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A CONVOLUTION FORMULA FOR TUTTE POLYNOMIALS OF ARITHMETIC MATROIDS AND OTHER COMBINATORIAL STRUCTURES

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Abstract. In this note we generalize the convolution formula for the Tutte polynomial of Kook, Reiner, and Stanton and of Etienne and Las Vergnas to a more general setting that includes both arithmetic matroids and delta-matroids. As corollaries, we obtain new proofs of two positivity results for pseudo-arithmetic matroids and a combinatorial interpretation of the arithmetic Tutte polynomial at infinitely many points in terms of arithmetic flows and colorings. We also exhibit connections with a decomposition of Dahmen–Micchelli spaces and lattice point counting in zonotopes.

1. Introduction

Matroids are combinatorial structures that capture and abstract the notion of independence. They were introduced in the 1930s, and since then they have become an important part of combinatorics and other areas of pure and applied mathematics. The Tutte polynomial is an important matroid invariant. Many invariants of graphs and hyperplane arrangements can be obtained as specializations of the Tutte polynomial (see [11]). Kook, Reiner, and Stanton [25] and Etienne and Las Vergnas [24] found a so-called convolution formula for the Tutte polynomial $\mathfrak{T}_M$ of a matroid M:

$$\mathfrak{T}_M(x, y) = \sum_{A \subseteq M} \mathfrak{T}_{M|A}(0, y)\mathfrak{T}_{M/A}(x, 0). \tag{1}$$

In this note we will generalize this formula to the far more general setting of ranked sets with multiplicities.

A *ranked set with multiplicities* is a finite set M, together with a rank function $\mathrm{rk} : 2^M \to \mathbb{Z}$ that satisfies $\mathrm{rk}(\emptyset) = 0$ and a multiplicity function $m : 2^M \to R$, where R denotes a commutative ring with 1.

This setting contains the following combinatorial structures as special cases:

- *Matroids*: if rk satisfies the rank axioms of a matroid, $R = \mathbb{Z}$, and $m \equiv 1$ (see *e.g.* [32]).

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- *Pseudo-arithmetic matroids*: if (M, rk) is a matroid and $m : 2^M \to \mathbb{R}_{\geq 0}$ satisfies certain positivity conditions (see [10]).
- *Quasi-arithmetic matroids*: if (M, rk) is a matroid and $m : 2^M \to \mathbb{Z}_{\geq 1}$ satisfies certain divisibility conditions (see [10]).
- *Arithmetic matroids*: if (M, rk, m) is both a pseudo-arithmetic matroid and a quasi-arithmetic matroid (see [10, 16]).
- *Integral polymatroids*: if rk is the submodular function that defines an integral polymatroid, $R = \mathbb{Z}$ and $m \equiv 1$ (see *e.g.* [33, Chapter 44]).
- *Rank functions of delta-matroids and ribbon graphs:* one can choose $m \equiv 1$ and $\mathrm{rk} = \rho$, the rank function of an even delta-matroid $(M, \mathcal{F})$ in the sense of Chun, Moffatt, Noble, and Rueckriemen (see [13, 26]). Ribbon graphs [8] define delta-matroids in a similar way as graphs define matroids [9, 13].

See Section 2 for definitions. Sometimes, we will write rk_M and m_M to denote the rank and multiplicity functions of M and we will occasionally write M instead of (M, rk_M, m_M) to denote the ranked set with multiplicities.

We will show that the convolution formula of Kook, Reiner, and Stanton and of Etienne and Las Vergnas holds in a very general setting. The only thing we require is that restriction and contraction are defined in the usual way: let $A \subseteq M$. The *restriction* $M|_A$ is the ranked set with multiplicities $(A, \mathrm{rk}|_A, m|_A)$, where $\mathrm{rk}|_A$ and $m|_A$ denote the restrictions of rk and m to A. The *contraction* M/A is the ranked set with multiplicities $(M \setminus A, \mathrm{rk}_{M/A}, m_{M/A})$, where $\mathrm{rk}_{M/A}(B) := \mathrm{rk}_M(B \cup A) - \mathrm{rk}_M(A)$ and $m_{M/A}(B) := m_M(B \cup A)$ for $B \subseteq M \setminus A$.

To a ranked set with multiplicities, we associate the *arithmetic Tutte function*

$$\mathfrak{M}_M(x, y) = \sum_{A \subseteq M} m(A)(x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} \in R(x, y) \tag{2}$$

and the *Tutte function*

$$\mathfrak{T}_M(x, y) = \sum_{A \subseteq M} (x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} \in R(x, y). \tag{3}$$

As usual, $R(x, y)$ denotes the ring of rational functions in x and y with coefficients in R. Note that $\mathfrak{M}_M(x+1, y+1)$ and $\mathfrak{T}_M(x+1, y+1)$ are Laurent polynomials in $R[x^{\pm 1}, y^{\pm 1}]$. If $\mathrm{rk}(A) \leq \mathrm{rk}(M)$ and $\mathrm{rk}(A) \leq |A|$ for all $A \subseteq M$, then both functions are polynomials in $R[x, y]$.

If M is a matroid, $\mathfrak{T}_M(x, y)$ is the usual Tutte polynomial. As far as we know, the Tutte Laurent polynomial $\mathfrak{T}_M(x+1, y+1)$ of a polymatroid M has not been studied yet. However, other Tutte invariants of polymatroids have appeared in the literature, cf. [12, 31]. If M is a (quasi/pseudo)-arithmetic matroid, $\mathfrak{M}_M(x, y)$ is the usual arithmetic Tutte polynomial [10, 16, 30]. The arithmetic Tutte polynomial appears in many different contexts, *e.g.* in the study of the combinatorics and topology of toric arrangements, of cell complexes, in the theory of vector partition functions, and in Ehrhart theory of zonotopes (see [2, 15, 27, 30, 35]).

If rk is the rank function of an even delta-matroid in the sense of Chun, Moffatt, Noble, and Rueckriemen [13, 26] (denoted by σ in [13]), then $\mathfrak{T}_M$ is the 2-variable Bollobás–Riordan polynomial of the delta-matroid (see [13, Eq. (6.2)] or [26, Eq. (41)]). A special case is the 2-variable Bollobás–Riordan polynomial of a ribbon graph (see [26, Eq. (47)]).

The following theorem is our main result.

Theorem 1. *Let (M, rk, m) be a ranked set with multiplicities (e.g. an arithmetic matroid). Let $\mathfrak{M}_M$ denote its arithmetic Tutte polynomial and let $\mathfrak{T}_M$ denote its Tutte polynomial. Then*

$$\mathfrak{M}_M(x, y) = \sum_{A \subseteq M} \mathfrak{M}_{M|A}(0, y)\mathfrak{T}_{M/A}(x, 0) \tag{4}$$

$$= \sum_{A \subseteq M} \mathfrak{T}_{M|A}(0, y)\mathfrak{M}_{M/A}(x, 0). \tag{5}$$

Kook, Reiner, and Stanton [25] proved this result in the case where (M, rk) is a matroid and $m \equiv 1$. Their result also follows easily from a theorem of Etienne and Las Vergnas on the decomposition of the ground set of a matroid that has a bijective proof (see [24, Theorem 5.1]). It would be interesting to give a bijective proof of our result in the case of arithmetic matroids. The result of Kook, Reiner, and Stanton can also be proved using Hopf algebras, see [22, 25].

In the case of even delta-matroids, our theorem specializes to a convolution formula for the 2-variable Bollobás–Riordan polynomial (see [26, Theorem 10(2)]).

Theorem 1 provides a new method to prove that the coefficients of the Tutte polynomial of a pseudo-arithmetic matroid are positive (see [16, Theorem 5.1] and [10, Theorem 4.5]).

Corollary 2. *The coefficients of the Tutte polynomial of a pseudo-arithmetic matroid are positive integers.*

Remark 3. Let M be an arithmetic matroid that is represented by a list of vectors X in some finitely generated abelian group. Let $\mathcal{V}(X)$ denote the set of vertices of the corresponding generalized toric arrangement (for definitions see [30]). If we set $x = 1$, the second expression for $\mathfrak{M}_M(x, y)$ in Theorem 1 is equivalent to [30, Lemma 6.1], which states that

$$\mathfrak{M}_M(1, y) = \sum_{p \in \mathcal{V}(X)} \mathfrak{T}_{M_p}(1, y). \tag{6}$$

Here, M_p denotes the matroid represented by the sublist of X that consists of all elements that define a hypersurface that contains p. This equivalence is explained in more detail in Section 3.

Equation (6) is related to two decomposition formulas in the theory of splines and vector partition functions: the decomposition of the discrete space $\mathrm{DM}(X)$ into continuous $\mathcal{D}$-spaces $\mathrm{DM}(X) = \bigoplus_{p \in \mathcal{V}(X)} e_p \mathcal{D}(X_p)$ by Dahmen and Micchelli [18] (see also [19, Theorem 49] and [20, Eq. (16.1)]) and dually, the decomposition of the periodic $\mathcal{P}$-spaces by the second author [27]. These decompositions could be a step towards a bijective proof of our result.

For two multiplicity functions $m_1, m_2 : 2^M \to R$, we will consider their product $m_1 m_2$, defined by $(m_1 m_2)(A) := m_1(A) m_2(A)$. The following generalization of our main theorem was suggested to us by Luca Moci.

Theorem 4. *Let (M, rk, m_1) and (M, rk, m_2) be two ranked sets with multiplicity. Then $(M, \mathrm{rk}, m_1 m_2)$ is a ranked set with multiplicity and its arithmetic Tutte polynomial is given by the convolution formula*

$$\mathfrak{M}_{(M,\mathrm{rk},m_1 m_2)}(x,y) = \sum_{A \subseteq M} \mathfrak{M}_{(M,\mathrm{rk},m_1)|_A}(0,y) \mathfrak{M}_{(M,\mathrm{rk},m_2)/A}(x,0). \tag{7}$$

Theorem 4 implies a generalized version of the key lemma of [21] (Lemma 2.7).

Corollary 5 (Positivity of products of multiplicity functions)**.** *Let (M, rk) be a matroid and let $m_1, m_2 : 2^E \to \mathbb{R}$ be two functions.*

If both m_1 and m_2 satisfy the positivity axiom (cf. (18)), so does their product $m_1 m_2$.

Since the first preprint version of our paper appeared, several authors found even more general convolution formulas, see [23, 29].

Remark 6. Delucchi and Moci [21] remarked that Corollary 5 implies that, if both (M, rk, m_1) and (M, rk, m_2) are arithmetic matroids, then $(M, \mathrm{rk}, m_1 m_2)$ is an arithmetic matroid as well. They used this to answer a question of Bajo, Burdick, and Chmutov [2] on cellular matroids of CW complexes.

Note that $(M, \mathrm{rk}, m_1 m_2)$ is not necessarily representable, even if both, (M, rk, m_1) and (M, rk, m_2) are representable. As an example, consider the arithmetic matroid (M, rk, m) represented by the list of vectors $X = ((1,0),(0,1),(1,1),(1,-1))$ and the arithmetic matroid (M, rk, m^2). The underlying matroid is uniform in both cases. Suppose there is a list of vectors X' that represents (M, rk, m^2). Since m^2 is equal to one for five of the six bases, one can assume without loss of generality that two of the vectors in X' are $(1,0)$ and $(0,1)$. Then it follows that the other two are of the form $(\pm 1, \pm 1)$. This implies that all bases have multiplicity one or two, which is a contradiction. Questions of this type are discussed in more detail in [28].

Zonotopes. It is easy to see that the number of integer points in a polytope is equal to the sum of the number of integer points in the interior of all of its faces. In the case of zonotopes, this statement is equivalent to the specialization of Theorem 1 to $(x,y) = (2,1)$.

Corollary 7. *Let $X = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^d$ be a list of vectors and let $Z(X) := \{\sum_{i=1}^N \lambda_i x_i : 0 \le \lambda_i \le 1\}$ be the zonotope defined by X. Then*

$$\begin{aligned} \left| Z(X) \cap \mathbb{Z}^d \right| = \mathfrak{M}(2,1) &= \sum_{A \subseteq X} \mathfrak{M}_{M|_A}(0,1) \mathfrak{T}_{M/A}(2,0) \\ &= \sum_{X \supseteq A \text{ flat}} \mathfrak{M}_{M|_A}(0,1) \mathfrak{T}_{M/A}(2,0) \\ &= \sum_F \left| \mathrm{relint}(F) \cap \mathbb{Z}^d \right|, \end{aligned} \tag{8}$$

where the last sum is over all faces of $Z(X)$.

Barvinok and Pommersheim proved a geometric convolution-like formula for the number of integer points in a lattice zonotope. It would be interesting to find a connection with our convolution formula.

Theorem 8 ([3, Section 7]). *Let $X \subseteq \mathbb{Z}^d$ be a list of N vectors. Then*

$$\left|Z(X) \cap \mathbb{Z}^d\right| = \sum_F \operatorname{vol}(F)\gamma(Z(X), F), \tag{9}$$

where the sum is over all faces F of the zonotope and $\gamma(Z(X), F)$ denotes the exterior angle of F at $Z(X)$. The volume of a face is measured intrinsically with respect to the lattice.

More specifically, the kth coefficient of the Ehrhart polynomial

$$E_X(q) = q^N \mathfrak{M}_X(1 + \frac{1}{q}, 1) \tag{10}$$

of the zonotope is equal to $\sum_{F:\dim F = k} \operatorname{vol}(F)\gamma(Z(X), F)$.

Flows and colorings. In this subsection we will give a combinatorial interpretation of the evaluation of the arithmetic Tutte polynomial and a closely related polynomial, the modified Tutte–Krushkal–Renardy polynomial, at infinitely many integer values in terms of arithmetic flows and colorings. This works for arbitrary representable arithmetic matroids.

D'Adderio and Moci [17] defined a class of "graphic arithmetic matroids" using graphs whose edges are labeled by positive integers. One can define so-called arithmetic flows and arithmetic colorings on these graphs. These notions of flows and colorings were extended by Brändén and Moci [10] to the setting where X is a finite list of elements from a finitely generated abelian group. These arithmetic flows and colorings are related to our convolution formula in a similar way as classical flows and colorings are related to the classical convolution formula [25, Theorem 2]. Arithmetic flows and colorings contain flows and colorings of CW complexes (see [4, 5]) as a special case, when the list of vectors is taken to be a boundary operator of CW complexes (see [21, Lemma 4]).

We briefly review the setup of Brändén and Moci. Let G be a finitely generated abelian group. Let X be a finite list (or sequence) of elements of G. We call $\phi \in \operatorname{Hom}(G, \mathbb{Z}_q)$ a *proper arithmetic q-coloring* if $\phi(x) \neq 0$ for all $x \in X$. We denote the number of proper arithmetic q-colorings of X by $\chi_X(q)$. As usual, $\mathbb{Z}_q$ denotes the cyclic group of cardinality q.

A *nowhere zero q-flow* on X is a function $\psi : X \to \mathbb{Z}_q \setminus \{0\}$ such that $\sum_{x \in X} \psi(x)x = 0$ in G/qG. We denote the number of such functions by $\chi^*_X(q)$.

For $B \subseteq X$, let G_B denote the torsion subgroup of the quotient $G/\langle\{x : x \in B\}\rangle$ and let $m(B) := |G_B|$.

Let $\operatorname{lcm}(X) := \operatorname{lcm}\{m(B) : B \subseteq X \text{ basis}\}$. We define the following two subsets of the set of positive integers:

$$\mathbb{Z}_M(X) := \{q \in \mathbb{Z}_{>0} : \gcd(q, \operatorname{lcm}(X)) = 1\} \tag{11}$$

$$\text{and } \mathbb{Z}_A(X) := \{q \in \mathbb{Z}_{>0} : qG_B = \{0\} \text{ for all bases } B \subseteq X\}. \tag{12}$$

Given a list of vectors X with associated arithmetic matroid $(M, \operatorname{rk}, m)$, let $\mathfrak{M}_X(x,y)$ denote the arithmetic Tutte polynomial $\mathfrak{M}_{(M,\operatorname{rk},m)}(x,y)$. Furthermore, we let $\mathfrak{M}_{X^2}(x,y)$ denote the arithmetic Tutte polynomial $\mathfrak{M}_{(M,\operatorname{rk},m^2)}(x,y)$. We recall that, by Corollary 5 (or by [21]), $(M, \operatorname{rk}, m^2)$ is indeed an arithmetic matroid. The polynomial $\mathfrak{M}_{X^2}(x,y)$ has a special significance for arithmetic matroids that arise from CW complexes. In this case, the *modified jth Tutte–Krushkal–Renardy polynomial*, that was introduced in [2], is equal to the arithmetic Tutte polynomial $\mathfrak{M}_{X^2}(x,y)$, where X is the list of vectors obtained from the jth boundary operator, see [21, Section 4]. In this setting, the modified jth Tutte–Krushkal–Renardy polynomial can be recovered from Corollary 11 below.

Theorem 9 (Brändén and Moci [10])**.** *Let G and X be as above.*

$$\text{If } q \in \mathbb{Z}_A(X), \text{ then } \chi_X(q) = (-1)^{\operatorname{rk}(X)} q^{\operatorname{rk}(G)-\operatorname{rk}(X)} \mathfrak{M}_X(1-q, 0) \tag{13}$$

$$\text{and } \chi^*_X(q) = (-1)^{|X|-\operatorname{rk}(X)} \mathfrak{M}_X(0, 1-q). \tag{14}$$

$$\text{If } q \in \mathbb{Z}_M(X), \text{ then } \chi_X(q) = (-1)^{\operatorname{rk}(X)} q^{\operatorname{rk}(G)-\operatorname{rk}(X)} \mathfrak{T}_X(1-q, 0) \tag{15}$$

$$\text{and } \chi^*_X(q) = (-1)^{|X|-\operatorname{rk}(X)} \mathfrak{T}_X(0, 1-q). \tag{16}$$

Example 10. Let $X = ((2,0), (-1,1), (1,1))$. Then $\operatorname{lcm}(X) = 2$, $\mathbb{Z}_M(X) = \{1,3,5,7,\ldots\}$, and $\mathbb{Z}_A(X) = \{2,4,6,8,\ldots\}$. The polynomials are $\chi_X(q)|_{\mathbb{Z}_A(X)} = q^2 - 4q + 4$, $\chi_X(q)|_{\mathbb{Z}_M(X)} = q^2 - 3q + 2$, $\chi^*_X(q)|_{\mathbb{Z}_A(X)} = 2q - 3$, and $\chi^*_X(q)|_{\mathbb{Z}_M(X)} = q - 1$. Hence there are two proper arithmetic 3-colorings ($[1,0]$ and $[2,0]$) and two nowhere zero 3-flows ($[1,1,2]$ and $[2,2,1]$).

Let $A \subseteq X$. We denote the sublist of X that is indexed by A by $X|_A$ (restriction) and the projection of $X|_{X\setminus A}$ to $G/A := G/\langle\{x : x \in A\}\rangle$ by X/A (contraction).

Corollary 11. *Let G and X be as above and $p, q \in \mathbb{Z}_A(X)$. Then*

$$\mathfrak{M}_{X^2}(1-p, 1-q) = p^{\operatorname{rk}(G)-\operatorname{rk}(X)} (-1)^{\operatorname{rk}(X)} \sum_{A \subseteq X} (-1)^{|A|} \chi^*_{X|_A}(q) \chi_{X/A}(p).$$

Corollary 12. *Let G and X be as above, $p \in \mathbb{Z}_A(X)$ and $q \in \mathbb{Z}_M(X)$. Then*

$$\mathfrak{M}_X(1-p, 1-q) = p^{\operatorname{rk}(G)-\operatorname{rk}(X)} (-1)^{\operatorname{rk}(X)} \sum_{A \subseteq X} (-1)^{|A|} \chi^*_{X|_A}(q) \chi_{X/A}(p).$$

The same statement holds if we instead take $p \in \mathbb{Z}_M(X)$ and $q \in \mathbb{Z}_A(X)$.

Remark 13. Suppose that the list X in Corollary 12 is the quotient of a scaled unimodular list, *i. e.* it satisfies the following conditions:

(1) There is a list $X_0 = (x_1, \ldots, x_N) \subseteq \mathbb{Z}^d$ (for some $d, N \in \mathbb{N}$) and $A_0 \subseteq X_0$ such that $X = X_0/A_0$.

(2) There is a sequence of integers $(b_1, \ldots, b_N)$ such that the scaled list $\tilde{X}_0 := (\frac{1}{b_1}x_1, \ldots, \frac{1}{b_N}x_N)$ is integral and totally unimodular.

Let $\tilde{A}_0$ be the subset of $\tilde{X}_0$ that corresponds to $A_0 \subseteq X_0$ and let $\tilde{X} := \tilde{X}_0/\tilde{A}_0$. Then $\mathfrak{M}_{\tilde{X}}(x,y) = \mathfrak{T}_X(x,y)$. Note that, due to total unimodularity, $\tilde{X}$ is contained in a free abelian group.

Therefore, we can interpret the arithmetic Tutte polynomial $\mathfrak{M}_X$ in terms of classical flows and arithmetic colorings, or vice versa. More specifically, in the previous corollary we obtain

$$\mathfrak{M}_X(1-p, 1-q) = p^{\operatorname{rk}(G)-\operatorname{rk}(E)}(-1)^{\operatorname{rk}(E)} \sum_{A \subseteq E} (-1)^{|A|} \chi^*_{X|_A}(q) \chi_{\tilde{X}/A}(p)$$

for any $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_A(X)$. For $p \in \mathbb{Z}_A(X)$ and any $q \in \mathbb{Z}$ we obtain

$$\mathfrak{M}_X(1-p, 1-q) = p^{\operatorname{rk}(G)-\operatorname{rk}(E)}(-1)^{\operatorname{rk}(E)} \sum_{A \subseteq E} (-1)^{|A|} \chi^*_{\tilde{X}|_A}(q) \chi_{X/A}(p).$$

Lists with these properties arise naturally when studying arithmetic matroids defined by labeled graphs, see [17]. In this case X is a list of vectors coming from a labeled graph and $\tilde{X}$ is the totally unimodular list of vectors that represents the underlying graphic matroid. Arithmetic matroids that can be represented by a quotient of a scaled unimodular list are studied in more detail in [28]. They can be characterized as arithmetic matroids that are regular and strongly multiplicative.

2. Background

2.1. **Matroids and polymatroids.** Let M be a finite set and let $\operatorname{rk} : M \to \mathbb{Z}_{\geq 0}$ be a function that satisfies the following axioms:

- $\operatorname{rk}(\emptyset) = 0$,
- $\operatorname{rk}(A) \leq \operatorname{rk}(B)$ for all $A \subseteq B \subseteq M$, and
- $\operatorname{rk}(A \cup B) + \operatorname{rk}(A \cap B) \leq \operatorname{rk}(A) + \operatorname{rk}(B)$ for all $A, B \subseteq M$.

Then the polytope

$$\left\{ x \in \mathbb{R}^M : 0 \leq \sum_{i \in S} x_i \leq \sum_{i \in S} \operatorname{rk}(x_i) \text{ for all } S \subseteq M \right\} \tag{17}$$

is called a *discrete polymatroid* and rk is its rank function (see [33, Chapter 44]).

A *matroid* is a pair (M, rk), where M denotes a finite set and the rank function $\operatorname{rk} : 2^M \to \mathbb{Z}_{\geq 0}$ satisfies the axioms of the rank function of a discrete polymatroid and in addition, $\operatorname{rk}(A \cup \{a\}) \leq \operatorname{rk}(A) + 1$ for all $A \subseteq M$ and $a \in M$ holds. See [32] for more details. Let $\mathbb{K}$ be a field. A matrix X with entries in $\mathbb{K}$ defines a matroid in a canonical way: M is the set of columns of the matrix and the rank function is the rank function from linear algebra. A matroid that can be represented in such a way is called *representable over* $\mathbb{K}$.

2.2. **Arithmetic matroids.**

Definition 14 (D'Adderio and Moci, Brändén and Moci [10, 16])**.** An *arithmetic matroid* is a triple (M, rk, m), where (M, rk) is a matroid and $m : 2^M \to \mathbb{Z}_{\geq 1}$ is the *multiplicity function* that satisfies certain axioms:

- (P) Let $R \subseteq S \subseteq M$. The set $[R, S] := \{A : R \subseteq A \subseteq S\}$ is called a *molecule* if S can be written as the disjoint union $S = R \cup F_{RS} \cup T_{RS}$ and for each $A \in [R, S]$, $\mathrm{rk}(A) = \mathrm{rk}(R) + |A \cap F_{RS}|$ holds. For each molecule $[R, S] \subseteq M$, the following inequality holds:
$$\rho(R, S) := (-1)^{|T_{RS}|} \sum_{A \in [R,S]} (-1)^{|S|-|A|} m(A) \geq 0. \tag{18}$$
- (A1) For all $A \subseteq M$ and $e \in M$: if $\mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A)$, then $m(A \cup \{e\}) \mid m(A)$. Otherwise $m(A) \mid m(A \cup \{e\})$.
- (A2) If $[R, S]$ is a molecule, then $m(R)m(S) = m(R \cup F_{RS})m(R \cup T_{RS})$.

A *pseudo-arithmetic matroid* is a triple (M, rk, m), where (M, rk) is a matroid and $m : 2^M \to \mathbb{R}_{\geq 0}$ satisfies (P). A *quasi-arithmetic matroid* is a triple (M, rk, m), where (M, rk) is a matroid and $m : 2^M \to \mathbb{Z}_{\geq 1}$ satisfies (A1) and (A2).

The prototypical example of an arithmetic matroid is defined by a list of vectors X in $\mathbb{Z}^d$. In this case, for a sublist S of d vectors that form a basis, we have $m(S) = |\det(S)|$ and in general $m(S) := \left| \langle x \in S \rangle_{\mathbb{R}} \cap \mathbb{Z}^d / \langle x \in S \rangle_{\mathbb{Z}} \right|$. As quotients of $\mathbb{Z}^d$ are in general not free groups, the following definition will use a slightly more general setting.

Definition 15. Let $\mathcal{A} = (M, \mathrm{rk}, m)$ be an arithmetic matroid. Let G be a finitely generated abelian group and let X be a finite list of elements of G that is indexed by M. For $A \subseteq M$, let G_A denote the maximal subgroup of G such that $|G_A / \langle A \rangle|$ is finite.

X is called a *representation* of $\mathcal{A}$ if the matroid defined by X is isomorphic to (M, rk) and $m(A) = |G_A / \langle A \rangle|$. The arithmetic matroid $\mathcal{A}$ is called *representable* if it has a representation X.

Given a representation $X \subseteq \mathbb{Z}^d$ of an arithmetic matroid, it is easy to calculate its multiplicity function (see [16, p. 344]): let $A \subseteq X$, then
$$m(A) = \gcd(\{m(B) : B \subseteq A \text{ and } |B| = \mathrm{rk}(B) = \mathrm{rk}(A)\}). \tag{19}$$
If A is independent, then $m(A)$ is the greatest common divisor of all minors of size $|A|$ of the matrix A (cf. [35, Theorem 2.2]).

2.3. **Arithmetic matroids defined by labeled graphs.** We call a graph $\mathcal{G} = (V, E)$ together with a map $\ell : E \to \mathbb{Z}_{\geq 1}$ *labeled graph*. The graph $\mathcal{G}$ is allowed to have multiple edges, but no loops. The set of edges is partitioned into a set R of *regular edges* and a set D of *dotted edges*. Such a graph defines a graphic arithmetic matroid, see [17]. Its definition extends the usual construction of the matrix representation of a graphic matroid by the oriented incidence matrix: let $V = \{v_1, \ldots, v_n\}$. We fix an arbitrary orientation θ of E such that each edge $e \in E$ can be identified with an ordered pair (v_i, v_j). To each edge

$e = (v_i, v_j)$, we associate the element $x_e \in \mathbb{Z}^n$ defined as the vector whose ith coordinate is $-\ell(e)$ and whose jth coordinate is $\ell(e)$. Then we define the list $X_R := (x_e)_{e\in R}$ and the group $G := \mathbb{Z}^n/\langle\{x_e : e \in D\}\rangle$. We denote by $\mathcal{A}(\mathcal{G}, \ell)$ the arithmetic matroid represented by the projection of X_R to G. The multiplicity function can easily be calculated: for any $A \subseteq R$, we have

$$m(A) = \gcd\left(\left\{\prod_{e\in T} \ell(e) : T \text{ maximal independent subset of } A \cup D\right\}\right).$$

2.4. **Delta-matroids and the Bollobás–Riordan polynomial.** A *delta-matroid* D is a pair $(E, \mathcal{F})$, where E denotes a finite set and $\emptyset \neq \mathcal{F} \subseteq 2^E$ satisfies the *symmetric exchange axiom*: for all $S, T \in \mathcal{F}$, if there is an element $u \in S \triangle T$, then there is an element $v \in S \triangle T$ such that $S \triangle \{u, v\} \in \mathcal{F}$. The elements of $\mathcal{F}$ are called *feasible sets*. If the sets in $\mathcal{F}$ all have the same cardinality, then $(E, \mathcal{F})$ satisfies the basis axioms of a matroid. Let $D = (E, \mathcal{F})$ be a delta-matroid and let $\mathcal{F}_{\max}$ and $\mathcal{F}_{\min}$ be the set of feasible sets of maximum and minimum cardinality, respectively. Define $D_{\max} := (E, \mathcal{F}_{\max})$ and $D_{\min} := (E, \mathcal{F}_{\min})$ to be the *upper matroid* and *lower matroid* for D, respectively, see [9]. Let $\mathrm{rk}_{\max}$ and $\mathrm{rk}_{\min}$ denote the corresponding rank functions. In [13], the following delta-matroid rank function was defined: $\rho(D) := \frac{1}{2}(\mathrm{rk}_{\max}(D) + \mathrm{rk}_{\min}(D))$, and $\rho(A) := \rho(D|_A)$ for $A \subseteq E$. This can be used to define the *(2-variable) Bollobás-Riordan polynomial*

$$\tilde{R}_D(x, y) := \sum_{A\subseteq E} (x-1)^{\rho(E)-\rho(A)}(y-1)^{|A|-\rho(A)}. \tag{20}$$

If D is a matroid, then ρ is its rank function. Note that the delta-matroid rank function ρ is different from Bouchet's birank [9]. A delta-matroid is even if all feasible sets have the same parity. A ribbon graph defines an even delta-matroid if and only if it is orientable, see [13, Proposition 5.3].

3. Proofs

To prove Theorem 1, we adapt the proof of Kook, Reiner, and Stanton [25] to our more general setting. We first define a convolution product and note some useful lemmas. Two ranked sets with multiplicity are isomorphic if there exists a bijection between the ground sets that preserves the rank and the multiplicity function. Let $\mathbb{M}$ be the set of all isomorphism classes of ranked sets with multiplicity, and let K be a commutative ring with 1. For any functions $f, g : \mathbb{M} \to K$, define the convolution $f \circ g : \mathbb{M} \to K$ by

$$(f \circ g)(M) = \sum_{A\subseteq M} f(M|_A)g(M/A). \tag{21}$$

Lemma 16. *The convolution* $\circ$ *is associative, with identity element* δ*, where*

$$\delta(M) := \begin{cases} 1 & \text{if } M = \emptyset \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

Note that there are infinitely many ranked sets with multiplicity on the empty set.

Proof of Lemma 16. It is easy to see that δ is the identity element.

Let $C, D \subseteq M$ and $C \cap D = \emptyset$. As in the case of matroids, we have $(M/C)|_D = M|_{C\cup D}/C$. Indeed, let $A \subseteq D$. Then by definition $\mathrm{rk}_{(M/C)|_D}(A) = \mathrm{rk}_M(A \cup C) - \mathrm{rk}_M(C) = \mathrm{rk}_{M|_{C\cup D}/C}(A)$. For the multiplicity function by definition $m_{(M/C)|_D}(A) = m_M(C \cup A) = m_{M|_{C\cup D}/C}(A)$.

Now let $f, g, h : \mathbb{M} \to K$. Then we have

$$\begin{aligned}((f \circ g) \circ h)(M) &= \sum_{A \subseteq M} (f \circ g)(M|_A) h(M/A) \\ &= \sum_{A \subseteq M} \sum_{C \subseteq A} f(M|_C) g(M|_A/C) h(M/A) \\ &= \sum_{C \subseteq A \subseteq M} f(M|_C) g(M|_A/C) h(M/A). \qquad (23)\end{aligned}$$

Now let $D := A \setminus C$. Hence $A = C \cup D$ and the above expression becomes

$$\begin{aligned}&\sum_{\substack{C,D \subseteq M \\ C \cap D = \emptyset}} f(M|_C) g(M|_{C\cup D}/C) h(M/(C \cup D)) \\ &= \sum_{C \subseteq M} f(M|_C) \sum_{D \subseteq M \setminus C} g((M/C)|_D) h((M/C)/D) \\ &= \sum_{C \subseteq M} f(M|_C)((g \circ h)(M/C)) \\ &= (f \circ (g \circ h))(M).\end{aligned}$$

□

Following Crapo [14], we consider $\zeta(x, y) : \mathbb{M} \to K$ defined by

$$\zeta(x, y)(M) := x^{\mathrm{rk}(M)} y^{|M| - \mathrm{rk}(M)},$$

where $K = R[x, y]$. Recall that R denotes the codomain of the multiplicity function, a commutative ring with 1.

The following simple lemma was proven for matroids in [25]. It is easy to verify that the same proof also works in our setting. Here, $\mathrm{rk}(\emptyset) = 0$ is required.

Lemma 17. *$\zeta(-x, -y)$ is the inverse of $\zeta(x, y)$ under the convolution $\circ$.*

Note that ζ only depends on the matroid, but not on the multiplicity function m. We will also need two weighted versions of ζ, namely

$$\xi(x, y)(M) := m_M(M) x^{\mathrm{rk}(M)} y^{|M| - \mathrm{rk}(M)} \qquad (24)$$

$$\text{and } \xi^*(x, y)(M) := m_M(\emptyset) x^{\mathrm{rk}(M)} y^{|M| - \mathrm{rk}(M)}. \qquad (25)$$

If M is an arithmetic matroid, then

$$\xi^*(x, y)(M) = m^*_M(M) x^{\mathrm{rk}(M)} y^{\mathrm{rk}(M^*)},$$

since $\mathrm{rk}(M^*) = |M| - \mathrm{rk}(M)$ and the dual multiplicity is defined by $m^*(A) := m(M \setminus A)$.

The following well-known description of the Tutte polynomial (see [14]) generalizes to our setting.

Lemma 18. *We have*

$$\mathfrak{T}_M(x+1,y+1) = (\zeta(1,y)\circ\zeta(x,1))(M). \tag{26}$$

Lemma 18 is actually a special case ($m \equiv 1$) of the next lemma.

Lemma 19. *We have*

$$\mathfrak{M}_M(x+1,y+1) = (\xi(1,y)\circ\zeta(x,1))(M) = (\zeta(1,y)\circ\xi^*(x,1))(M).$$

Proof. We calculate

$$\begin{aligned}
(\xi(1,y)\circ\zeta(x,1))(M) &= \sum_{A\subseteq M} m_M(A) y^{|(M|_A)|-\operatorname{rk}(M|_A)} x^{\operatorname{rk}(M/A)} \\
&= \sum_{A\subseteq M} m_M(A) x^{\operatorname{rk}(M)-\operatorname{rk}(A)} y^{|A|-\operatorname{rk}(A)} \\
&= \mathfrak{M}_M(x+1,y+1) \\
(\zeta(1,y)\circ\xi^*(x,1))(M) &= \sum_{A\subseteq M} y^{|(M|_A)|-\operatorname{rk}(M|_A)} m_{(M/A)}(\emptyset) x^{\operatorname{rk}(M/A)} \qquad (27)\\
&= \sum_{A\subseteq M} m_M(A) x^{\operatorname{rk}(M)-\operatorname{rk}(A)} y^{|A|-\operatorname{rk}(A)} \\
&= \mathfrak{M}_M(x+1,y+1).
\end{aligned}$$

□

Proof of Theorem 1. Lemma 19 implies

$$\mathfrak{M}_M(x+1,0) = (\zeta(1,-1)\circ\xi^*(x,1))(M) \tag{28}$$

$$\text{and } \mathfrak{M}_M(0,y+1) = (\xi(1,y)\circ\zeta(-1,1))(M). \tag{29}$$

Using Lemma 17 and Lemma 18 we obtain

$$\begin{aligned}
&\sum_{A\subseteq M} \mathfrak{M}_{M|_A}(0,y+1)\mathfrak{T}_{M/A}(x+1,0) \\
&\quad = ((\xi(1,y)\circ\zeta(-1,1))\circ(\zeta(1,-1)\circ\zeta(x,1)))(M) \\
&\quad = (\xi(1,y)\circ(\zeta(-1,1)\circ\zeta(1,-1))\circ\zeta(x,1))(M) \\
&\quad = (\xi(1,y)\circ\zeta(x,1))(M) = \mathfrak{M}(x+1,y+1) \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{A\subseteq M} \mathfrak{T}_{M|_A}(0,y+1)\mathfrak{M}_{M/A}(x+1,0) \\
&\quad = ((\zeta(1,y)\circ\zeta(-1,1))\circ(\zeta(1,-1)\circ\xi^*(x,1)))(M) \\
&\quad = (\zeta(1,y)\circ(\zeta(-1,1)\circ\zeta(1,-1))\circ\xi^*(x,1))(M) \\
&\quad = (\zeta(1,y)\circ\xi^*(x,1))(M) = \mathfrak{M}(x+1,y+1).
\end{aligned}$$

□

Proof of Corollary 2. Using Theorem 1 twice, we obtain

$$\begin{aligned}\mathfrak{M}_M(x,y) &= \sum_{A\subseteq M} \mathfrak{M}_{M|_A}(0,y)\mathfrak{T}_{M/A}(x,0)\\ &= \sum_{A\subseteq M} \mathfrak{T}_{M/A}(x,0)\left(\sum_{B\subseteq A}\mathfrak{T}_{M|_B}(0,y)\mathfrak{M}_{M|_A/B}(0,0)\right). \qquad (31)\end{aligned}$$

It is well-known that the coefficients of the Tutte polynomial are positive integers. Hence it is sufficient to show that

$$\mathfrak{M}_M(0,0) = \sum_{A\subseteq M}(-1)^{\mathrm{rk}(M)-|A|}m(A) \geq 0 \qquad (32)$$

for any pseudo-arithmetic matroid M. This can be shown in various ways:

(i) induction.

(ii) It is known that 2^M can be partitioned into molecules $[R,S]$ with $\mathrm{rk}(R) = |R|$ and $\mathrm{rk}(S) = \mathrm{rk}(M)$ (see [10, Proposition 4.4], see also [6, 14]). For each such molecule $\mathrm{rk}(M) = |R| + |F_{RS}| = |S \setminus T_{RS}|$ holds. Hence we obtain

$$\mathfrak{M}_M(0,0) = \sum_{\substack{[R,S]\\ \text{molecule}}} \underbrace{(-1)^{|T_{RS}|}\sum_{R\subseteq A\subseteq S}(-1)^{|S|-|A|}m(A)}_{\rho(R,S)\geq 0} \geq 0. \qquad (33)$$

(iii) In the case of an arithmetic matroid that is represented by a list of vectors it follows from the interpretation of $\mathfrak{M}(0,q)$ in [27]. □

Proof of Remark 3. We will now prove that, if we set $x = 1$, the second expression for $\mathfrak{M}_M(x,y)$ in Theorem 1 is equivalent to [30, Lemma 6.1].

Using [30, Lemma 6.1], *i. e.* formula (6), and the classical convolution formula we obtain

$$\begin{aligned}\mathfrak{M}_M(1,y) &= \sum_{p\in\mathcal{V}(X)}\mathfrak{T}_{M_p}(1,y) \qquad (34)\\ &= \sum_{p\in\mathcal{V}(X)}\sum_{A\subseteq M_p}\mathfrak{T}_{(M_p)|_A}(0,y)\mathfrak{T}_{M_p/A}(1,0)\\ &= \sum_{A\subseteq M}\mathfrak{T}_{M|_A}(0,y)\left(\sum_{p:A\subseteq M_p}\mathfrak{T}_{M_p/A}(1,0)\right) \qquad (35)\\ &= \sum_{A\subseteq M}\mathfrak{T}_{M|_A}(0,y)\left(\sum_{\bar{p}\in\mathcal{V}(M/A)}\mathfrak{T}_{(M/A)_{\bar{p}}}(1,0)\right) \qquad (36)\\ &= \sum_{A\subseteq M}\mathfrak{T}_{M|_A}(0,y)\mathfrak{M}_{M/A}(1,0).\end{aligned}$$

Recall that the vertices of the generalized toric arrangement are contained in the generalized real torus $\hom(G,S^1)$, where G denotes a finitely generated abelian group. To verify the equality of (35) and (36), note that

$$\{p \in \mathcal{V}(X) \subseteq \hom(G, S^1) : A \subseteq M_p\} = \{p \in \mathcal{V}(X) : p(A) = \{1\}\} \\ \leftrightarrow \mathcal{V}(X/A) \subseteq \hom(G/\langle A \rangle, S^1)$$

and $M_p/A = (M/A)_{\bar{p}}$ since restriction and contraction commute. We have also used that, since $A \subseteq M_p$, $\mathfrak{T}_{M_p|A}(x,y) = \mathfrak{T}_{M|A}(x,y)$.

For the other direction, note that $m(A) = |\{p \in \mathcal{V}(X) : A \subseteq X_p\}|$ holds by [30, Lemma 5.4]. Hence

$$\mathfrak{M}_X(1,0) = \sum_{p \in \mathcal{V}(X)} \sum_{\substack{A \subseteq X_p \\ \mathrm{rk}(A) = \mathrm{rk}(X)}} (-1)^{|A| - \mathrm{rk}(A)} = \sum_{p \in \mathcal{V}(X)} \mathfrak{T}_{X_p}(1,0). \tag{37}$$

Now [30, Lemma 6.1] follows using essentially the same calculation as above. □

Proof of Theorem 4. The proof is very similar to the proof of Theorem 1. As we want to be able to work with arithmetic Tutte polynomials with different multiplicity functions, we will work with ranked sets, *i. e.* sets that are equipped with a rank function (but *a priori* no multiplicity function).

A *formal minor* of a ranked set (M, rk) is a tuple $M_{A,B} := ((M, \mathrm{rk}), A, B)$ with $B \subseteq A \subseteq M$. The formal minor $M_{A,B}$ defines a ranked set on $A \setminus B$ with rank function $\mathrm{rk}_{M_{A,B}}(S) = \mathrm{rk}(S \cup B) - \mathrm{rk}(B)$. This is nothing else than the ranked set $(M|_A)/B = (M/B)|_A$. However, the formal minor carries additional information on how it was constructed. This will be important as we want to add in the multiplicity later on. For example, consider the list of vectors $X = ((1,0),(0,2))$. This defines an arithmetic matroid whose multiplicity function is given by $m(X) = m((0,2)) = 2$ and $m(\emptyset) = m((1,0)) = 1$. If we contract the second vector or restrict to the first vector, respectively, we obtain the same ranked set in both cases. On the other hand, the two ranked sets with multiplicity that we obtain are different.

Restriction and contraction of formal minors are defined the usual way, *i. e.* $M_{A,B}|_{A'} := M_{A' \cup B, B}$ for $A' \subseteq A \setminus B$ and $M_{A,B}/B' := M_{A', B \cup B'}$ for $B' \subseteq A \setminus B$.

For a ranked set (M, rk), let $\mathbb{M}(M, \mathrm{rk}) := \{M_{A,B} : A \subseteq B\}$ be the set of all formal minors of the ranked set (M, rk). One can proceed as in the proof of Theorem 1, replacing $\mathbb{M}$ by $\mathbb{M}(M, \mathrm{rk})$. For example, for any functions $f, g : \mathbb{M}(M, \mathrm{rk}) \to K$, we define the convolution $f \circ g : \mathbb{M}(M, \mathrm{rk}) \to K$ as above. Let $m : M \to R$ be a multiplicity function. The functions $\zeta(x,y), \xi_m(x,y), \xi_m(x,y)^* : \mathbb{M}(M, \mathrm{rk}) \to K = R[x,y]$ are defined by

$$\zeta(x,y)((M|_A)/B) := x^{\mathrm{rk}((M|_A)/B)} y^{|M \setminus (A \cup B)| - \mathrm{rk}((M|_A)/B)}$$
$$\xi_m(x,y)((M|_A)/B) := m(A \cup B) x^{\mathrm{rk}((M|_A)/B)} y^{|M \setminus (A \cup B)| - \mathrm{rk}((M|_A)/B)}$$
$$\text{and } \xi^*_m(x,y)((M|_A)/B) := m(B) x^{\mathrm{rk}((M|_A)/B)} y^{|M \setminus (A \cup B)| - \mathrm{rk}((M|_A)/B)}.$$

Let (M, rk) be a ranked set and let $m_1, m_2 : 2^M \to R$ be two multiplicity functions. Using essentially the same proof as for Lemma 19, one can show that

$$\mathfrak{M}_{(M, \mathrm{rk}, m_1 m_2)}(x+1, y+1) = (\xi_{m_1}(1,y) \circ \xi^*_{m_2}(x,1))(M) \tag{38}$$

and deduce using (28), (29), and Lemma 17

$$\begin{aligned}\sum_{A\subseteq M} &\mathfrak{M}_{(M,\mathrm{rk},m_1)|_A}(0,y+1)\mathfrak{M}_{(M,\mathrm{rk},m_2)/A}(x+1,0)\\ &= ((\xi_{m_1}(1,y)\circ\zeta(-1,1))\circ(\zeta(1,-1)\circ\xi^*_{m_2}(x,1)))(M)\\ &= (\xi_{m_1}(1,y)\circ\xi^*_{m_2}(x,1))(M) = \mathfrak{M}_{(M,\mathrm{rk},m_1m_2)}(x+1,y+1).\end{aligned}$$ □

Proof of Corollary 5. Let $[R,S]$ be a molecule. We need to show that $\rho(R,S)$ is nonnegative for the multiplicity function m_1m_2. Note that the positivity axiom is closed under minors: for deletions it is obvious and for contractions it follows from the fact that $[R,S]$ is a molecule in the contraction M/e if and only if $[R\cup\{e\}, S\cup\{e\}]$ is a molecule in M.

It is known that $\rho(R,S)$ is the constant coefficient of the arithmetic Tutte polynomial obtained by restricting to S and contracting the elements in R. This was observed in the proof of [10, Lemma 4.5] using [10, Lemma 4.3][1].

Hence by Theorem 4 and Corollary 2

$$\begin{aligned}\rho(R,S) &= \mathfrak{M}_{((M,\mathrm{rk},m_1m_2)|_S)/R}(0,0)\\ &= \sum_{A\subseteq S\setminus R}\underbrace{\mathfrak{M}_{((M,\mathrm{rk},m_1)/R)|_A}(0,0)}_{\geq 0}\underbrace{\mathfrak{M}_{(M,\mathrm{rk},m_2)/(R\cup A)}(0,0)}_{\geq 0}\geq 0.\end{aligned}$$ □

Proof of Corollary 7. It is known that $\left|Z(X)\cap\mathbb{Z}^d\right| = \mathfrak{M}(2,1)$ and $\left|\mathrm{relint}\, Z(X)\cap\mathbb{Z}^d\right| = \mathfrak{M}(0,1)$ (see [15, 34, 35]). The second equality is Theorem 1. The third follows from the fact that $\mathfrak{T}_{M/A}(2,0)=0$ if A is not a flat since, in this case, M/A contains a loop. Furthermore, the number of vertices of the zonotope is equal to the number of regions of the central hyperplane arrangement defined by X (see [7, Proposition 2.2.2]). This number equals $\mathfrak{T}_M(2,0)$ (see [36]). For a flat A, there is a canonical bijection between the vertices of $Z(X/A)$ and the faces of $Z(X)$ that correspond to A. □

Proof of Corollary 11.

$$\begin{aligned}\mathfrak{M}_{X^2}(1-p,1-q) &= \sum_{A\subseteq X}\mathfrak{M}_{X|_A}(0,1-q)\mathfrak{M}_{X/A}(1-p,0)\\ &= \sum_{A\subseteq X}(-1)^{|A|-\mathrm{rk}(A)}\chi^*_{X|_A}(q)(-1)^{\mathrm{rk}(X/A)}p^{\mathrm{rk}(G/A)-\mathrm{rk}(X/A)}\chi_{X/A}(p)\\ &= p^{\mathrm{rk}(G)-\mathrm{rk}(X)}\sum_{A\subset X}(-1)^{|A|-\mathrm{rk}(A)+\mathrm{rk}(X/A)}\chi^*_{X|_A}(q)\chi_{X/A}(p)\\ &= p^{\mathrm{rk}(G)-\mathrm{rk}(X)}(-1)^{\mathrm{rk}(X)}\sum_{A\subseteq X}(-1)^{|A|}\chi^*_{X|_A}(q)\chi_{X/A}(p).\end{aligned}\tag{39}$$

[1]Note that [10, Lemma 4.5] contains a small error: the factor $(y-1)^{|R|-\mathrm{rk}(R)}$ is missing on the right-hand side of the first equation.

The first two steps use Theorems 4 and 9. The third uses $\mathrm{rk}(X/A) - \mathrm{rk}(G/A) = \mathrm{rk}(X) - \mathrm{rk}(G)$. The last equality holds because

$$(-1)^{\mathrm{rk}(X/A)-\mathrm{rk}(A)} = (-1)^{\mathrm{rk}(X)}. \qquad \square$$

Proof of Corollary 12. This follows by the same argument as in the proof of Corollary 11, using Theorem 1 instead of Theorem 4 in the first step. □

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