(q,t)-Foulkes Conjecture

François Bergeron, LaCIM

ADVANCES IN APPLIED MATHEMATICS 13, 390–403 (1992)

Plethysm and Vertex Operators

CHRISTOPHE CARRE AND JEAN-YVES THIBON

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We derive some stability properties and recurrence relations for plethysm coefficients.

PLETHYSM

DUDLEY ERNEST LITTLEWOOD
(1903 – 1979)

POLYNOMIAL CONCOMITANTS AND INVARIANT MATRICES

J. LONDON MATH. SOC., 1936

A wider problem here suggests itself. Any invariant matrix of an invariant matrix must be an invariant matrix of the original matrix, and thus expressible as the direct sum of irreducible invariant matrices.

Thus $$[A^{[\lambda]}]^{[\mu]} = \Sigma\, k_{\lambda\mu\nu} A^{[\nu]}.$$

Hence we may define a new type of multiplication of S-functions

$$\{\lambda\} \otimes \{\mu\} = \Sigma\, k_{\lambda\mu\nu} \{\nu\}.$$

$$\{\lambda\} \otimes \{\mu\} = s_\mu[s_\lambda] = s_\mu \circ s_\lambda$$

Rules of Plethysm

$$(f+g)[☁] = f[☁] + g[☁]$$

$$(f\cdot g)[☁] = f[☁]\cdot g[☁]$$

$$p_k[X\,Y] = p_k[X]\; p_k[Y]$$

$$p_k[X/Y] = p_k[X] \div p_k[Y]$$

$$p_k[X \pm Y] = p_k[X] \pm p_k[Y]$$

$$p_k[x] = x^k \qquad p_k[cte] = cte$$

HERBERT OWEN FOULKES
(1907 – 1977)

FOULKES CONJECTURE

CONCOMITANTS OF THE QUINTIC AND SEXTIC UP TO DEGREE FOUR IN THE COEFFICIENTS OF THE GROUND FORM

H. O. FOULKES*.

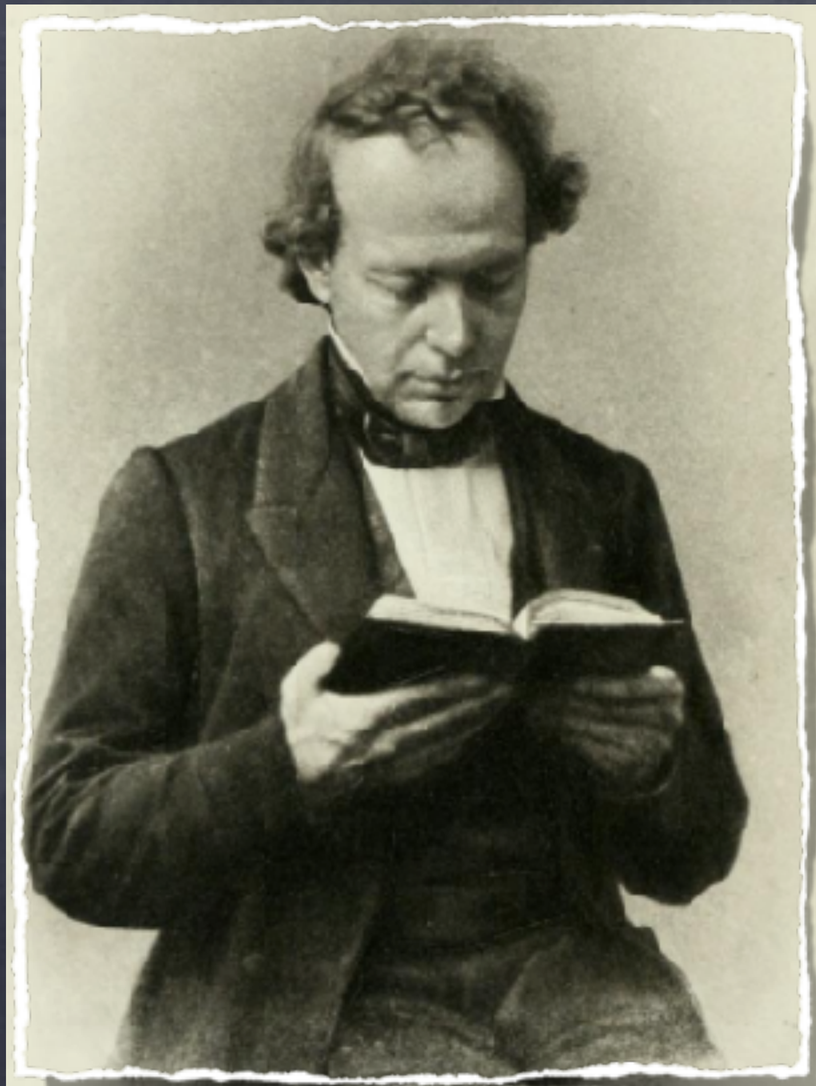
1. D. E. Littlewood (1) has shown that there is an exact correspondence between the S-functions appearing in the "new multiplication" $\{\lambda\}\otimes\{q\}$ and those concomitants, reducible or otherwise, of a ground form of type $\{\lambda\}$ which are of degree q in the coefficients of the ground form. He has also given several methods of computing such products and has obtained the number and types of concomitant for the cubic up to degree six in the coefficients, and for the quartic up to degree five in the coefficients†.

As the degrees of the ground form and the concomitants increase, it

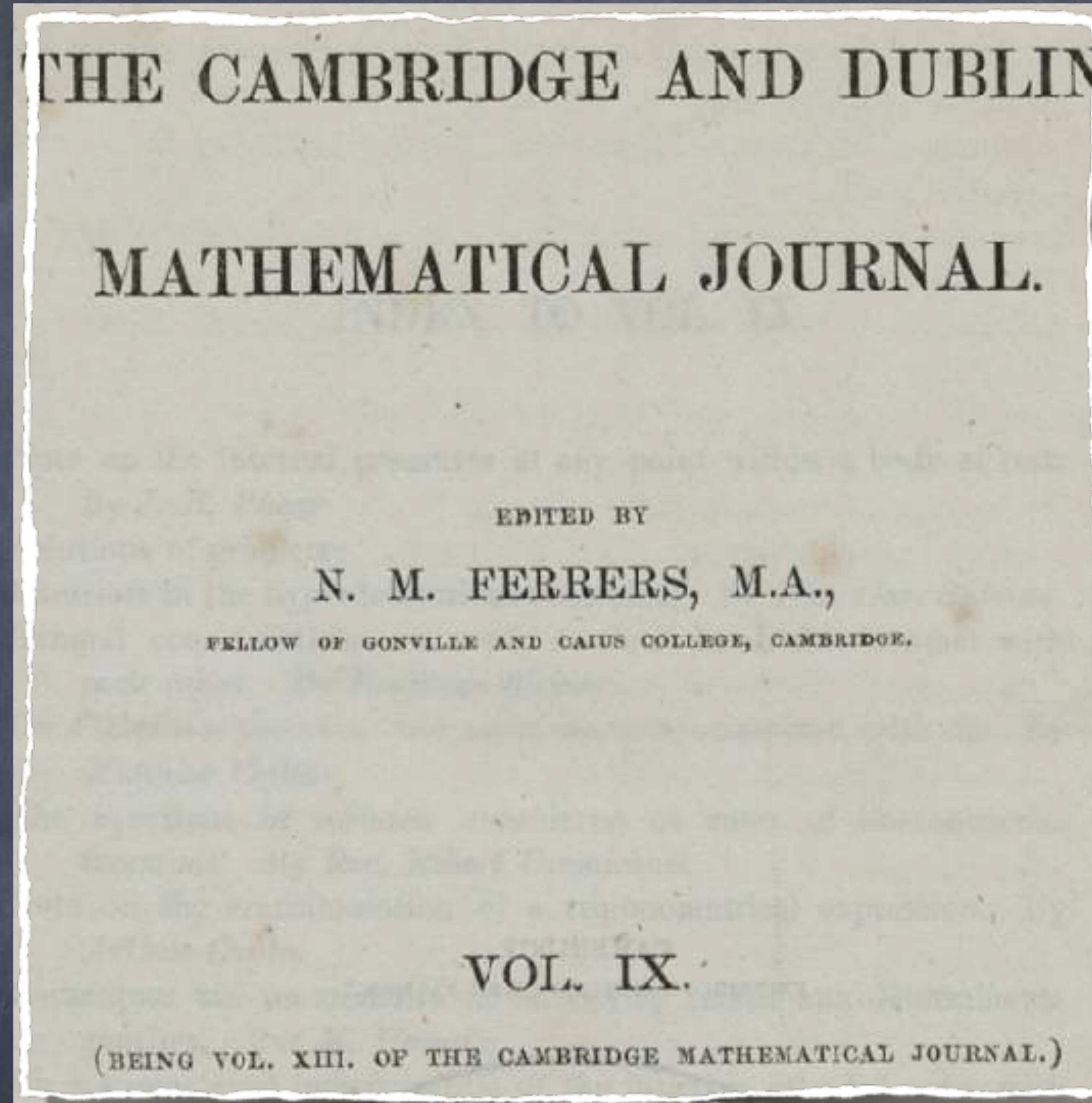
J. of London Math. Soc.

1950

A proof by S-functions of the general theorem underlying this assumption has not yet been obtained. The theorem is that for integers m, n, where $n > m$, the product $\{m\}\otimes\{n\}$ includes all terms of $\{n\}\otimes\{m\}$.



CHARLES HERMITE
(1822–1901)



THE CAMBRIDGE AND DUBLIN

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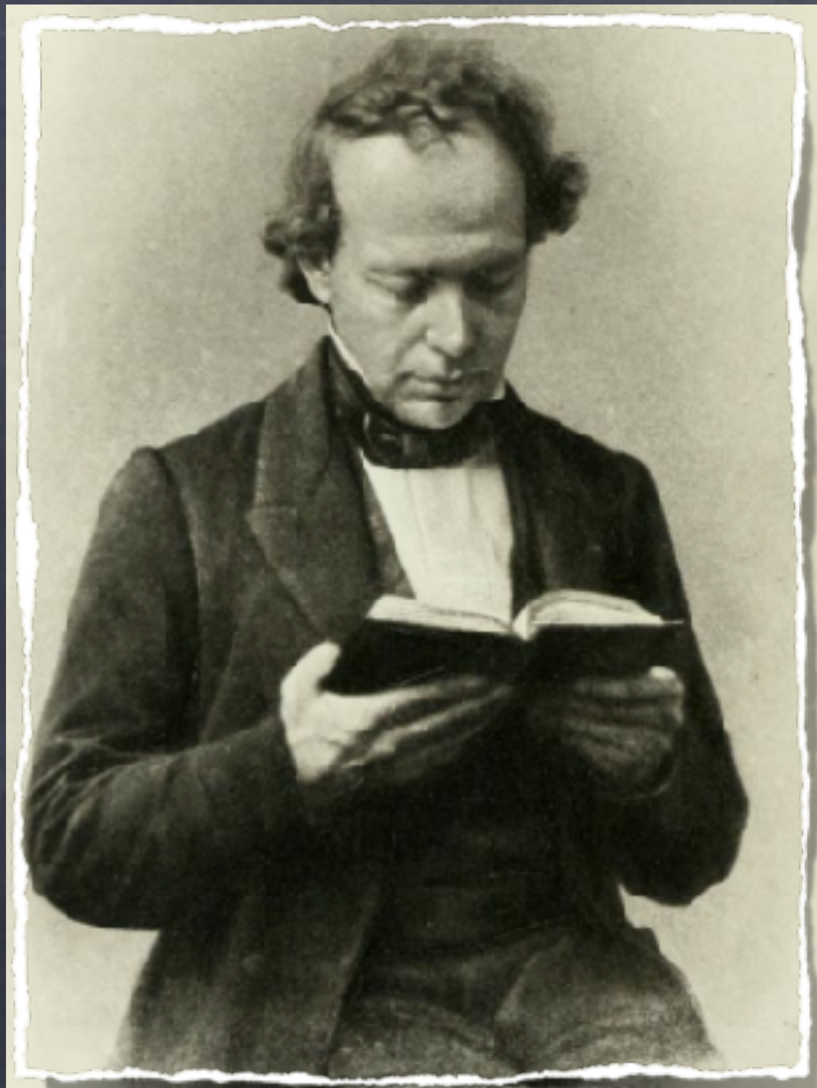
$$\Delta_b \circ \Delta_a = \Delta_a \circ \Delta_b$$

$$X = x_1, x_2$$



CHARLES HERMITE
(1822–1901)

SUR LA THEORIE DES FONCTIONS HOMOGENES À DEUX INDETERMINEES.

Par M. Hermite.

Mes premières recherches sur la théorie des formes à deux indéterminées, ont pour objet la démonstration de cette proposition arithmétique élémentaire, *que les formes à coefficients entiers et en nombre infini, qui ont les mêmes invariants, ne donnent qu'un nombre essentiellement limité de classes distinctes.*

Section I.—Loi de Réciprocité.

Elle est contenue dans le théoréme: *A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant du degré m par rapport aux coefficients, d'une forme du degré p.*

1854

$$\Delta_b \circ \Delta_a = \Delta_a \circ \Delta_b \qquad X = x_1, x_2$$

Jacques Hadamard
(1865 – 1963)

Acta Math.
Volume 20 (1897), 201-238.

Mémoire sur l'élimination

J. Hadamard

MÉMOIRE SUR L'ÉLIMINATION

PAR

J. HADAMARD
A BORDEAUX.

1\. La méthode des fonctions symétriques apprend à éliminer les inconnues $x_1, x_2, \ldots, x_n$ entre les équations

$$(1) \qquad f_1 = 0, \quad f_2 = 0, \quad \ldots, \quad f_{n+1} = 0$$

en formant le produit $\Pi f_{n+1}(x_1, x_2, \ldots, x_n)$, étendu aux systèmes de valeurs de $x_1, x_2, \ldots, x_n$ qui vérifient les n premières équations (1).

Two Formulations $a < b$

Symmetric Function Formulation

$s_b \circ s_a - s_a \circ s_b$ is Schur positive

Functorial Formulation

$$S^a \circ S^b \overset{\exists}{\hookrightarrow} S^b \circ S^a$$

$F : \mathcal{V}ect_{\mathbb{k}} \longrightarrow \mathcal{V}ect_{\mathbb{k}}$

EXAMPLES

$$\Delta_3 \circ \Delta_2 - \Delta_2 \circ \Delta_3 = \Delta_{222}$$

$$\Delta_4 \circ \Delta_2 - \Delta_2 \circ \Delta_4 = \Delta_{422} + \Delta_{2222}$$

$$\Delta_5 \circ \Delta_2 - \Delta_2 \circ \Delta_5 = \Delta_{622} + \Delta_{442} + \Delta_{4222} + \Delta_{22222}$$

$$\Delta_4 \circ \Delta_3 - \Delta_3 \circ \Delta_4 = \Delta_{732} + \Delta_{6222} + \Delta_{5421}$$

MORE THAN 3 VARIABLES

$$\Delta_\lambda(x_1, x_2, \ldots, x_n) = 0$$

WHEN $n < \ell(\lambda)$

Examples

$$\Delta_5 \circ \Delta_3 - \Delta_3 \circ \Delta_5 = \Delta_{10,3,2} + \Delta_{942} + \Delta_{9222} + \Delta_{843} + \Delta_{8421} + \Delta_{8322} + \Delta_{762} + \Delta_{7521} + \Delta_{7431} + \Delta_{7422} + \Delta_{72222} + \Delta_{6522} + \Delta_{6441} + \Delta_{64221} + \Delta_{55311} + \Delta_{5442}$$

SCHUR POSITIVITY

$f \preccurlyeq_s g$ IFF

$g - f$ is SCHUR POSITIVE

CLOSED UNDER ALL USUAL OPERATIONS

Schur-Positivity is Rare

Among positive coefficient homogeneous degree d symmetric functions

$d = 6$

Proportion of Schur-positive

$1/1027458432000$

$$\prod_{\mu \vdash d} \Big(\sum_{\lambda} K_{\lambda\mu}\Big)^{-1}$$

Vic Reiner

Rebecca Patrias

Ian G. Macdonald

Combinatorial Macdonald Polynomials

Publ. I.R.M.A. Strasbourg, 1988, 372/S–20
Actes 20e Séminaire Lotharingien, p. 131–171

A NEW CLASS OF SYMMETRIC FUNCTIONS

BY

I. G. MACDONALD

Contents.

1988

$H_\mu(x;q,t)$ COMBINATORIAL MACDONALD POLYNOMIALS

SCHUR POSITIVE

EXAMPLES

$H_n(x;q,t) = H_n(x;q)$

$$H_2 = \Delta_2 + q\,\Delta_{11}$$

$$H_3 = \Delta_3 + (q+q^2)\,\Delta_{21} + q^3\,\Delta_{111}$$

$$\begin{aligned} H_{31} = \Delta_4 &+ (q+q^2+t)\,\Delta_{31} \\ &+ (q^2+qt)\,\Delta_{22} \\ &+ (q^3+q^2t+qt)\,\Delta_{211} \\ &+ q^3t\,\Delta_{1111} \end{aligned}$$

EXAMPLES

$H_n(x;q,t) = H_n(x;q)$

$$H_{22} = \Delta_4 + (qt + q + t)\Delta_{31} + (q^2 + t^2)\Delta_{22} + (q^2t + qt^2 + qt)\Delta_{211} + q^2t^2\Delta_{1111}$$

$$H_{31} = \Delta_4 + (q + q^2 + t)\Delta_{31} + (q^2 + qt)\Delta_{22} + (q^3 + q^2t + qt)\Delta_{211} + q^3t\Delta_{1111}$$

SPECIALIZATIONS

$$H_\mu(x;0,0) = \Delta_n(x)$$

$$H_\mu(x;1,1) = \Delta_1^n(x)$$

$$H_\mu(x;0,t)\Big|_{t^{\text{MAXDEG}}} = \Delta_\mu(x)$$

$$H_\mu(x;q,t) = H_{\mu'}(x;t,q)$$

$$H_n(x;q) = \prod_{i=1}^{n}(1-q^i)\cdot s_n\left[\frac{X}{1-q}\right]$$

PLETHYSM

$$H_{\underbrace{11\cdots1}_{n}}(x;t) = \prod_{i=1}^{n}(1-t^i)\cdot s_n\left[\frac{X}{1-t}\right]$$

$$H_\mu(x;q,1) = H_{\mu_1}(x;q)\, H_{\mu_2}(x;q) \cdots H_{\mu_\ell}(x;q)$$

$$H_\mu(x;1,t) = \prod_{k\in\mu'} H_{\underbrace{11\cdots1}_{k}}(x;t)$$

François Bergeron, LACIM

$H_\mu(x;0,t)$

GRADED CHARACTER OF THE COHOMOLOGY RING OF SPRINGER VARIETIES

GARSIA

PROCESI

Foulkes Conjecture

Generalizations

Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 277 (2004) 579–614
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized Foulkes' Conjecture and tableaux construction

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JOURNAL OF PURE AND APPLIED ALGEBRA
Journal of Pure and Applied Algebra 130 (1998) 85–98

On Foulkes' conjecture

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Communicated by P.J. Freyd; received 11 October 1996

$a \leq c, d \leq b$ $\qquad$ $ab = cd$

$s_c \circ s_d - s_a \circ s_b$ is Schur positive

A Purely Combinatorial Problem

For all

$$a \leq c, d \leq b \qquad ab = cd$$

Show that

$$h_c \circ h_d[1+q] - h_a \circ h_b[1+q] \in \mathbb{N}[q]$$

$$\begin{bmatrix} c+d \\ c \end{bmatrix}_q - \begin{bmatrix} a+b \\ a \end{bmatrix}_q \in \mathbb{N}[q]$$

Two New Extensions

$a \leq c, d \leq b \qquad ab = cd \qquad k$

Conjecture 1

$$\Delta_a \circ \Delta_b^k \leq_\Delta \Delta_c \circ \Delta_d^k$$

AND

$$\Delta_a \circ \Delta_{bb\cdots b} \leq_\Delta \Delta_c \circ \Delta_{dd\cdots d}$$

(q,t)-Foulkes Conjecture

Conjecture

$a \leq c, d \leq b$ $\qquad ab = cd \qquad k$

$$\frac{H_c \circ H_{dd\cdots d} - H_a \circ H_{bb\cdots b}}{1-q}$$

is (q,t)-Schur Positive

$H_r \circ H_{dd\cdots d} - H_a \circ H_{bb\cdots b}$ is divisible by $1-q$

$\Big\downarrow q=1$

$$\Delta_1^r \circ H_{\underbrace{11\cdots 1}_{k}}(x;t)^d - \Delta_1^a \circ H_{\underbrace{11\cdots 1}_{k}}(x;t)^b = 0$$

$$H_{\underbrace{11\cdots 1}_{k}}(x;t)^{rd} - H_{\underbrace{11\cdots 1}_{k}}(x;t)^{ab} = 0$$

Specialization

$$\mathcal{J}_{a,b}(x;q) := \frac{H_b \circ H_a - H_a \circ H_b}{1-q}$$

$$\begin{aligned}\mathcal{J}_{23}(x;q) = {} & q^2(q+1)^2 s_{33} + q(q^2+1)(q+1)^2 s_{321} \\ & + q^2(q+1)^2 s_{3111} + (q+1)(q^2+1) s_{222} \\ & + q(q+1)(q^2+1)(q^2+q+1) s_{2211} \\ & + q^2(q+1)(2q^2+q+1) s_{21111} \\ & + q^3(q+1)(q^2+1) s_{111111}\end{aligned}$$

SPECIALIZATIONS

$$\Delta_b \circ \Delta_a - \Delta_a \circ \Delta_b$$

$$\Delta_c \circ \Delta_d - \Delta_a \circ \Delta_b$$

$$\Delta_c \circ \Delta_{dd\cdots d} - \Delta_a \circ \Delta_{bb\cdots b}$$

$$\Delta_c \circ \Delta_d^k - \Delta_a \circ \Delta_b^k$$

$$\frac{H_c \circ H_d^k - H_a \circ H_b^k}{1-q} \qquad (t=1)$$

AND MORE

Supporting Evidence

- Computer algebra calculations

Conjecture holds for all cases when total degree ≤ 18

- At $q=1$ and $t=1$ the (q,t)-Foulkes conjecture holds

$$\mathcal{F}_{a,b}(x;q) := \frac{H_b \circ H_a - H_a \circ H_b}{1-q}$$

$$\mathcal{F}_{23}(x,1) = 4\Delta_{33} + 8\Delta_{321} + 4\Delta_{3111} + 4\Delta_{222} + 12\Delta_{2211} + 8\Delta_{21111} + 4\Delta_{111111}$$

$$= 4\Delta_{11}^{3}$$

EXAMPLES

$$\mathcal{F}_{a,b}(x,1) \in \mathbb{N}[\Delta_1, \Delta_{11}, \Delta_2]$$

$a < b$

$$\mathcal{F}_{23}(x,1) = 4\Delta_{11}^3$$

$$\mathcal{F}_{24}(x,1) = 8\Delta_{11}^4 + 16\Delta_{11}^3\Delta_2$$

$$\mathcal{F}_{34}(x,1) = 24\Delta_1^4\Delta_{11}^3\Delta_2$$

$$\mathcal{F}_{25}(x,1) = 16\Delta_{11}^5 + 40\Delta_{11}^4\Delta_2 + 40\Delta_{11}^3\Delta_2^2$$

$$\mathcal{F}_{45}(x,1) = 16\Delta_1^{10}\Delta_{11}^5 + 80\Delta_1^{10}\Delta_{11}^3\Delta_2^2$$

$$\mathcal{F}_{56}(x,1) = 120\Delta_1^{18}\Delta_{11}^5\Delta_2 + 200\Delta_1^{18}\Delta_{11}^3\Delta_2^3$$

THEOREM

WE HAVE $\boxed{\mathcal{F}_{a,b}(x,1) \in \mathbb{N}[\Delta_1, \Delta_{11}, \Delta_2]}$ $a < b$

EXPLICITLY GIVEN BY THE FORMULA:

$$\mathcal{F}_{a,b}(x,1) = \frac{\Delta_1^{(a-2)b}}{2}\Big(ab(b-a)\Delta_1^{2(b-1)}\Delta_{11} + \binom{a}{2}(P^b - Q^b) - \binom{b}{2}(P^a - Q^a)\Big)$$

WHERE

$$P := \Delta_2 + \Delta_{11} \qquad Q := \Delta_2 - \Delta_{11}$$

THEOREM

$a \leq c, d \leq b \qquad n = abk = cdk$

$$\lim_{q\to 1} \frac{H_c \circ H_{dd\cdots d} - H_a \circ H_{bb\cdots b}}{1-q}\Bigg|_{t=1} =$$

$$\left(ak\binom{b}{2} - ck\binom{d}{2}\right) s_1^{n-2} s_{11} + \binom{a}{2} s_1^{(a-2)bk}\,\theta_{bk}$$
$$- \binom{c}{2} s_1^{(c-2)dk}\,\theta_{dk}$$

$$\theta_m = \frac{1}{2}\left((s_2+s_{11})^m - (s_2-s_{11})^m\right)$$

STABILITY

Michel Brion

manuscripta math. 80, 347 – 371 (1993)

manuscripta mathematica

Stable properties of plethysm : on two conjectures of Foulkes

Michel BRION

Two conjectures made by H.O. Foulkes in 1950 can be stated as follows.
1) Denote by V a finite-dimensional complex vector space, and by $S_m V$ its m-th symmetric power. Then the GL(V)-module $S_n(S_m V)$ contains the GL(V)-module $S_m(S_n V)$ for $n > m$.
2) For any (decreasing) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, denote by $S_\lambda V$ the associated simple, polynomial GL(V)-module. Then the multiplicity of $S_{(\lambda_1+np,\lambda_2,\lambda_3,\ldots)}V$ in the GL(V)-module $S_n(S_{m+p}V)$ is an increasing function of p. We show that Foulkes' first conjecture holds for n large enough with respect to m (Corollary 1.3). Moreover, we state and prove two broad generalizations of Foulkes' second conjecture. They hold in the framework of representations of connected reductive groups, and they lead e.g. to a general analog of Hermite's reciprocity law (Corollary 1 in 3.3).

Foulkes Conjecture thrue for $a \ll b$

Michel Brion

Theorem

Functorial Formulation

$$S^a \circ S^b \twoheadleftarrow S^b \circ S^a$$

$S^a(V)$ symmetric power

Foulkes Conjecture thrue for $a \ll b$

$\overline{\mu}$: REMOVE LARGEST PART FROM μ

$$\overline{\Delta_\mu} := \Delta_{\overline{\mu}}$$

$$\overline{43221} = 3221$$

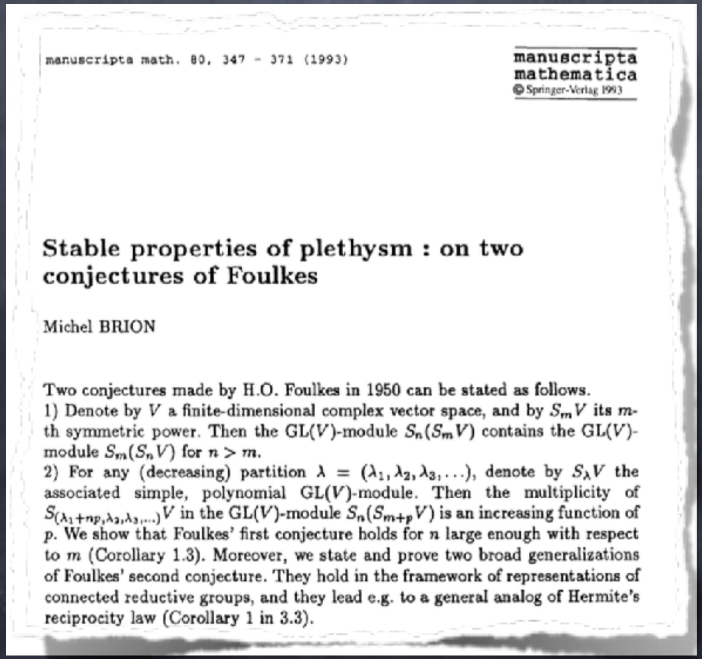
Conjecture 4

$$\overline{\mathcal{F}_{a,b+1}} - \overline{\mathcal{F}_{a,b}} \text{ is Schur positive}$$

and coefficients stabilize

Theorem

Michel Brion



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Stable properties of plethysm : on two conjectures of Foulkes

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$g = 0$

THEOREM $q=1$

$$\mathcal{F}_{a,b+1}(x,1) = e_1^a \mathcal{F}_{a,b}(x,1) + \Delta_{a,b}$$

WHERE $\Delta_{a,b}$ IS SCHUR POSITIVE.

THIS IMPLIES STABILITY AT $q=1$.

Questions/Problems

- Can we extend Brion's approach?
- $s_r \circ h_\mu \circ s_d - s_a \circ h_\mu \circ h_b$ Schur positive for all μ with small parts?

...

FIN

BON 60^{e} JEAN-YVES