au))$. Since $\mathrm{Lie} = L^{\mathrm{conn}}$ (see (2.29)), this means that

$$\underline{\textit{words over } A} \quad \text{are} \quad \underline{\textit{multisets of “connected” words over } A}. \tag{3.36}$$

This fact was first stated by de Bruijn and Klarner [deBK82]. In their paper, connected words over A are called *aperiodic cycles* (see also [FS91]). Since Lyndon words over A are totally ordered by the induced lexicographic order, multisets of Lyndon words can be canonically written as weakly decreasing sequences of Lyndon words. This corresponds to the classical Lyndon theorem.

Theorem 3.7 (LYNDON [Lyn54]). *Every word ω over a totally ordered alphabet A has a unique factorization*

$$\omega = \lambda_1 \lambda_2 \cdots \lambda_k, \tag{3.37}$$

where λ_i is a Lyndon word over A, $1 \le i \le k$, and the sequence $(\lambda_i)_{i=1,\dots,k}$ is lexicographically weakly decreasing

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k. \tag{3.38}$$

The number k of Lyndon words in factorization (3.37) is called the *Lyndon index* of the word ω and is denoted $ind(\omega)$.

Since $B(-au, -1) = (1 - au)^{-1}$ is a geometric series, it immediately follows that

$$\binom{-au, -1}{n} = a^n u^n \quad \text{and} \quad \binom{-a, -1}{n} = a^n. \tag{3.39}$$

Example 3.2. Using combinatorial arguments, independent from Pólya theory, Strehl [Str92] proved an unexpected duality between the alphabet size, a, and a Lyndon index counter, t, about words on a totally ordered alphabet. As a by-product of this fact, he obtained a more general "symmetric version" of the cyclotomic identity:

$$\prod_{n\ge 1} \left(\frac{1}{1 - au^n}\right)^{M(t,n)} = \prod_{n\ge 1} \left(\frac{1}{1 - tu^n}\right)^{M(a,n)}. \tag{3.40}$$

Algebraically speaking, (3.40) can independently be checked by noting that its left-hand side can be expanded in the form

$$\prod_{n\ge 1} \left(\frac{1}{1 - au^n}\right)^{M(t,n)} = \exp \sum_{i,j,k \ge 1} a^i t^j \mu(k) \frac{u^{ijk}}{ijk}, \tag{3.41}$$

which is obviously symmetric in a, t. Identity (3.33) corresponds to the case where $t = 1$. From the point of view of species and Pólya theory, the left-hand side of (3.40) follows from the combinatorial equation

$$\left(\frac{1}{1 - aX}\right)^{\uparrow} tY = L(aX)^{\uparrow} tY = B(-aX, -tY) = E \circ tY \operatorname{Lg}\left(\frac{1}{1 - aX}\right) = E(tY \operatorname{Lie}(aX)), \tag{3.42}$$

where a is a complex variable and t is a weight variable. Then upon evaluation of (3.42) at $X = u = u \cdot 1$ and $Y = 1$, this means (see (2.29)) that

$$\underline{\textit{words over } A \textit{ with index } k} \quad \text{are} \quad \underline{k\textit{-multisets of Lyndon words over } A}. \tag{3.43}$$

Equivalently, this corresponds to the following "enrichment" of the cyclotomic identity:

$$\left(\frac{1}{1 - au}\right)^{\uparrow} t = \prod_{n \ge 1} \left(\frac{1}{1 - tu^n}\right)^{M(a,n)} = \sum_{n \ge 0} s_n(a, t) u^n, \tag{3.44}$$

where the $s_n(a,t)$ are the symmetric polynomials in a and t defined by Strehl in [Str92].[24] In fact, we have

$$\binom{-au,-t}{n} = s_n(a,t), \quad n \geq 0. \tag{3.45}$$

We now illustrate Proposition 3.6 in very specific simple cases. Let x, y be auxiliary variables and u, v be weight variables, and consider the four possibilities

$$(1+x)^{\uparrow}y, \quad (1+u)^{\uparrow}y, \quad (1+x)^{\uparrow}v, \quad (1+u)^{\uparrow}v, \tag{3.46}$$

according to the values assigned to α and β.

Example 3.3. Let x and y be auxiliary variables. Then

$$(1+x)^{\uparrow}y = (1+x)^y = \sum_{n\geq 0}\binom{x,y}{n}, \quad \binom{x,y}{n} = \binom{y}{n}x^n, \quad \binom{x,y}{0} = 1, \tag{3.47}$$

$$\binom{x,y}{n} = \frac{1}{n}\left[xy\binom{x,y}{n-1} + \cdots + (-1)^{\ell-1}x^{\ell}y\binom{x,y}{n-\ell} + \cdots\right], \; n > 0. \tag{3.48}$$

Proof. Let $\alpha = x$ and $\beta = y$ in (3.23)–(3.26). Since $\alpha_1 = x$, $\beta_1 = y$, $\alpha_k = 0$ and $\beta_k = 0$ for $k > 1$, it follows that $\theta(\ell) = (-1)^{\ell-1}x^{\ell}y$. This corresponds to the classical version of the formal binomial expansion in the variables x and y. □

Example 3.4. Let u be a weight variable and y be an auxiliary variable. Then,

$$(1+u)^{\uparrow}y = \prod_{k\geq 1}(1+u^k)^{y\mu(k)/k} = e^{u(1-u)y} = \sum_{n\geq 0}\binom{u,y}{n}, \tag{3.49}$$

$$\binom{u,y}{0} = 1, \; \binom{u,y}{1} = uy, \; \binom{u,y}{n} = \frac{1}{n}\left[uy\binom{u,y}{n-1} - 2u^2y\binom{u,y}{n-2}\right], \quad n > 1. \tag{3.50}$$

Proof. Let $\alpha = u$ and $\beta = y$ in (3.23)–(3.26). Since $\alpha_k = u^k$ for $k \geq 1$ and $\beta_k = 0$ for $k > 1$, the first equality in (3.49) follows from the fact that $\frac{1}{k}\sum_{d|k}\mu(d)\beta_{k/d} = y\mu(k)/k$. Taking the analytic logarithm, the second equality in (3.49) is equivalent to

$$\sum_{k\geq 1} y\frac{\mu(k)}{k}\log(1+u^k) = y\sum_{k,m\geq 1}(-1)^{m-1}\frac{u^{km}}{km} = (u-u^2)y,$$

which is equivalent to the last equality in (2.14b) of Corollary 2.2. Recurrence (3.50) can be proved via (3.24), or, more simply, by using the fact that

$$u\frac{\partial}{\partial u}e^{u(1-u)y} = y(1-2u)e^{u(1-u)y}.$$ □

Example 3.5. Let x be an auxiliary variable and v be a weight variable. Then,

$$(1+x)^{\uparrow}v = (1+x)^v = \sum_{n\geq 0}\binom{x,v}{n}, \quad \binom{x,v}{n} = \binom{v}{n}x^n, \quad \binom{x,v}{0} = 1, \tag{3.51}$$

$$\binom{x,v}{n} = \frac{1}{n}\left[xv\binom{x,v}{n-1} + \cdots + (-1)^{\ell-1}x^{\ell}v\binom{x,v}{n-\ell} + \cdots\right], \; n > 0. \tag{3.52}$$

[24] In his paper, Strehl used the variables q, λ instead of a, t.

Proof. Let $\alpha = x$ and $\beta = v$ in (3.23)–(3.26). Since $\alpha_k = 0$ for $k > 1$ and $\beta_k = v^k$ for $k \geq 1$, it follows that $\theta(\ell) = (-1)^{\ell-1}x^\ell v$. This again corresponds to the classical version of the formal binomial expansion in x and v. □

Example 3.6. Let u and v be weight variables. Then,

$$(1+u)^{\uparrow}v = \prod_{k\geq 1}(1+u^k)^{\frac{1}{k}\sum_{d|k}\mu(d)v^{k/d}} = \frac{1-u^2v}{1-uv} = \sum_{n\geq 0}\binom{u,v}{n}, \tag{3.53}$$

$$\binom{u,v}{0} = 1,\ \binom{u,v}{1} = uv,\ \binom{u,v}{n} = u^n(v^n - v^{n-1}), \quad n > 1. \tag{3.54}$$

Proof. Let $\alpha = u$ and $\beta = v$ in (3.23)–(3.26). Since $\alpha_k = u^k$ and $\beta_k = v^k$, it follows that

$$\begin{aligned}\theta(\ell) &= \sum_{ijk=\ell}(-1)^{i-1}\mu(j)u^{ijk}v^k = u^\ell\sum_{ijk=\ell}(-1)^{i-1}\mu(j)v^k \\ &= \begin{cases} u^\ell v^\ell, & \ell \text{ odd} \geq 1, \\ u^\ell(v^\ell - 2v^{\ell/2}), & \ell \text{ even} \geq 2. \end{cases}\end{aligned} \tag{3.55}$$

This implies (3.54), which, in turn, implies (3.53). □

Example 3.7. Of course, (q,t)-analogues of the above four examples follow by suitable substitutions. For example, the substitution $v := v/(1-t)$ in (3.53) gives

$$B_{0,t}(u,v) = (1+u)^{\uparrow}(v/(1-t)) = \prod_{i\geq 0}\frac{1-u^2vt^i}{1-uvt^i} = \sum_{n\geq 0}\binom{u,v}{n}_{0,t}. \tag{3.56}$$

Example 3.8. More generally, let $B_{q,t}(X,Y) = B(\frac{1}{1-q}X, \frac{1}{1-t}Y) = \sum_{n\geq 0}\binom{X,Y}{n}_{q,t}$ be the (q,t)-analogue of the binomial species. Then making the substitutions $\alpha := \alpha/(1-q)$, $\beta := \beta/(1-t)$ in Proposition 3.6, it immediately follows that

$$B_{q,t}(\alpha,\beta) = \left(1+\frac{\alpha}{1-q}\right)^{\uparrow}\left(\frac{\beta}{1-t}\right) = \prod_{k\geq 1}\left(1+\frac{\alpha_k}{1-q^k}\right)^{\frac{1}{k}\sum_{d|k}\mu(d)\frac{\beta_{k/d}}{1-t^{k/d}}} = \sum_{n\geq 0}\binom{\alpha,\beta}{n}_{q,t}, \tag{3.57}$$

$\binom{\alpha,\beta}{0}_{q,t} = 1$, and, for $n > 0$,

$$\binom{\alpha,\beta}{n}_{q,t} = \frac{1}{n}\left[\theta_{q,t}(1)\binom{\alpha,\beta}{n-1}_{q,t} + \cdots + \theta_{q,t}(\ell)\binom{\alpha,\beta}{n-\ell}_{q,t} + \cdots\right], \tag{3.58}$$

$$\theta_{q,t}(\ell) = \sum_{ijk=\ell}(-1)^{i-1}\mu(j)\left(\frac{\alpha_{jk}}{1-q^{jk}}\right)^i\frac{\beta_k}{1-t^k}, \quad \ell \geq 1. \tag{3.59}$$

Example 3.9. The classical q-analogue $\binom{n}{k}_q$ appears in expressions of the form $(1+\alpha)^{\uparrow}\beta$ in various "disguises". For example, let u, v, q be weight variables. Then one can easily check that

$$E_q(u)^{\uparrow}(1+v) = \left(1+E_+\left(\frac{u}{1-q}\right)\right)^{\uparrow}(1+v) = \sum_{n,k\geq 0}\binom{n}{k}_q\frac{u^nv^k}{(q;q)_n}. \tag{3.60}$$

Also, a variant of the q-binomial expansion can be written in the form

$$(E_n)_q(u+v) = \sum_{k=0}^{n} \frac{u^k v^{n-k}}{(q;q)_k (q;q)_{n-k}}. \tag{3.61}$$

Example 3.10. Since $B(x,y) = (1+x)^y = \sum_{n,k\geq 0} b_{n,k}\, x^n y^k / n!\, k!$, where $b_{n,k} = k!\, c(n,k)$, and $c(n,k)$ are the unsigned Stirling numbers of the first kind. In view of (1.62), the coefficients $b_{n,k}(q,t)$ in the expansion

$$B_{q,t}(u,v) = \left(1 + \frac{u}{1-q}\right)^{\uparrow}\left(\frac{v}{1-t}\right) = \sum_{n,k\geq 0} b_{n,k}(q,t) \frac{u^n}{(q;q)_n} \cdot \frac{v^k}{(t,t)_k} \tag{3.62}$$

can be considered as (q,t)-analogues of the numbers $k!\, c(n,k)$.

Remark 1. One must be careful when manipulating equalities involving substitutions of series into species. Each resulting equality must follow from a corresponding combinatorial equality about species. For example, given three weight variables, u, v, w,

$$(1+u)^{\uparrow} v = \frac{1-u^2 v}{1-uv} \quad \text{does not imply} \quad (1+u)^{\uparrow}(v+w) = \frac{1-u^2(v+w)}{1-u(v+w)}. \tag{3.63}$$

The correct equations are

$$(1+u)^{\uparrow}(v+w) = \frac{1-u^2 v}{1-uv} \cdot \frac{1-u^2 w}{1-uw} = (1+u)^{\uparrow} v \cdot (1+u)^{\uparrow} w, \tag{3.64}$$

since $B(X, Y+Z) = B(X,Y) \cdot B(X,Z)$. However, of course, if $F(X,Y,Z) = \frac{1-X^2(Y+Z)}{1-X(Y+Z)}$, then $F(u,v,w) = \frac{1-u^2(v+w)}{1-u(v+w)}$.

3.4. **The computation of $B(X,Y)$ to arbitrary large degrees in X.** We now describe an efficient method to compute the species $B(X,Y)$ up to arbitrary large degrees in X by "lifting" (3.24) up to a combinatorial scheme which expresses

$$\binom{X,Y}{n} \text{ in terms of } \binom{X,Y}{n-1}, \binom{X,Y}{n-2}, \ldots,\ n > 0. \tag{3.65}$$

Proposition 3.8. *The following recursive scheme holds*: $\binom{X,Y}{0} = 1$, *and, for* $n > 1$,

$$\binom{X,Y}{n} = \frac{1}{n}\left[\Theta(1)\binom{X,Y}{n-1} + \Theta(2)\binom{X,Y}{n-2} + \cdots + \Theta(k)\binom{X,Y}{n-k} + \cdots\right], \tag{3.66}$$

where the coefficients $\Theta(k) = \Theta(k,X,Y)$ *are species independent of* n *that are given by*

$$\Theta(k) = \sum_{d|k} d\, P_{k/d} \circ (Y\Omega_d(X)). \tag{3.67}$$

Proof. We have $(1+sX)^{\uparrow} Y = \sum_{n\geq 0} \binom{X,Y}{n} s^n$, where s is an extra weight variable. Also,

$$(1+sX)^{\uparrow} Y = E(Y\, \mathrm{Lg}(1+sX)) = \exp \sum_{m\geq 1,\, \ell\geq 1} \frac{s^{m\ell}}{m} P_m \circ (Y\Omega_\ell(X)), \tag{3.68}$$

by (2.40) and plethystic linearity of P_m. Taking the analytic logarithm, we get the identity

$$\sum_{k\geq 1}\left(\sum_{m\ell=k}\frac{1}{m}P_m\circ(Y\Omega_\ell(X))\right)s^k = \log\sum_{n\geq 0}\binom{X,Y}{n}s^n, \tag{3.69}$$

and we conclude by applying the differential operator $s\frac{\partial}{\partial s}$ on both sides of (3.69).[25] □

Example 3.11. Let v be a weight variable and consider the 1-sort species $(1+X)^{\uparrow v}$. This special weighted virtual species has been considered first by Pierre Leroux and the author in 1996 and was denoted by $\Lambda^{(v)}(X)$ in their paper [LabLer96]. They used it to put a weight v on each connected component of structures (see Example 4.3 below). Making the substitution $Y := v$ in Proposition 3.8, we obtain the new recursive scheme

$$\Lambda^{(v)}(X) = (1+X)^{\uparrow}v = \sum_{n\geq 0}\binom{X,v}{n}, \qquad \binom{X,v}{0} = 1, \tag{3.70}$$

$$\binom{X,v}{n} = \frac{1}{n}\left[V(1)\binom{X,v}{n-1} + V(2)\binom{X,v}{n-2} + \cdots + V(k)\binom{X,v}{n-k} + \cdots\right], \quad n>1, \tag{3.71}$$

where the coefficients $V(k) = V(k,X,v)$ are species independent of n that are given by

$$V(k) = \sum_{d|k} d\,v^{k/d}P_{k/d}\circ\Omega_d(X), \tag{3.72}$$

since $P_{k/d}\circ(v\Omega_d(X)) = v^{k/d}P_{k/d}\circ\Omega_d(X)$ due to plethystic linearity (2.41).

Example 3.12. Completely different species arise from the substitutions $Y := y$ and $Y := c$ in the binomial species $B(X,Y)$, where y is an auxiliary variable and $c\in\mathbb{C}$ is a scalar symbol. A careful analysis shows that

$$(1+X)^{\uparrow}y = \exp(y\Omega(X)) = \sum_{n\geq 0}\tfrac{1}{n!}y^n\Omega^n(X) \in \mathbb{Q}_y||X||, \tag{3.73}$$

$$(1+X)^{\uparrow}c = \sum_{n\geq 0}\tfrac{c(c-1)(c-2)\cdots(c-n+1)}{n!}X^n \in \mathbb{C}[[X]]. \tag{3.74}$$

Example 3.13. Further examples of new species are worth mentioning. They arise from the substitutions of special symbols for the sort X, instead of Y, in the binomial species $B(X,Y)$. For example, let u be a weight variable. Then the substitution $X := u$ in

[25] The author implemented the above computational scheme in Maple to expand $(1+X)^{\uparrow}Y$ up to degree 20 in X. It took less than 5 seconds on a MacPro. The expansion contains 131834 terms. The corresponding expansion for the classical $(1+x)^y$ contains only 211 terms.

Proposition 3.8 gives rise to the species (compare with Example 3.6 above)

$$(1+u)^{\uparrow}Y = E(Y\,\mathrm{Lg}(1+u)) = E((u-u^2)Y) = E(uY)/E(u^2Y) = \sum_{n\geq 0}\binom{u,Y}{n}, \quad (3.75)$$

$$\binom{u,Y}{n} = \frac{1}{n}\left[U(1)\binom{u,Y}{n-1} + U(2)\binom{u,Y}{n-2} + \cdots + U(k)\binom{u,Y}{n-k} + \cdots\right], \quad n>1, \quad (3.76)$$

where the coefficients $U(k) = U(k,u,Y)$ are species independent of n that are given by

$$U(k) = \begin{cases} u^k P_k(Y), & k \text{ odd}, \\ u^k(P_k(Y) - 2P_{k/2}(Y)), & k \text{ even}. \end{cases} \quad (3.77)$$

Indeed, by (2.13), $\Omega_n(u) = \Omega_n(X)|_{X:=u} = [u^n]\,\mathrm{Lg}(1+u)$ are *polynomials* in u satisfying

$$\Omega_n(u) = \begin{cases} u, & n=1, \\ -u^2, & n=2, \\ 0, & n>2, \end{cases} \quad (3.78)$$

and (3.77) follows from the fact that, by (3.67) and (2.41), the coefficients $U(k) = \Theta(k) = \Theta(k,u,Y)$ are species in Y given by

$$U(k) = \sum_{d|k} d\,\Omega_d(u^{k/d})P_{k/d}(Y). \quad (3.79)$$

Example 3.14. Note that since $(1+u)^{\uparrow}Y = E(uY)/E(u^2Y)$, Corollary 3.4 implies via "telescopic multiplicative cancellations" the remarkable combinatorial equality

$$\left(\frac{1}{1-u}\right)^{\uparrow} Y = E(uY). \quad (3.80)$$

Stated otherwise, this means that the species $E(uY)$ of finite sets in which each element has weight u can be expressed via the binomial species in the somewhat unexpected form

$$E(uY) = B(-u,-Y). \quad (3.81)$$

4. Combinatorial exponentiation of species

We now use the (virtual) binomial species $B(X,Y) = (1+X)^{\uparrow}Y$ to define a new combinatorial exponentiation operation, $F^{\uparrow}G$, between species F and G. Recall that if F is a species for which $F(\mathbf{0}) = 1$, the species F^{conn} of "connected" F-structures is defined by $F^{\mathrm{conn}} = \mathrm{Lg}(F) = \mathrm{Lg}(1+F_+)$, where $F = 1+F_+$, $F_+(\mathbf{0}) = 0$.

Definition 4.1. Let F be a species satisfying $F(\mathbf{0}) = 1$ and G be an arbitrary species. The species $F^{\uparrow}G$, called F *raised to the combinatorial power* G (or simply F *uparrow* G) is defined by

$$F^{\uparrow}G \underset{\mathrm{def}}{=} B(X,Y)|_{X:=F_+,Y:=G} = E(G\cdot \mathrm{Lg}(F)) = E(G\cdot F^{\mathrm{conn}}). \quad (4.1)$$

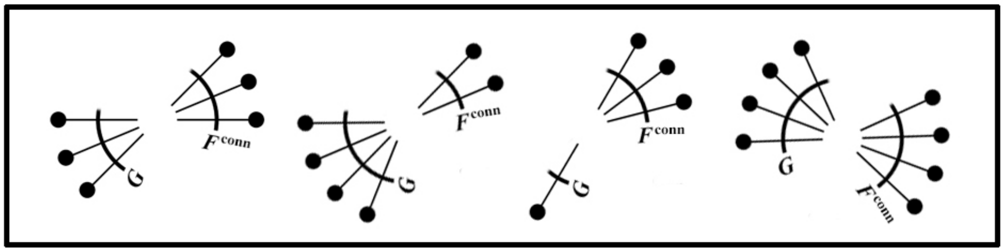


FIGURE 6. Illustration of an $F^{\uparrow}G$-structure (that is, an assembly of $G \cdot F^{\text{conn}}$-structures) on a set of 24 elements, where F and G are "ordinary" species

Proposition 4.1. *For species F, G, H, we have*

$$F^{\uparrow}(G+H) = (F^{\uparrow}G)\cdot(F^{\uparrow}H), \qquad F^{\uparrow}(G\cdot H) = (F^{\uparrow}G)^{\uparrow}H, \quad \textit{if } F(\mathbf{0}) = 1, \tag{4.2}$$

$$(F\cdot G)^{\uparrow}H = (F^{\uparrow}H)\cdot(G^{\uparrow}H), \qquad F^{\uparrow}G^{\text{conn}} = G^{\uparrow}F^{\text{conn}}, \qquad \textit{if } F(\mathbf{0}) = G(\mathbf{0}) = 1, \tag{4.3}$$

$$\frac{\partial}{\partial X}F^{\uparrow}G = (F^{\uparrow}G)\cdot\left(F^{\text{conn}}\frac{\partial G}{\partial X} + G\frac{\partial F/\partial X}{F}\right), \qquad \textit{if } F(\mathbf{0}) = 1. \tag{4.4}$$

Proof. Formulas (4.2) and (4.3 left) are easy consequences of Proposition 3.1. Formula (4.3 right) follows from Definition 4.1, since

$$F^{\uparrow}G^{\text{conn}} = E(G^{\text{conn}}\cdot F^{\text{conn}}) = E(F^{\text{conn}}\cdot G^{\text{conn}}) = G^{\uparrow}F^{\text{conn}}.$$

Formula (4.4) is a consequence of the combinatorial chain rule and the fact that $\partial\,\mathrm{Lg}(1+X)/\partial X = 1/(1+X)$:

$$\begin{aligned}
\frac{\partial}{\partial X}F^{\uparrow}G &= \frac{\partial}{\partial X}E(G\cdot\mathrm{Lg}(1+F_+)) \\
&= E'(G\cdot\mathrm{Lg}(1+F_+))\cdot(G'\cdot\mathrm{Lg}(1+F_+) + G\cdot\mathrm{Lg}'(1+F_+)\cdot(F_+)') \\
&= (F^{\uparrow}G)\cdot\left(G'\cdot\mathrm{Lg}(F) + G\cdot\frac{(F_+)'}{1+F'}\right) \\
&= (F^{\uparrow}G)\cdot\left(F^{\text{conn}}\frac{\partial G}{\partial X} + G\frac{\partial F/\partial X}{F}\right),
\end{aligned}$$

where H' denotes $\frac{\partial H}{\partial X}$ and $(F_+)' = (F-1)' = F' = \frac{\partial F}{\partial X}$. □

It is important to realize that

$$F^{\uparrow}G \neq F^{G}, \text{ in general}, \tag{4.5}$$

where F^G denotes the "ordinary" or "analytic" exponentiation of species defined by the summable series

$$F^G = (1+F_+)^G = \sum_{n\geq 0}\tfrac{G\cdot(G-1)\cdot(G-2)\cdots(G-n+1)}{n!}F_+^n. \tag{4.6}$$

We now introduce a new "plethystic" exponential operation, $f^{\uparrow}g$, between cycle index series that reflects the combinatorial exponentiation between species, extending list (1.21)–(1.22).

Definition 4.2. Let $f = f(x_1, x_2, \ldots; y_1, y_2, \ldots)$ and $g = g(x_1, x_2, \ldots; y_1, y_2, \ldots)$ be two (formal) cycle index series, with $f = f(0, 0, \ldots; 0, 0, \ldots) = 1$. The series $f^{\uparrow}g$, called f *raised to the plethystic power* g (or simply f *uparrow* g) is defined by

$$f^{\uparrow}g = \prod_{n\geq 1} f_n^{\frac{1}{n}\sum_{d|n}\mu(d)g_{n/d}} \tag{4.7}$$

$$= f^{\,g} \cdot f_2^{\frac{1}{2}(g_2-g_1)} \cdot f_3^{\frac{1}{3}(g_3-g_1)} \cdot f_4^{\frac{1}{4}(g_4-g_2)} \cdot f_5^{\frac{1}{5}(g_5-g_1)} \cdot f_6^{\frac{1}{6}(g_6-g_3-g_2+g_1)} \cdots, \tag{4.8}$$

where $\psi^{\omega} = \exp(\omega\log(\psi))$ denotes the *ordinary analytic power*, and where, as usual,

$$\psi_n = \psi(x_n, x_{2n}, x_{3n}, \ldots; y_n, y_{2n}, y_{3n}, \ldots), \quad n = 1, 2, \ldots, \tag{4.9}$$

for a cycle index series $\psi = \psi(x_1, x_2, x_3, \ldots; y_1, y_2, y_3, \ldots)$.

Corollary 4.2. *Let $F = F(X, Y, \ldots)$ be a species satisfying $F(\mathbf{0}) = 1$ and $G = G(X, Y, \ldots)$ be an arbitrary species. Then the cycle index series $Z_{F^{\uparrow}G}$ depends only on the cycle index series Z_F and Z_G and is given by*

$$Z_{F^{\uparrow}G} = Z_F{}^{\uparrow}Z_G. \tag{4.10}$$

In particular, if $\alpha, \beta, \ldots$ are power series in the auxiliary variables $x, y, \ldots$ and weight variables $u, v, \ldots,$ then

$$(F^{\uparrow}G)(\alpha, \beta, \ldots) = \prod_{n\geq 1} F(\alpha_n, \beta_n, \ldots)^{\frac{1}{n}\sum_{d|n}\mu(d)G(\alpha_{n/d}, \beta_{n/d}, \ldots)}, \tag{4.11}$$

$$(F^{\uparrow}G)(u, v, \ldots) = \prod_{n\geq 1} F(u^n, v^n, \ldots)^{\frac{1}{n}\sum_{d|n}\mu(d)G(u^{n/d}, v^{n/d}, \ldots)}, \tag{4.12}$$

$$(F^{\uparrow}G)(x, y, \ldots) = F(x, y, \ldots)^{G(x, y, \ldots)}. \tag{4.13}$$

Proof. It is sufficient to prove (4.10). To simplify the computations, let $f = Z_F$ and $g = Z_G$. Then, using Tables 1 and 2, we have

$$\begin{aligned} Z_{F^{\uparrow}G} &= Z_{E(G\cdot \mathrm{Lg}(F))} = Z_E \circ (Z_G \cdot Z_{\mathrm{Lg}(F)}) = Z_E \circ (g \cdot \textstyle\sum_{k\geq 1} \frac{\mu(k)}{k}\log(f_k)) \\ &= \exp(\textstyle\sum_{\ell,k\geq 1} \frac{1}{\ell}\, g_\ell \frac{\mu(k)}{k}\log(f_{\ell k})) = \exp(\sum_{n\geq 1}(\sum_{d|n}\mu(d)g_{n/d})\frac{\log(f_n)}{n}) \\ &= \prod_{n\geq 1} f_n{}^{\frac{1}{n}\sum_{d|n}\mu(d)g_{n/d}} = f^{\uparrow}g = Z_F{}^{\uparrow}Z_G. \end{aligned}$$ □

As in Section 3, special choices of species in Proposition 4.1 and Corollary 4.2 and/or specializations of variables can be used to obtain a great variety of new formulas and identities between species and/or power series.

Example 4.1. Let $C = C(X)$, $S = S(X) = E(C(X))$, $L = L(X)$, $A = A(X)$, $\mathrm{Inv} = \mathrm{Inv}(X)$ be the 1-sort species of *oriented cycles, permutations, linear orders, rooted trees, involutions*, respectively. Let $\Phi = \Phi(X, Y)$ and $\mathrm{Bij} = \mathrm{Bij}(X, Y)$ be the 2-sort species of *functions* and *bijections* mapping sets of elements of sort X to sets of elements of sort Y. The following small sample of equalities hold.

a) $E^{\uparrow}E = E(XE) =$ species of *equivalence relations with a system of representatives.*

Proof. Since connected sets are singletons, $E^{\uparrow}E = E(E \cdot E^{\mathrm{conn}}) = E(E \cdot X)$. Moreover, XE is the species of pointed sets, and assemblies of pointed sets are equivalence relations with a system of representatives (the pointed elements). $\square$

b) $S^{\uparrow}X = E^{\uparrow}C = E(XC)$ is the species of *assemblies of oriented "wheels"*.

Proof. Since oriented cycles are connected permutations and XC-structures can be thought of as oriented wheels (the X-structure being the center of the wheel, and the C-structure being the circumference of the wheel), $S^{\uparrow}X = E(X \cdot S^{\mathrm{conn}}) = E(XC) =$ species of assemblies of oriented wheels. Moreover, $E(XC) = E(C \cdot E^{\mathrm{conn}}) = E^{\uparrow}C$. $\square$

c) $(A/X)^{\uparrow}(X + E_2) = \mathrm{Inv}^{\uparrow}A$.

Proof. Since involutions are assemblies of fixed points or unordered pairs of interchanged points, $\mathrm{Inv} = E(X + E_2)$. Moreover, rooted trees are characterized by the well-known equation $A = XE(A)$. Hence, $(A/X)^{\uparrow}(X + E_2) = E((X + E_2) \cdot (A/X)^{\mathrm{conn}}) = E((X + E_2) \cdot A) = E(A \cdot \mathrm{Inv}^{\mathrm{conn}}) = \mathrm{Inv}^{\uparrow}A$. $\square$

d) $\frac{\partial}{\partial X}(S^{\uparrow}E) = (S^{\uparrow}E) \cdot (C + L) \cdot E$.

Proof. This is an example of (4.4) with $F = S$ and $G = E$, since $\frac{\partial}{\partial X}E = E$ and $\frac{\partial}{\partial X}S = S \cdot L$. $\square$

e) $\Phi(X, Y) = E(Y)^{\uparrow}E(X)$.

Proof. Since a function from a set of elements of sort X to a set of elements of sort Y can be thought of as an assembly of $E(X)\cdot Y$-structures (each $E(X)$-structure being interpreted as the possibly empty inverse image of a Y-singleton), we can write $\Phi(X, Y) = E(E(X) \cdot Y)$. On the other hand, $E(Y)^{\uparrow}E(X) = E(E(X) \cdot (E(Y))^{\mathrm{conn}}) = E(E(X) \cdot Y)$. $\square$

f) $\mathrm{Bij}(X, Y) = \Phi(X, Y)^{\uparrow}(X/E(X))$.

Proof. Since a bijection from a set of elements of sort X to a set of elements of sort Y can be thought of as an assembly of ordered pairs (x, y), where x is a singleton of sort X and y is a singleton of sort Y, we can write $\mathrm{Bij}(X, Y) = E(XY)$. On the other hand, $\Phi(X, Y)^{\uparrow}(X/E(X)) = E((X/E(X)) \cdot \Phi^{\mathrm{conn}}(X, Y)) = E((X/E(X)) \cdot E(X)Y) = E(XY)$.[26] $\square$

Of course, the underlying series of **a)**-**f)** can also be analyzed as well as their various weighted versions. The following example illustrates this by assigning weights to the species $\Phi(X, Y)$ of functions in case **e)** above.

[26] Note that $X/E(X) = A^{<-1>}(X) =$ the inverse of the species of rooted trees under substitution.

Example 4.2. Consider the (q,t)-analogue $\Phi_{q,t}(X,Y) = \Phi(\frac{X}{1-q}, \frac{Y}{1-t})$ of the species $\Phi(X,Y) = E(Y)^{\uparrow}E(X)$ of functions mapping sets of elements of sort X to sets of elements of sort Y. Since $\Phi(X,Y) = E(E(X)\cdot Y)$ and $E_q(u) = \sum_{i\geq 0} u^i/(q;q)_i$, we have, using Table 1,

$$\Phi_{q,t}(u,v) = E\left(\frac{v}{1-t}\right)^{\uparrow} E\left(\frac{u}{1-q}\right) = E\left(E\left(\frac{u}{1-q}\right)\cdot\frac{v}{1-t}\right) \tag{4.14}$$

$$= \exp\left(\sum_{i,j\geq 1}\frac{1}{i}\cdot\frac{u^{ij}v^i}{(1-t^i)(1-q^i)(1-q^{2i})(1-q^{3i})\cdots(1-q^{ij})}\right) \tag{4.15}$$

$$= \sum_{n,k\geq 0}\phi_{n,k}(q,t)\frac{u^n}{(q;q)_n}\cdot\frac{v^k}{(t;t)_k}, \tag{4.16}$$

where $\phi_{n,k}(q,t)$ is a polynomial in (q,t) which is a (q,t)-analogue of the number $\phi_{n,k} = k^n$ of functions from $[n]$ to $[k]$, i.e., $\phi_{n,k}(1,1) = k^n$. On the other hand, $\phi_{n,k}(0,0)$ is the number $\widetilde{\phi_{n,k}}$ of "unlabelled" functions from n indistinguishable white dots to k indistinguishable black dots given by

$$\prod_{m\geq 0}\frac{1}{1-u^m v} = \sum_{n,k\geq 0}\widetilde{\phi_{n,k}}\,u^n v^k. \tag{4.17}$$

Example 4.3. Let $F = F(X,Y,\dots)$ be a species satisfying $F(\mathbf{0}) = 1$ and v be an extra weight variable. Then we can consider the species $F^{\uparrow}v = F^{\uparrow}Z|_{Z=v}$, which can be interpreted (when F possesses connected components) as the species of F-structures whose connected components are each weighted by v (see Figure 6 in which each G-structure is replaced by a $v = v\cdot 1$-structure living on the empty set). This kind of species was considered, using distinct notation, by Leroux and the author (see [LabLer96]). By (3.4), we have

$$F^{\uparrow}v = (1+F_+)^{\uparrow}v = \sum_{n\geq 0}\binom{Z,v}{n}_{Z:=F_+}, \tag{4.18}$$

and the binomial coefficients $\binom{Z,v}{n}$ can be recursively computed as in Example 3.11 above. This provides an alternate efficient way to compute $F^{\uparrow}v$.

Example 4.4. The 2-sort weighted species $G(X,Y)$ of simple graphs on black and white nodes of Example 1.1 in the introduction can be written in the form

$$G(X,Y) = \mathcal{G}(X)^{\uparrow}(u-t)\cdot\mathcal{G}(Y)^{\uparrow}(v-t)\cdot\mathcal{G}(X+Y)^{\uparrow}t, \tag{4.19}$$

where $\mathcal{G}$ denotes the species of simple graphs. This is a consequence of the more direct combinatorial equation

$$G(X,Y) = E(u\mathcal{G}^{\mathrm{conn}}(X))E(v\mathcal{G}^{\mathrm{conn}}(Y))E(t(\mathcal{G}^{\mathrm{conn}}(X+Y)-\mathcal{G}^{\mathrm{conn}}(X)-\mathcal{G}^{\mathrm{conn}}(Y))), \tag{4.20}$$

where $\mathcal{G}^{\mathrm{conn}}$ is the species of connected simple graphs. All underlying counting series for $G(X,Y)$ can be computed from combinatorial equation (4.19).

Example 4.5. Let q be a weight variable. Making the substitution $X := q$ in the simple combinatorial equation

$$S(X)^{\uparrow}C(X) = E(C^2(X)), \tag{4.21}$$

using (4.12) (with $F = S, G = C, u = q$) together with the fact that $C(q) = q/(1-q)$ and $S(q) = \prod_{n\geq 1}(q^n;q^n)_\infty^{-1}$, we arrive at the identity

$$\prod_{n,\,k\geq 1}(q^n;q^n)_\infty^{-M(q^k,\,n)} = \exp\left(\sum_{i\geq 1}\frac{1}{i}\frac{q^{2i}}{(1-q^i)^2}\right), \tag{4.22}$$

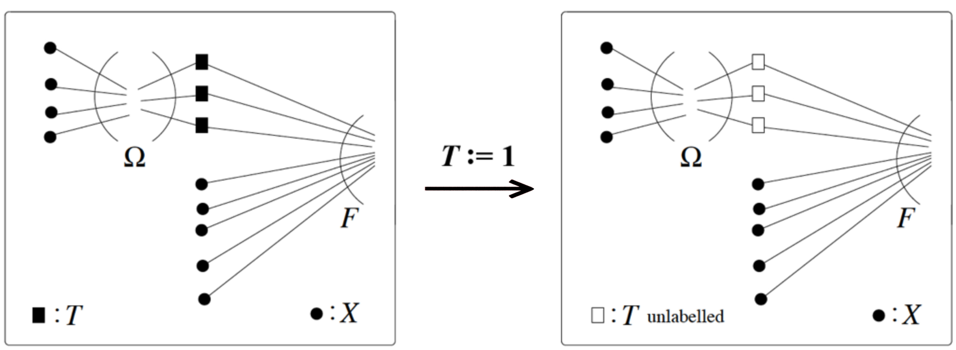
where the $M(q,n) = \frac{1}{n}\sum_{d|n}\mu(d)q^{n/d}$, $n \geq 1$, are the necklace polynomials evaluated at q.

Example 4.6. The operation $^{\uparrow}$ can be used to define an exponentiation operation between the very general combinatorial differential operators introduced recently by Cédric Lamathe and the author [LabLam09]. In their simplest form, these differential operators are of the form $\Omega(X,D)$, where $\Omega(X,T)$ is any two-sort species and $D = d/dX$.[27] These operators act on species $F(X)$ via

$$\Omega(X,D)F(X) \underset{\text{def}}{=} \Omega(X,T) \times_T F(X+T)|_{T:=1}, \tag{4.23}$$

where $\times_T$ denotes the Cartesian product with respect to sort T, and $T := 1$ means unlabelling the elements of sort T.

Figure 7 describes this action. The black dots and black squares in this figure are elements of sorts X and T, respectively, while the white squares are *unlabelled* elements of sort T. In [LabLam09], it is shown how to compose (i.e., apply successively) such



$\Omega(X,T) \times_T F(X+T)$ $\Omega(X,D)F(X)$

FIGURE 7. Illustration of an $\Omega(X,D)F(X)$-structure on a 9-element set.

operators, how to take their adjoint, and how they behave with respect to the classical

[27]The classical linear analytic differential operators $A_0(X) + A_1(X)\frac{\partial}{\partial X} + A_2(X)\frac{\partial^2}{\partial X^2} + \cdots$, where $A_0(X), A_1(X), A_2(X), \cdots \in \mathbb{C}[[X]]$ are power series in X in the usual sense, are special cases of these combinatorial operators. The combinatorial differential operators $\Omega(X,D)$ can take a great variety of forms. For example, $C(X + D + E(XD))$ is such an operator, where $C(X)$ and $E(X)$ are the species of oriented cycles and finite sets, respectively.

combinatorial operations. The adjoint of $\Omega(X, D)$ is the operator $\Omega^*(X, D) = \Omega(D, X)$, which is associated to the species $\Omega(T, X)$. In particular, the adjoint of the "pure" differential operator $G(D)$ is the operator $G(X)$ (= multiplication by $G(X)$). This generalizes the well-known fact that the adjoint of the "annihilation operator", D, is the "creation operator", X.

The cycle index series of the species $\Omega(X, D)F(X)$ can be computed as follows.

Proposition 4.3 ([LabLam09])**.** *Let $G(X) = \Omega(X, D)F(X)$. Then*

$$Z_G(x_1, x_2, x_3, \dots) = Z_\Omega(x_1, x_2, x_3, \dots; \frac{\partial}{\partial x_1}, 2\frac{\partial}{\partial x_2}, 3\frac{\partial}{\partial x_3}, \dots)Z_F(x_1, x_2, x_3, \dots). \quad (4.24)$$

The combinatorial exponentiation, $\Omega^{\uparrow}\Lambda$, of operators $\Omega = \Omega(X, D)$ and $\Lambda = \Lambda(X, D)$ is defined in the obvious way.

Definition 4.3. Let $\Omega = \Omega(X, D)$ and $\Lambda = \Lambda(X, D)$ be combinatorial differential operators associated to species $\Omega(X, T)$ and $\Lambda(X, T)$, with $\Omega(0, 0) = 1$. The combinatorial differential operator $(\Omega^{\uparrow}\Lambda)(X, D)$ is associated to the species

$$\Omega(X, T)^{\uparrow}\Lambda(X, T) = E(\Lambda(X, T)\,\mathrm{Lg}(\Omega(X, T))) = E(\Lambda(X, T) \cdot \Omega^{\mathrm{conn}}(X, T)). \quad (4.25)$$

A general study of the combinatorial differential operators $\Omega^{\uparrow}\Lambda$ should be developed.

5. Conclusion

The main goal of combinatorics is to create, manipulate, combine, analyze, classify, and enumerate finite discrete structures by making use of a variety of mathematical operations and tools. The binomial species and combinatorial exponentiation are two such tools to be exploited further.

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Appendix A. The general substitution formulas for $\mathbb{C}_{u,v,\dots}\|X, Y, \dots\|$

We start by analyzing substitution of ordinary species in the 1-sort case. Let $u, v, \dots$ be a family of weight variables and F, G be ordinary 1-sort weighted species (i.e., elements of $\mathbb{N}_{u,v,\dots}\|X\|$). To do this, we first write the molecular species $\neq 1$ as a list according to weakly increasing degrees in X,

$$M', M'', M''', M^{(4)}, M^{(5)}, M^{(6)}, M^{(7)}, \dots, \quad (A.1)$$

where $M' = X$, $M'' = X^2$, $M''' = E_2(X)$, $M^{(4)} = X^3$, $M^{(5)} = XE_2(X)$, $M^{(6)} = C_3(X)$, $M^{(7)} = E_3(X)$, etc. We have two cases to consider: $G(0) = 0$ and $G(0) \neq 0$.

Case 1: $G(0) = 0$. Assume first that $G(0) = 0$ (that is, there is no G-structure on the empty set). In this context, an $F \circ G$-structure is an F-assembly of G-structures. This means that such a structure on a finite set U is a triple

$$s = (\pi, \phi, \gamma), \tag{A.2}$$

where

1) π is a partition of U,
2) ϕ is an F-structure on the set of classes of π,
3) $\gamma = (\gamma_p)_{p\in\pi}$, where for each class p of π, γ_p is a G-structure on p.

The weight of s is defined by the product of the weights of ϕ and of all γ_p, $p \in \pi$:

$$w(s) = w(\phi)\Pi_{p\in\pi} w(\gamma_p). \tag{A.3}$$

Now, write G and F as combinatorial power series in $\mathbb{N}_{u,v,\ldots}\| X \|$:

$$G = G(X) = \alpha' M' + \alpha'' M'' + \cdots, \quad \alpha', \alpha'', \cdots \in \mathbb{N}_{u,v,\ldots}, \tag{A.4}$$

$$F = F(X) = \beta + \beta' M' + \beta'' M'' + \cdots, \quad \beta, \beta', \beta'', \cdots \in \mathbb{N}_{u,v,\ldots}. \tag{A.5}$$

By (A.4), each G-structure γ_p in (A.2) is an M'-structure or an M''-structure, etc., with corresponding weight distribution described by α' or α'', etc. In other words, the following combinatorial equations (due to Yeh) hold:

$$\begin{aligned} F \circ G &= F(\alpha' M' + \alpha'' M'' + \cdots) \\ &= F(\alpha' X_1 + \alpha'' X_2 + \cdots)|_{X_1 := M',\, X_2 := M'', \ldots} \\ &= F(X_1 + X_2 + \cdots) \times E(\alpha' X_1 + \alpha'' X_2 + \cdots)|_{X_1 := M',\, X_2 := M'', \ldots}, \end{aligned} \tag{A.6}$$

where $X_1, X_2, \ldots$ are extra sorts of singletons, $(\times)$ denotes the Cartesian "superposition" product, and E is the species of finite sets. This is essentially due to the fact that the weight assigned to each structure γ_p can be canonically associated to its underlying set p instead of being associated to γ_p itself. In order to explicitly expand (A.6) in terms of $\alpha'_1, \alpha'_2, \ldots, \alpha''_1, \alpha''_2, \ldots$, etc., and $\beta, \beta', \beta'', \ldots$, we need an auxiliary result.

Lemma A.1 ([Lab08]). *For every $\alpha \in \mathbb{C}_{u,v,\ldots}$ we have an expansion of the form*

$$E(\alpha X) = \sum_{\lambda} \epsilon_\lambda(\alpha_1, \alpha_2, \ldots) E_\lambda(X), \tag{A.7}$$

where λ runs through all partitions $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ of all integer $n \geq 0$, $\epsilon_\lambda(\alpha_1, \alpha_2, \ldots)$ is a polynomial in $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with rational coefficients, and

$$E_\lambda(X) = E_{\lambda_1}(X) \cdot E_{\lambda_2}(X) \cdots E_{\lambda_n}(X). \tag{A.8}$$

Proof. By (2.45) and (2.46), we have, in analogy with the theory of symmetric functions,

$$\begin{aligned} E(\alpha X) &= \exp \sum_{m\geq 1} \frac{\alpha_m}{m} P_m(X) = \sum_{k_1,k_2,k_3,\ldots} \frac{\alpha_1^{k_1}\alpha_2^{k_2}\cdots}{1^{k_1}k_1!\,2^{k_2}k_2!\cdots} P_1(X)^{k_1} P_2(X)^{k_2} \cdots \\ &= \sum_{\lambda} \frac{\alpha_\lambda}{z_\lambda} P_\lambda(X), \end{aligned}$$

where $\alpha_\lambda = \alpha_{\lambda_1}\alpha_{\lambda_2}\cdots$, $P_\lambda(X) = P_{\lambda_1}(X)P_{\lambda_2}(X)\cdots$, $z_\lambda = 1^{d_1}d_1!\,2^{d_2}d_2!\cdots$ if λ consists of d_i parts i, $i = 1, 2, \ldots$. We conclude by expanding the $P_\lambda(X)$'s in terms of the $E_\lambda(X)$'s. □

By Lemma A.1, we can write

$$E(\alpha' X_1 + \alpha'' X_2 + \cdots) = E(\alpha' X_1)E(\alpha'' X_2)\cdots \tag{A.9}$$

$$= \sum_{\lambda',\lambda'',\ldots} \epsilon_{\lambda',\lambda'',\ldots} E_{\lambda'}(X_1)E_{\lambda''}(X_2)\cdots\,, \tag{A.10}$$

where each $\lambda', \lambda'', \ldots$ runs through all integer partitions, and

$$\epsilon_{\lambda',\lambda'',\ldots} = \epsilon_{\lambda'}(\alpha'_1, \alpha'_2, \ldots)\epsilon_{\lambda''}(\alpha''_1, \alpha''_2, \ldots)\cdots \tag{A.11}$$

is a polynomial in a finite number of $\alpha'_1, \alpha'_2, \ldots, \alpha''_1, \alpha''_2, \ldots, \ldots$ with rational coefficients. By (A.6), $F \circ G$ can be expanded in the form

$$F \circ G = F(\alpha' M' + \alpha'' M'' + \cdots) = \sum_{\lambda',\lambda'',\ldots} \epsilon_{\lambda',\lambda'',\ldots} F_{\lambda',\lambda'',\ldots}(M', M'', \ldots), \tag{A.12}$$

$$F_{\lambda',\lambda'',\ldots}(X_1, X_2, \ldots) = F(X_1 + X_2 + \cdots) \times E_{\lambda'}(X_1)E_{\lambda''}(X_2)\cdots\,. \tag{A.13}$$

Note that each term in the expansion of (A.13) is of total degree $n' + n'' + \cdots < \infty$ in $X_1, X_2, \ldots$, where $\lambda' \vdash n', \lambda'' \vdash n'', \ldots$. Hence each $F_{\lambda',\lambda'',\ldots}(M', M'', \ldots)$ in (A.12) involves only a finite number of the *unweighted* species $M', M'', \ldots$, and the weight distribution arising from G in $F \circ G$ is included in the coefficients $\epsilon_{\lambda',\lambda'',\ldots}$. Moreover, the lower degree in X in $F_{\lambda',\lambda'',\ldots}(M', M'', \ldots)$ is $\geq n' + n'' + \cdots$ since $\deg M' > 0, \deg M'' > 0, \ldots$. It follows that each coefficient $\gamma, \gamma', \gamma'', \ldots$ in the full expansion of (A.12),

$$F \circ G = (F \circ G)(X) = \gamma + \gamma' M' + \gamma'' M'' + \cdots\,, \tag{A.14}$$

is a polynomial expression with rational coefficients involving a finite number of coefficients $\beta, \beta', \beta'', \ldots$ appearing in the expansion of F and a finite number of $\alpha'_1, \alpha'_2, \ldots, \alpha''_1, \alpha''_2, \ldots$, etc., appearing in G. This implies that (A.12) is formally summable, and $F \circ G$ is a well-defined species in $\mathbb{N}_{u,v,\ldots}\|\,X\,\|$.

Case 2: $G(0) \neq 0$. Assume now that $G(0) = \alpha \neq 0$. In this context, we assume that F is of finite degree in X for summability reasons. Writing $G(X) = \alpha + G_+(X)$, where $G_+(X)$ is of the form (A.4) with $G_+(0) = 0$, and introducing an extra sort T of singletons, we have

$$F \circ G = F(\alpha + G_+(X)) = F(\alpha T + G_+(X))|_{T:=1} \tag{A.15}$$

$$= F(T + G_+(X)) \times E(\alpha T + X)|_{T:=1} \tag{A.16}$$

$$= \sum_\lambda \epsilon_\lambda(\alpha_1, \alpha_2, \ldots)F(T + G_+(X)) \times E_\lambda(T)E(X)|_{T:=1} \tag{A.17}$$

$$= \delta + \delta' M' + \delta'' M'' + \cdots\,, \tag{A.18}$$

where (since $\deg F < \infty$) each coefficient $\delta, \delta', \delta'', \ldots$ in (A.18) is a polynomial expression with rational coefficients involving a finite number of coefficients $\beta, \beta', \beta'', \ldots$ appearing in the expansion of F and a finite number of $\alpha_1, \alpha_2, \ldots, \alpha'_1, \alpha'_2, \ldots, \alpha''_1, \alpha''_2, \ldots$, etc., appearing in G. This implies that (A.17) is formally summable, and $F \circ G$ is again a well-defined species in $\mathbb{N}_{u,v,\ldots}\| X \|$.

More generally, replacing $\mathbb{N}_{u,v,\ldots}$ by $\mathbb{C}_{u,v,\ldots}$, we *define* $F \circ G$ for $F, G \in \mathbb{C}_{u,v,\ldots} \| X \|$, with $G(0) = 0$ or $\deg F < \infty$, by formula (A.14) or (A.18) by making use of the *same* expressions for $\gamma, \gamma', \gamma'', \ldots$ or $\delta, \delta', \delta'', \ldots$ as polynomials with rational coefficients involving a finite number of $\beta, \beta', \beta'', \ldots$ appearing in the expansion of F and a finite number of $\alpha_1, \alpha_2, \ldots, \alpha_1', \alpha_2', \ldots, \alpha_1'', \alpha_2'', \ldots$, etc., appearing in G .

General substitutions in the case of species on sorts $X, Y, \ldots$ are treated in a similar way.

Appendix B. Formal summability in $\mathbb{C}_{x,y,\ldots;u,v,\ldots} \| X, Y, \ldots \|$

In what follows, the groups H run through *fixed* systems of representatives of conjugacy classes of subgroups of $S_{n,k,\ldots}$, where $n, k, \cdots \geq 0$. A species $F \in \mathbb{C}_{x,y,\ldots;u,v,\ldots} \| X, Y, \ldots \|$ can be canonically fully expanded in the form

$$F = F(X, Y, \ldots) = \sum_{\mu,H} c_{\mu,H}\, \mu\, X^n Y^k \cdots /H, \tag{B.1}$$

where $\mu = x^p y^q \cdots u^i v^j \cdots$ runs through all power products in $\mathbb{C}_{x,y,\ldots;u,v,\ldots}$. By analogy with the usual notation for classical power series, the coefficient $c_{\mu,n,k,\ldots,H}$ in (B.1) is also denoted by

$$[\, \mu X^n Y^k \cdots /H\,] F = c_{\mu,H}. \tag{B.2}$$

Definition B.1. Let $(F_\iota)_{\iota \in I}$ be a family of species indexed by a (finite or infinite) set I, where

$$F_\iota = \sum_{\mu,H} c_{\iota,\mu,H}\, \mu\, X^n Y^k \cdots /H \in \mathbb{C}_{x,y,\ldots;u,v,\ldots} \| X, Y, \ldots \|, \quad \iota \in I. \tag{B.3}$$

For every μ and H, denote by

$$I_{\mu,H} = \{\iota \in I : [\, \mu\, X^n Y^k \cdots /H\,] F_\iota \neq 0\} \tag{B.4}$$

the subset of indices ι for which the coefficient $c_{\iota,\mu,H}$ of F_ι is not 0. The family $(F_\iota)_{\iota \in I}$ is *formally summable* if

$$I_{\mu,h} \text{ is finite for every } \mu \text{ and } H. \tag{B.5}$$

Note. This definition of summability for families of species includes the special case of families of formal power series (simply take $c_{\iota,\mu,H} = 0$ whenever $X^n Y^k \cdots \neq 1 = X^0 Y^0 \cdots$). Summability in the case of families of formal cycle index series is defined similarly by adding the extra variables $x_1, x_2, \ldots; y_1, y_2, \ldots; \ldots$.

Example B.1. A special case of summability is implicit in (B.1): the family

$$(c_{\mu,H}\, \mu\, X^n Y^k \cdots /H)_{\mu,H} \tag{B.6}$$

of its terms is summable with sum F.

Example B.2. Formula (1.12) above, for the multiplicative inverse $1/F$ of a species F,

$$1/F = 1 - F_+ + F_+^2 - F_+^3 + \cdots + (-1)^n F_+^n + \cdots, \tag{B.7}$$

is obviously summable since $F_+(\mathbf{0}) = \mathbf{0}$.

Example B.3. The expansion of a species $F = F(X, Y, \dots)$ into its homogeneous components provides another instance of summable series:

$$F = \sum_{n,k,\dots \geq 0} F_{n,k,\dots}, \tag{B.8}$$

where $F_{n,k,\dots}$ is the (finite) sum of the terms of degrees n in X, k in Y, $\dots$ in the full expansion of F. Other important cases are

$$F = \sum_{n\geq 0} F_{n,-,-,\dots}, \qquad F = \sum_{k\geq 0} F_{-,k,-,\dots}, \quad \dots, \tag{B.9}$$

where $F_{n,-,-,\dots}$ is the sum of terms of degree n in X in the expansion of F, $F_{-,k,-,\dots}$ is the sum of terms of degree k in Y in the expansion of F, etc. The generalized binomial coefficients (3.4) are special instances of such a notation:

$$\binom{X, Y}{n} = B_{n,-}(X, Y) = [(1 + X)^{\uparrow} Y]_{n,-}. \tag{B.10}$$

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