Walks in the quadrant: differential algebraicity

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with

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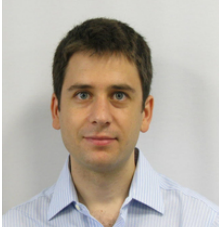


Counting quadrant walks... at the séminaire lotharingien

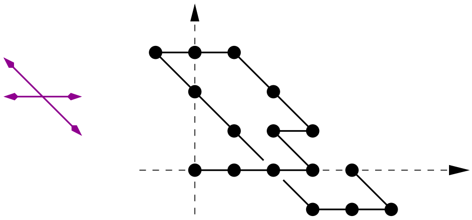
SLC 74, March 2015, Ellwangen: Three lectures by Alin Bostan
"Computer Algebra for Lattice Path Combinatorics"

SLC 77, September 2016, Strobl: Three lectures by Kilian Raschel
"Analytic and Probabilistic Tools for Lattice Path Enumeration"

Counting quadrant walks

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^2$ (set of steps) and $p_0 \in \mathbb{N}^2$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$



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- What is the number $q(n)$ of n-step walks starting at p_0 and contained in $\mathbb{N}^2$?
- For $(i,j) \in \mathbb{N}^2$, what is the number $q(i,j;n)$ of such walks that end at (i,j)?

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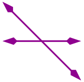

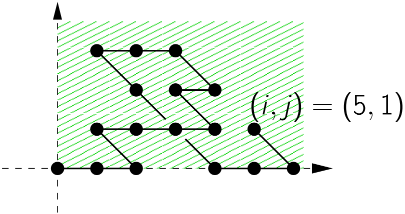

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The associated generating function:

$$Q(x, y; t) = \sum_{n \geq 0} \sum_{i,j \geq 0} q(i, j; n) x^i y^j t^n.$$

What is the nature of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0$$

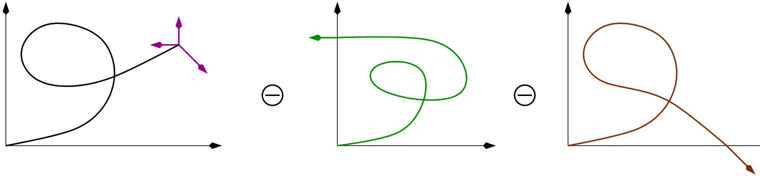
Multi-variate series: one DE per variable

1. Write a functional equation

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$

$$Q(x,y;t) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$

with $\bar{x} = 1/x$ and $\bar{y} = 1/y$.



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- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation
- The equation is linear, with two catalytic variables x and y (tautological at $x = 0$ or $y = 0$) [Zeilberger 00]

Equations with one catalytic variable are much easier!

Theorem [mbm-Jehanne 06]

Let $P(t, y, S(y; t), A_1(t), \dots, A_k(t))$ be a polynomial equation in one catalytic variable y that defines uniquely $S(y; t), A_1(t), \dots, A_k(t)$ as formal power series. Then each of this series is algebraic.

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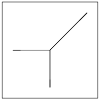
Example: for $S(y;t) = Q(0, y; t)$,

$$\frac{t}{y^2} - \frac{1}{y} - ty = t\left(tyS(y;t) + \frac{1}{y}\right)^2 - \left(tyS(y;t) + \frac{1}{y}\right) - 2t^2 S(0;t).$$



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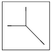
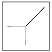
⇒ A special case of an Artin approximation theorem with "nested" conditions [Popescu 86]

Equations with two catalytic variables are harder...



D-finite transcendental

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)xyA(x,y) = xy - tyA(0,y) - tx^2A(x,0)$$



Algebraic

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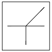

Not D-finite

$$(1 - t(x + \bar{x} + \bar{y} + xy))xyA(x,y) = xy - tyA(0,y) - txA(x,0)$$

But why?

2. The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$P(x, y) = \bar{x} + y + x\bar{y}$$

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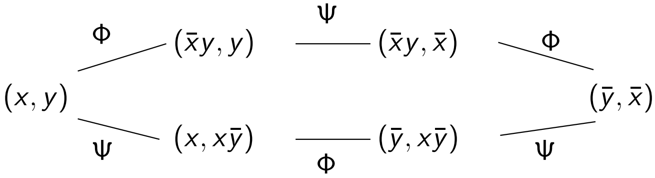
They are involutions, and generate a finite dihedral group G:



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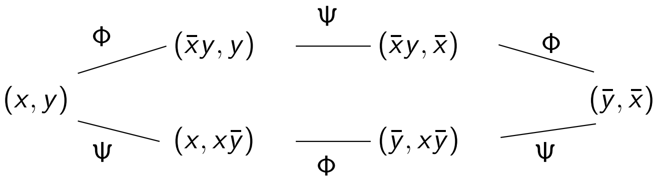


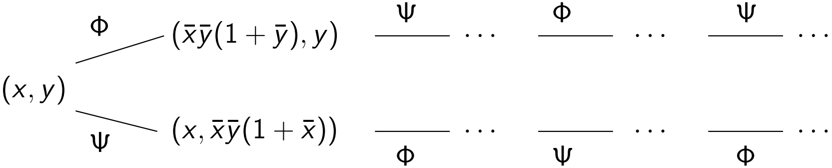
Remark. G can be defined for any quadrant model *with small steps*

The group is not always finite

- If $\mathcal{S} = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $P(x, y) = \bar{x}(1 + \bar{y}) + \bar{y} + xy$ and

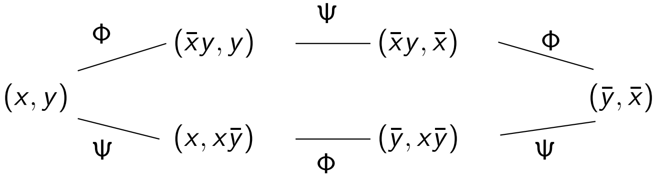
$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + \bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1 + \bar{x}))$$

generate an infinite group:



3. When G is finite: the orbit sum

Example. If $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, the orbit of (x, y) is



and the (alternating) orbit sum is

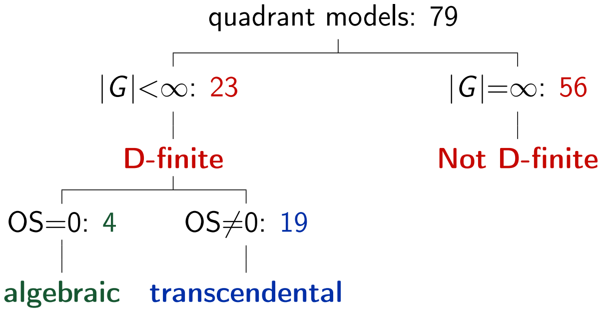
$$OS = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

Classification of quadrant walks with small steps

Theorem

The series $Q(x, y; t)$ is D-finite iff the group G is finite.
It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10] — D-finite
[Kurkova-Raschel 12] — non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13] — singular non-D-finite



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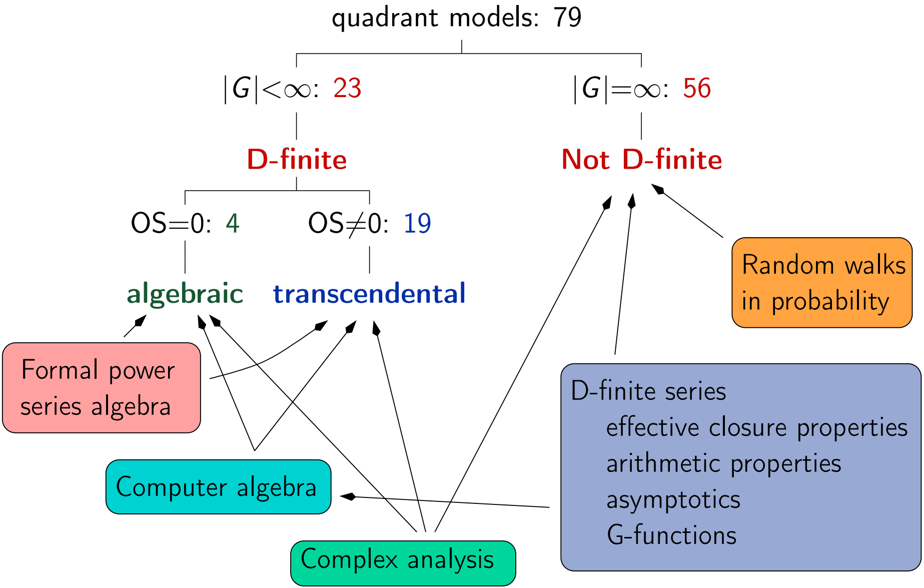
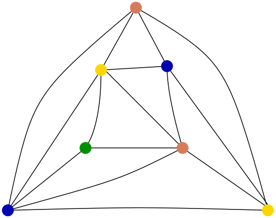


An old equation [Tutte 73]

- Properly coloured triangulations (q colours):

$$T(x,y;t) \equiv T(x,y) = x(q-1) + xyt\,T(x,y)T(1,y) + xt\,\frac{T(x,y)-T(x,0)}{y} - x^2yt\,\frac{T(x,y)-T(1,y)}{x-1}$$



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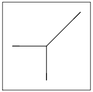
Isn't this reminiscent of quadrant equations?

$$Q(x,y;t) \equiv Q(x,y) = 1 + txy\,Q(x,y) - t\,\frac{Q(x,y)-Q(0,y)}{x} - t\,\frac{Q(x,y)-Q(x,0)}{y}$$



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Theorem [Tutte 73-84]

- For $q = 4\cos^2\frac{\pi}{m}$, $q \neq 0, 4$, the series $T(1,y)$ satisfies an equation with one catalytic variable y.

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• For any q, the generating function of properly q-coloured planar triangulations is differentially algebraic:

$$2(1-q)w + (w + 10H - 6wH')H'' + (4-q)(20H - 18wH' + 9w^2H'') = 0$$

with $H(w) = wT(1, 0; \sqrt{w})$.

In this talk

I. Adapt Tutte's method to quadrant walks: new and uniform proofs of algebraicity.

II. Extension to an analytic context: some walks with an infinite group (hence not D-finite) are still D-algebraic.

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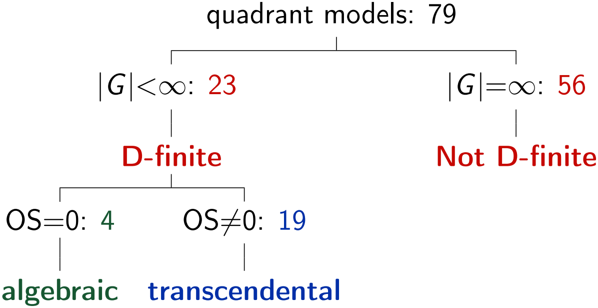

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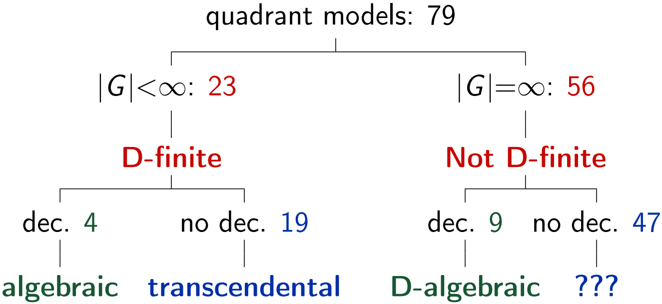

I. New proofs for algebraic models

[In the world of formal power series]

Kreweras' algebraic model

- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):

$$\begin{aligned}\big(1 - t(\bar{x} + \bar{y} + xy)\big) xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ &= xy - R(x) - S(y)\end{aligned}$$

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$$= xy - R(x) - S(y)$$

- If we take $x = t + ut^2$, both roots of the kernel

$$Y_{0,1} = \frac{x - t \pm \sqrt{(x-t)^2 - 4t^2x^3}}{2tx^2}$$

are series in t with rational coefficients in u, and can be legally substituted for y in $Q(x,y)$.

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$$xY_0 = R(x) + S(Y_0), \qquad xY_1 = R(x) + S(Y_1),$$

so that

$$\boxed{S(Y_0) - S(Y_1) = xY_0 - xY_1.}$$

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- Are there rational solutions to this equation?

Decoupling functions

Def. A rational function $D(y;t) \equiv D(y)$ is a decoupling function if, for $Y_{0,1}$ the roots of the kernel,

$$D(Y_0) - D(Y_1) = xY_0 - xY_1.$$

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Example: For Kreweras' model, $D(y) = -1/y$ is a decoupling function.

Proof:

$$\frac{1}{t} = P(x, Y_i) = \frac{1}{x} + \frac{1}{Y_0} + xY_0 = \frac{1}{x} + \frac{1}{Y_1} + xY_1$$

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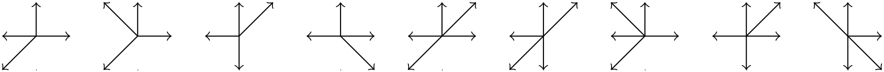
Theorem [Bernardi-mbm-Raschel]

- A quadrant model with finite group admits a decoupling function if and only if its orbit sum is zero (exactly 4 models).
- Exactly 9 quadrant models with an infinite group admit a decoupling function.



Back to Kreweras' model

- The equation

$$S(Y_0) - S(Y_1) = xY_0 - xY_1,$$

with $S(y) = tyQ(0, y)$, now reads

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Def. A rational function $I(y; t) \equiv I(y)$ is an invariant if, the roots Y_0, Y_1 of the kernel satisfy

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Theorem [Bernardi-mbm-Raschel]

A quadrant model admits a rational invariant if and only if the associated group is finite.

Back to Kreweras' model: combining decoupling functions and invariants

We have

$$S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1) \qquad \text{and} \qquad I(Y_0) = I(Y_1)$$

with

$$S(y) - D(y) = tyQ(0,y) + \frac{1}{y} \qquad \text{and} \qquad I(y) = \frac{t}{y^2} - \frac{1}{y} - ty.$$

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There are few invariants: $I(y)$ must be a polynomial in $S(y) - D(y)$ whose coefficients are series in t.

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$$\frac{t}{y^2} - \frac{1}{y} - ty = t\left(tyQ(0,y) + \frac{1}{y}\right)^2 - \left(tyQ(0,y) + \frac{1}{y}\right) + c \qquad .$$

Expanding at $y = 0$ gives the value of c.

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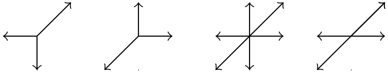
Algebraic models: a uniform approach

All models with a finite group and a zero orbit sum have a rational invariant and a decoupling function $\Rightarrow$ uniform solution via the solution of an equation with one catalytic variable



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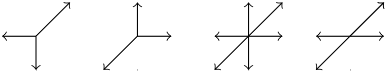
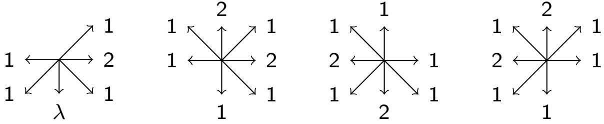
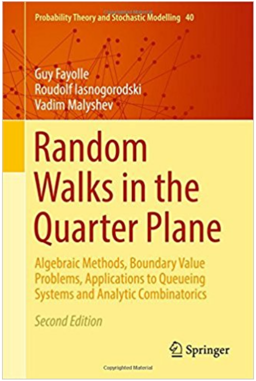


This applies as well to weighted algebraic models [Kauers, Yatchak 14(a)]:



II. Infinite groups: some differentially algebraic models

[An excursion in the world of analytic functions]

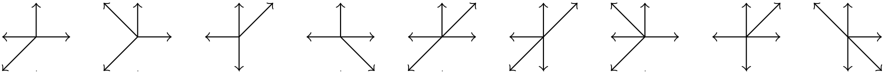


Fayolle, Iasnogorodski, Malyshev [1999]

The role of decoupling functions

Theorem [Bernardi-mbm-Raschel]

For the 9 models with an infinite group and a decoupling function, the series $Q(x, y; t)$ is D-algebraic.
That is, it satisfies a DE in t (and a DE in x, and a DE in y) with polynomial (or even constant) coefficients.



A weaker (and analytic) notion of invariants

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- meromorphicity condition in a domain

Can we find weak invariants?

Theorem [Fayolle et al. 99, Raschel 12]

For each non-singular model, there exists an (explicit) weak invariant of the form

$$I(y;t) = \wp(\mathcal{R}(y;t), \omega_1(t), \omega_3(t))$$

where

- $\wp$ is Weierstrass elliptic function
- its periods ω_1 and ω_3 are elliptic integrals
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$$\omega_1 = i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-\delta(x)}}, \quad \omega_3 = \int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{\delta(x)}}.$$

$$\mathcal{R}(y;t) = \int_{f(y_2)}^{f(y)} \frac{\mathrm{d}z}{\sqrt{4z^3 - g_2 z - g_3}}$$

g_2, g_3 polynomials in t, $f(y)$ rational in y and algebraic in t.

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Proposition [Bernardi-mbm-Raschel]

$I(y;t)$ is D-algebraic in y and t.

Combining decoupling functions and invariants

For a model with decoupling function $D(y)$ we have, for $x \in (x_1, x_2)$:

$$S(Y_0) - D(Y_0) = S(Y_1) - D(Y_1) \qquad \text{and} \qquad I(Y_0) = I(Y_1)$$

where $S(y) = K(0, y)Q(0, y)$ and $I(y)$ is the weak invariant.

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The invariant lemma [Litvinchuk 00]

There are few invariants: $S(y) - D(y)$ must be a rational function in $I(y)$. The value of this rational function is found by looking at the poles and zeroes of $S(y) - D(y)$.

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Example:  is decoupled with $D(y) = -1/y$ and

$$S(y) + \frac{1}{y} = t(1+y)Q(0,y) + \frac{1}{y} = \frac{I'(0)}{I(y) - I(0)} - \frac{I'(0)}{I(-1) - I(0)} - 1$$

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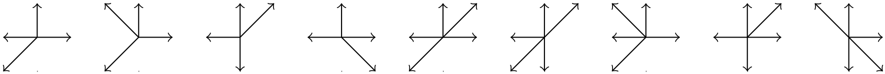
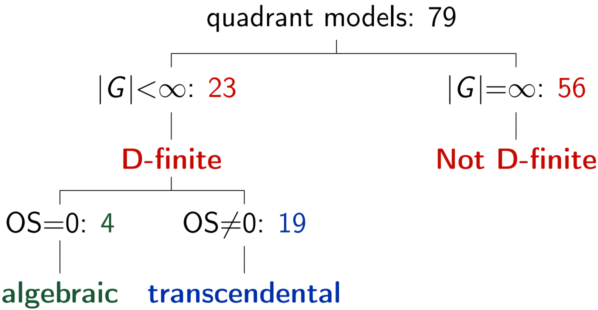
Corollary

For the 9 models with an infinite group and a decoupling function, the series $Q(x, y; t)$ is D-algebraic.



Conclusion


quadrant models: 79
$|G|<\infty$: 23
$|G|=\infty$: 56
D-finite
Not D-finite
OS=0: 4
OS≠0: 19
algebraic
transcendental

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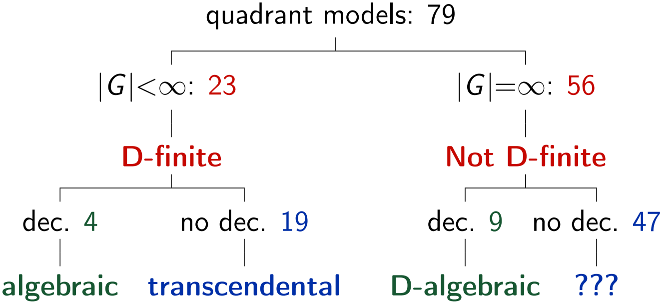

To do:

- find explicit DEs (done for y)
- Nature of $Q(x, y; t)$ when no decoupling function exists? [Dreyfus, Hardouin, Roques, Singer 17(a)]