ν-Tamari lattices via subword complexes

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(joint with Arnau Padrol and Camilo Sarmiento)

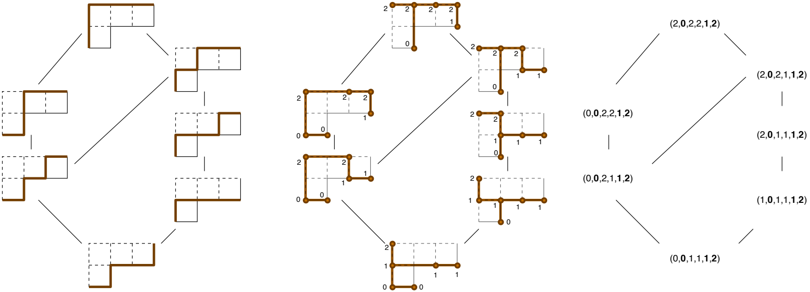
universität wien

The 78th Séminaire Lotharingien de Combinatoire
Ottrott, March 28, 2016

In this talk

Theorem

The ν-Tamari lattice is the dual of a well chosen subword complex.



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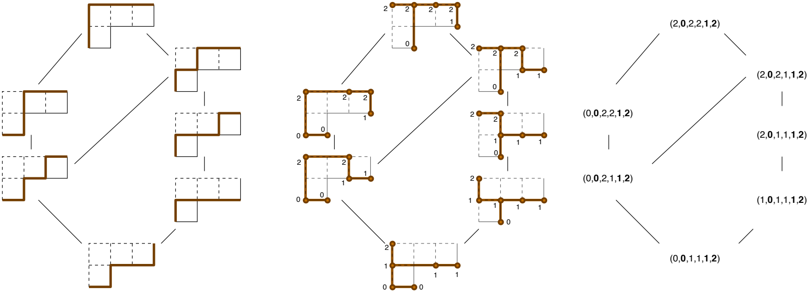
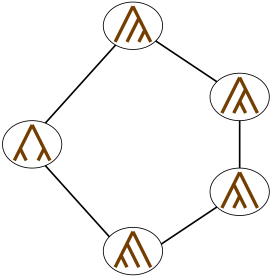
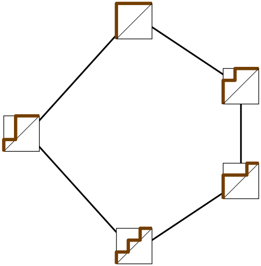


The picture actually contains three theorems and one corollary.
Please remember the picture!

Tamari lattices

The Tamari-lattice: partial order on Catalan objects.





Tamari. Monoïdes préordonnés et chaînes de Malcev. Doctoral Thesis, Paris 1951.
Associahedra, Tamari Lattices and Related Structures. Birkhäuser/Springer, 2012.

Tamari lattices

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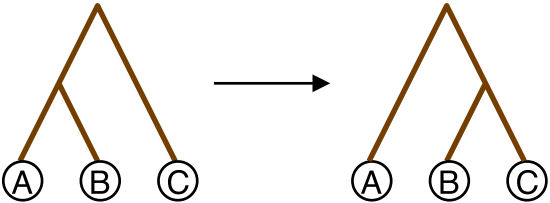

Rotation on binary trees

Tamari lattices

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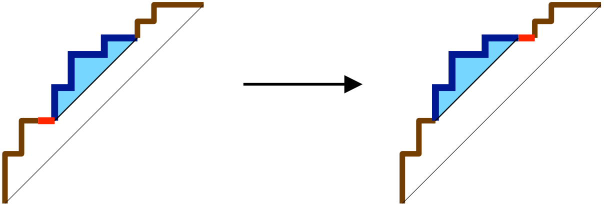
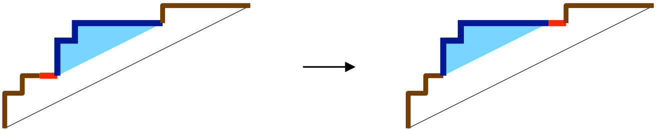


Interchanging operation on Dyck paths

m-Tamari lattices

Motivated by trivariate diagonal harmonics, F. Bergeron Introduced the m-Tamari lattice on Fuss-Catalan paths.



F. Bergeron–Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. J. Comb 3(3), 2012.

m-Tamari lattices: nice enumerative properties

- Number of elements: Fuss Catalan number $\frac{1}{mn+1}\binom{(m+1)n}{n}$
- Number of intervals: $\frac{m+1}{n(mn+1)}\binom{(m+1)^2 n+m}{n-1}$

  Chapoton. Sur le nombre d'intervalles dans les treillis de Tamari. Sém. Lothar. Combin., 55, 2005/07. (m=1)

  F. Bergeron–Préville-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. J. Comb 3(3), 2012. (conjectured)

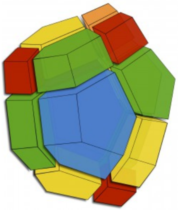
  Bousquet-Mélou–Fusy–Préville-Ratelle. The number of intervals in the m-Tamari lattices. Electron. J. Combin., 18(2), 2011. (proof)
- Number of "decorated" intervals: $(m+1)^n(mn+1)^{n-2}$

  Bousquet-Mélou–Chapuy–Préville-Ratelle. The representation of the symmetric group on m-Tamari intervals. Adv. Math., 2013.

Conjecture (F. Bergeron (Haiman for $m=1$))

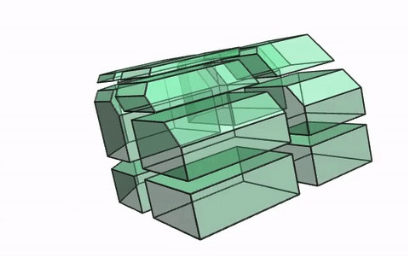
The number of intervals is conjecturally interpreted as the dimension of the alternating component of a space in trivariate diagonal harmonics. Decorated intervals correspond to the entire space.

m-Tamari lattices: nice geometry



The 2-Tamari lattice for $n = 4$

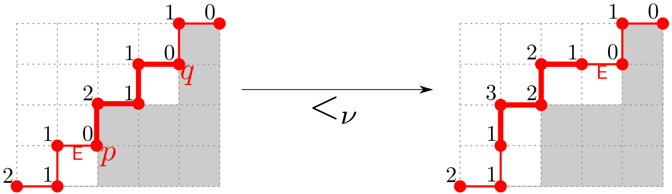
C.–Padrol–Sarmiento, 2016: The Hasse diagram of m-Tamari lattices are the edge graphs of (tropical) polytopal subdivisions of associahedra.



ν-Tamari lattices

Préville-Ratelle–Viennot:
Introduced the ν-Tamari lattice on lattice paths weakly above ν.

Covering relation:



Theorem (Préville-Ratelle–Viennot)

This partial order defines a lattice structure on ν-Dyck paths.

Préville-Ratelle–Viennot. An extension of Tamari lattices. To appear in Trans. AMS.

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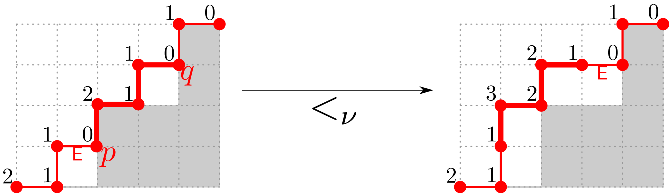

They also have nice enumerative and geometric properties.

Fang–Préville-Ratelle. The enumeration of generalized Tamari intervals.
European Journal of Combinatorics 61, 2017.

C.–Padrol–Sarmiento. Geometry of ν-Tamari lattices in types A and B.
arXiv:1611.09794, 2016.

First theorem

Theorem 1

The Hasse diagram of the ν-Tamari lattice is the facet adjacency graph of a well chosen subword complex .

This generalizes a known result by Woo (2004), Pilaud–Pocchiola (2010), Stump (2010), and Stump-Serrano (2010) in the classical case.

Subword complexes

$W = \mathfrak{S}_{n+1}$ group of permutations of $[n+1]$

$S = \{s_1, \ldots, s_n\}$ the set of simple generators $s_i = (i \;\; i+1)$

$Q = (q_1, \ldots, q_m)$ a word in S

$\pi \in W$

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$\pi \in W$

Definition (Knutson–Miller, 2004)

The subword complex $\Delta(Q, \pi)$ is the simplicial complex whose

faces $\longleftrightarrow$ subwords P of Q such that $Q \setminus P$ contains a reduced expression of π

Knutson–Miller. Gröbner geometry of Schubert polynomials. Ann. Math., 161(3), '05
Knutson–Miller. Subword complexes in Coxeter groups. Adv. Math., 184(1), '04

Subword complexes - Example modify s_3

In type A_2:

$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

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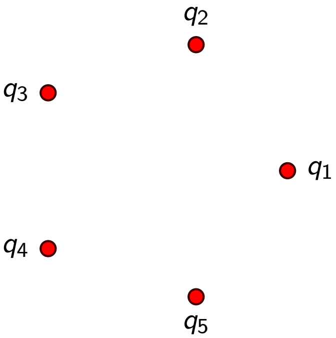

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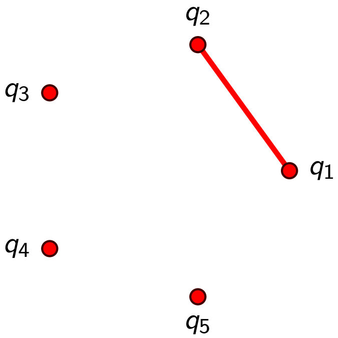

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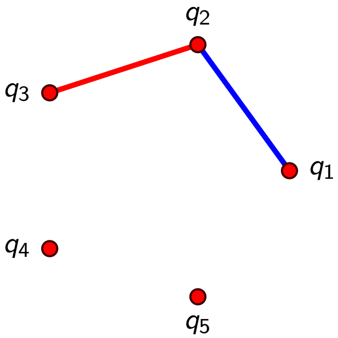

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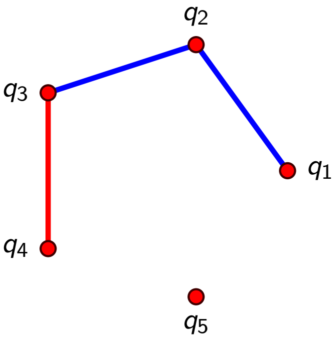

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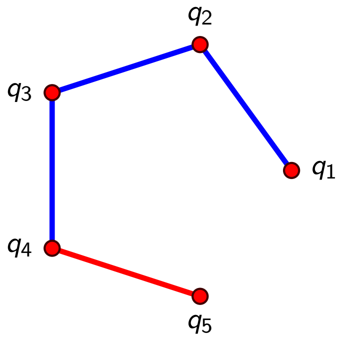

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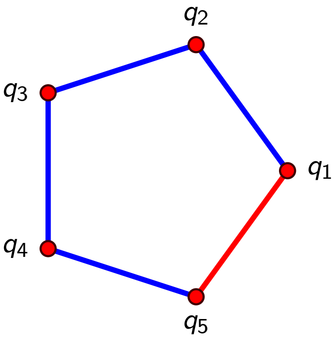

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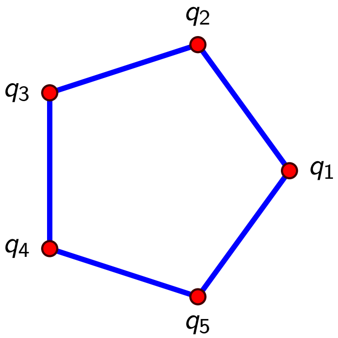
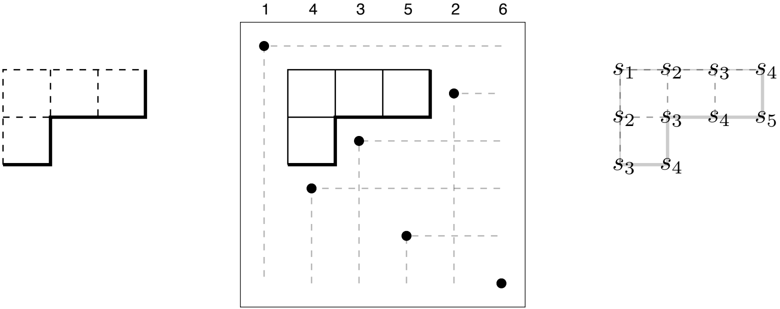


The subword complex result

Theorem 1

The Hasse diagram of the ν-Tamari lattice is the facet adjacency graph of a well chosen subword complex $\Delta(Q_\nu, \pi_\nu)$.



$Q_\nu = (s_3, s_2, s_1, s_4, s_3, s_2, s_4, s_3, s_5, s_4)$
$\pi_\nu = [1, 4, 3, 5, 2, 6]$

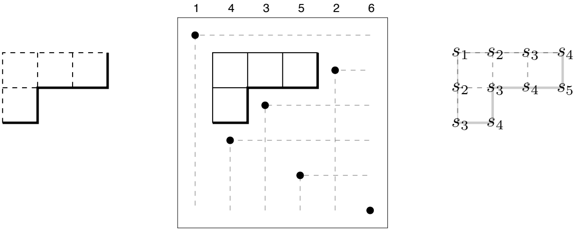
The subword complex result

These objects keep showing up in independent places:

Serrano–Stump. Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials. Electron. J. Combin., 19(1), 2012.

Mészáros. Root polytopes, triangulations, and the subdivision algebra. I. Trans. Amer. Math. Soc., 363(8), 2011.

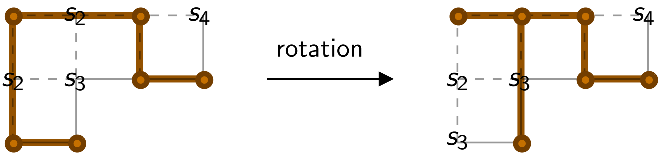
Escobar–Mészáros. Subword complexes via triangulations of root polytopes. arXiv:1502.03997.



They are special but still some what mysterious.

Facets and ν-trees

The facets of $\Delta(Q_\nu, \pi_\nu)$ are given by ν-trees.
Two facets are adjacent $\leftrightarrow$ the trees are related by rotation.



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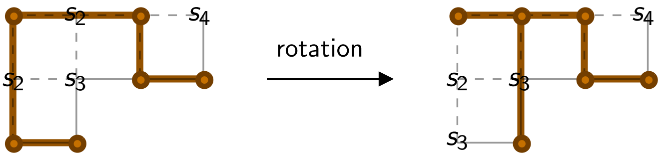


$s_2 s_3 s_2 s_4 = [1, 4, 3, 5, 2, 6]$ $\qquad$ $s_3 s_2 s_3 s_4 = [1, 4, 3, 5, 2, 6]$

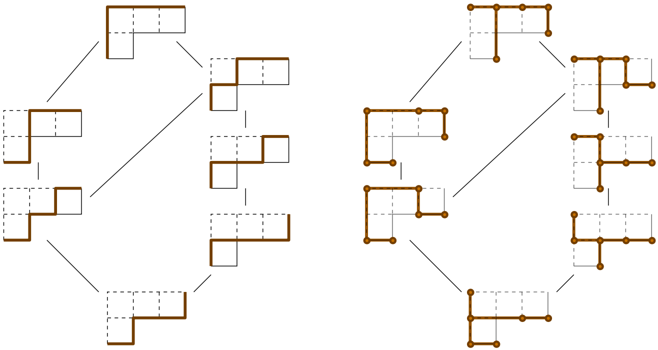
ν-tree:
(Serrano–Stump) Maximal sets of lattice points above ν avoiding north-east increasing chains p, q such that $p \lrcorner q$ is above ν.
(This talk) some "maximal" binary trees fitting above ν.

The rotation lattice of ν-trees

Theorem 1 follows from:

Theorem 2

The ν-Tamari lattice is isomorphic to the rotation lattice on ν-trees.



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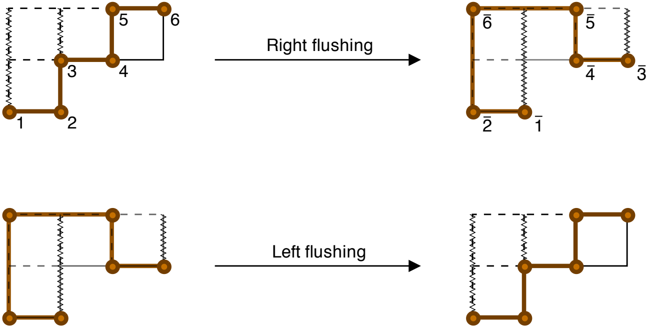

The lattice of ν-bracket vectors

The meet and join: very simple on ν-trees.

The lattice of ν-bracket vectors

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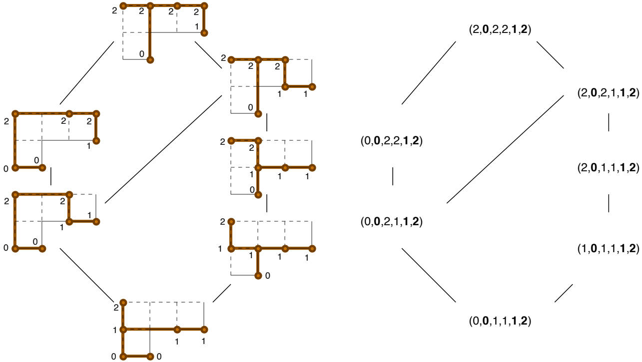
Theorem 3

The ν-Tamari lattice is isomorphic to the lattice of ν-bracket vectors under componentwise order.



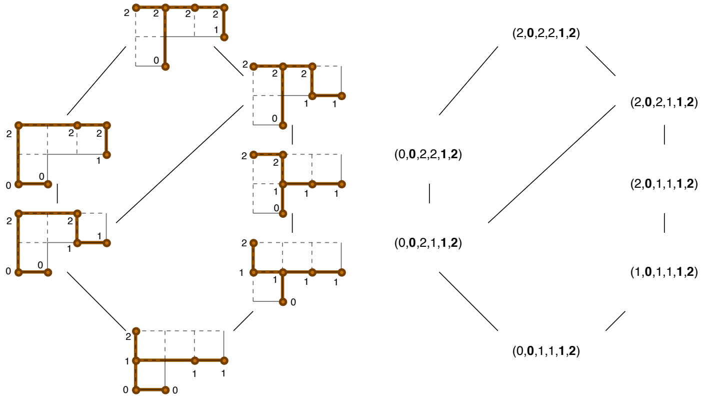
$b(T)$ = read y-coordinates of the nodes in in-order.

The lattice of ν-bracket vectors

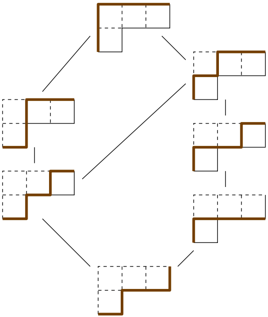
ν-bracket vectors are easily characterized.
Their meet is obtained by taking componentwise minimum.

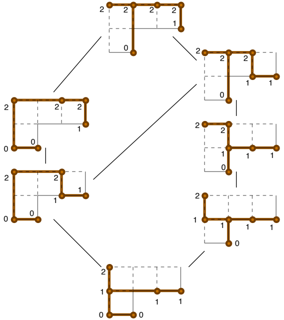
Corollary

Simple proof of the lattice property.

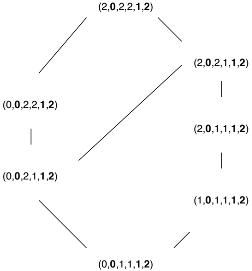


In summary





Thm 1 & Thm 2



Thm 3 & Cor

Multi ν-Tamari complexes

For $k \geq 1$, define the (k, ν)-Tamari complex

faces $\leftrightarrow$ sets of points above ν avoiding $(k+1)$-north-east incr. chains.

- $\nu = (NE)^n$: simplicial multiassociahedron $\Delta_{n+2,k}$. Conjectured to be realizable as a polytope (Jonsson 2004).
- $k = 1$, ν without consecutive north steps: facet adjacency graph $=$ edge graph of a polytopal subdivision of an associahedron.

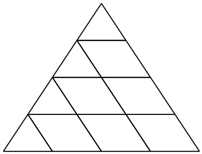
Question

Is the facet adjacency graph of the (k, ν)-Tamari complex the edge graph of a polytopal subdivision of a multiassociahedron?

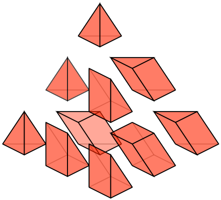
Multi ν-Tamari complexes

Proposition

Let $m \geq k$ and $\nu = (NE^m)^{k+1}$. The facet adjacency graph $G_{k,\nu}$ of the Fuss-Catalan (k,ν)-Tamari complex is the edge graph of a polytopal subdivision of the multi-associahedron $\Delta_{2k+2,k}$.



$k = 2$ and $\nu = (NE^5)^3$



$k = 3$ and $\nu = (NE^5)^4$

$\Delta_{2k+2,k}$: a k-dimensional simplex
Subdivision: *staircase subdivision* of its $(m - k + 1)$ dilation.

My birthday present!

Is this true in general?

Merci!