A superplancherel measure associated to set partitions and its limit

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Call such elements *irreducible characters* and the basis $Irr(G)$.

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Measure on $Irr(G)$

$$\mathrm{Pl}_G(\chi) = \frac{\chi(1)^2}{|G|}$$

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Classifying the irreducible representations of $U_n(\mathbb{F}_q)$ is a “wild” problem

OK, no irreducible characters. Now what?

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- thus, ψ are class functions (constant on conjugacy classes);
- $Irr(G)$ is a basis for the algebra of class functions, so each ψ must be a linear combination of irreducible characters;

Supercharacter theory (Diaconis and Isaacs)

A *supercharacter theory* is a pair $(\mathcal{K}, \mathcal{H})$ where $\mathcal{K}$ is a set partition of G and $\mathcal{H}$ is an orthogonal set of characters such that

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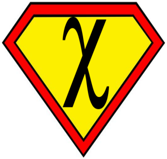
Given a suitable $\mathcal{K}$ then $\mathcal{H}$ is fixed (and viceversa).



(Bergeron and Thiem) A supercharacter theory for $U_n(\mathbb{F}_q)$

$$h = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 4 & 0 & 3 \\ & 1 & 5 & 2 & 0 & 3 & 6 & 0 \\ & & 1 & 0 & 0 & 0 & 4 & 3 \\ & & & 1 & 0 & 0 & 3 & 0 \\ & & & & 1 & 0 & 4 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{bmatrix},$$

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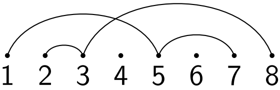

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4. the algebra of superclass functions is isomorphic to the algebra of symmetric functions in noncommutative variables;
5. Nice decomposition of the supercharacter table (Bergeron and Thiem).

Set partitions notation

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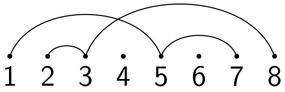

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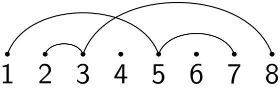

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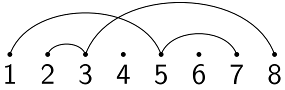

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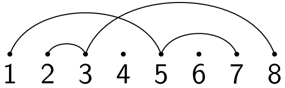

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- $crs(\pi) = \sharp\text{crossings of } \pi = 1$;

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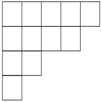
3. let $n \to \infty$;

First step to look for a limit



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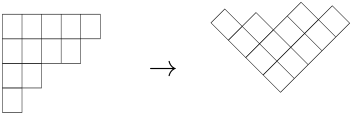

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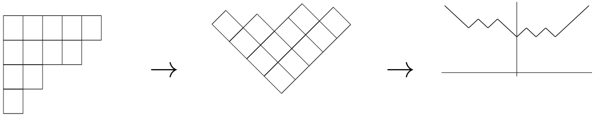

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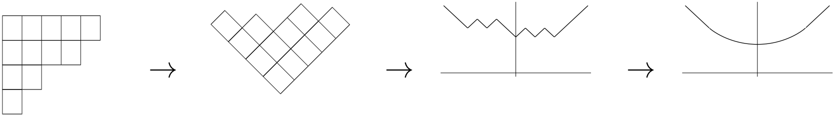

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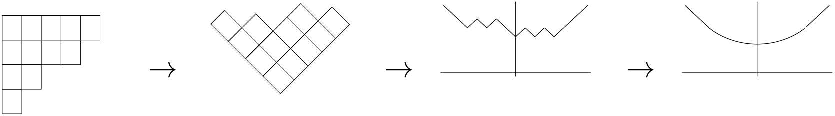

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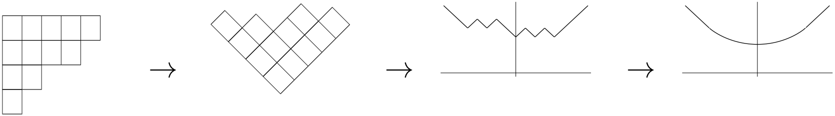

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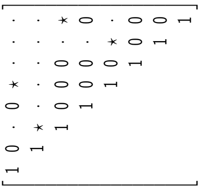

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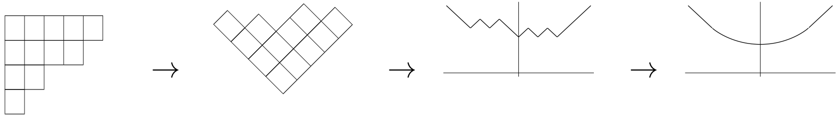

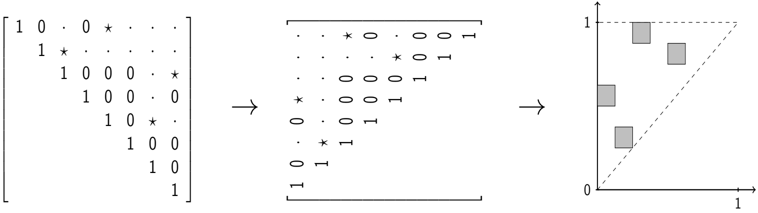

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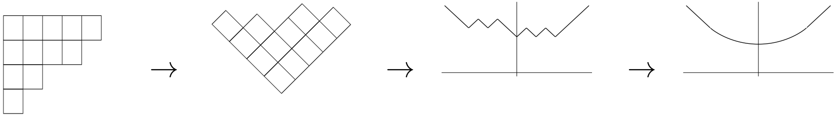
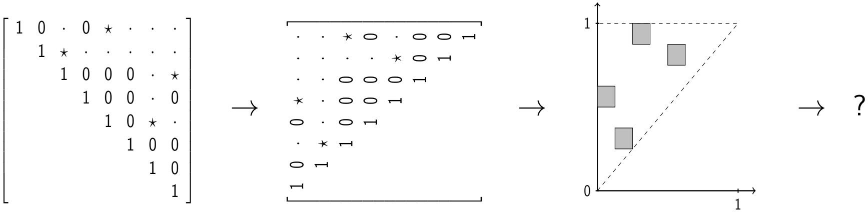
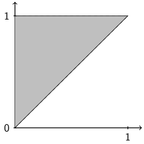




Our setting

$\Delta =$ 

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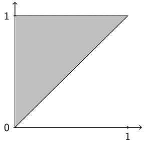
$\Delta =$ 

$$\Gamma = \left\{ \begin{array}{c} \text{measures } \mu \text{ on } \Delta \text{ s.t.} \\ \int_\Delta \mu \leq 1 \text{ (\textit{subprobability})} \\ \mu \text{ has \textit{sub-uniform marginals}} \end{array} \right\}$$

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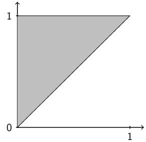
$\Delta =$ 

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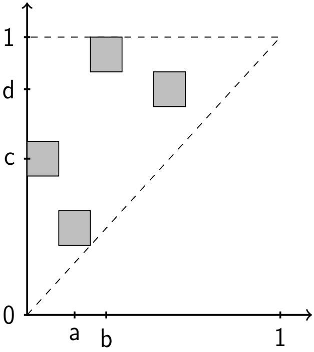

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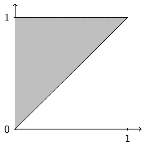
$\Delta =$ 

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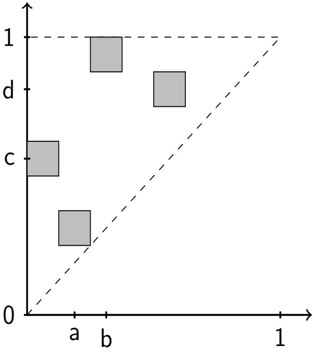

$$\int_{x=0}^{x=1} \int_{y=c}^{y=d} d\mu \leq d - c \quad \text{SUB-UNIFORM}$$

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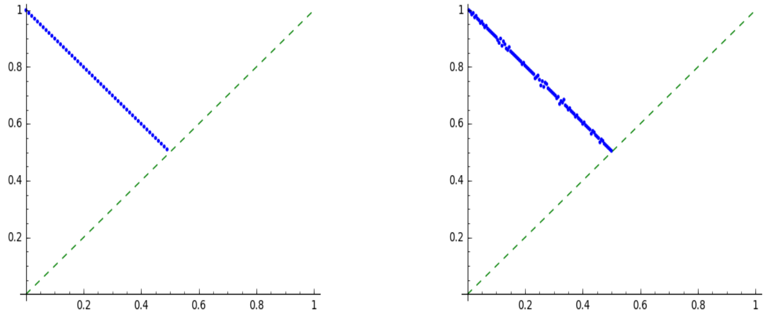

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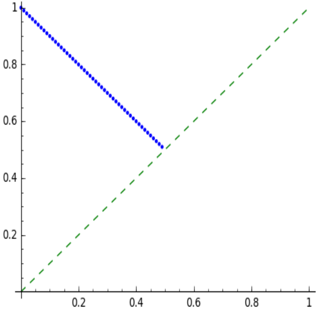

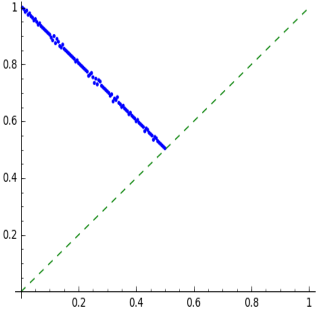

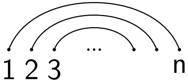

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$$\dim(\pi) \to I_{\dim}(\mu_\pi) := \int_\Delta (y-x)\, d\mu$$

$$\mathsf{SPI}_n(\chi^\pi) = \frac{1}{q^{\frac{n(n-1)}{2}}} \frac{q^{2\dim(\pi)-2a(\pi)}}{(q-1)^{a(\pi)} q^{crs(\pi)}} =$$

$$\exp\left(-n^2 \log q \left(\frac{1}{2} - 2I_{\dim}(\mu_\pi) + I_{crs}(\mu_\pi)\right) + O(n)\right)$$

$$\dim(\pi) \to I_{\dim}(\mu_\pi) := \int_\Delta (y-x)\, d\mu$$

$$crs(\pi) \to \int_{\Delta^2} \mathbb{1}[x_1 < x_2 < y_1 < y_2]\, d\mu_\pi(x_1, y_1)\, d\mu_\pi(x_2, y_2)$$

$$\mathrm{SPl}_n(\chi^\pi) = \frac{1}{q^{\frac{n(n-1)}{2}}} \frac{q^{2\dim(\pi)-2a(\pi)}}{(q-1)^{a(\pi)} q^{crs(\pi)}} =$$

$$\exp\left(-n^2 \log q \left(\frac{1}{2} - 2I_{\dim}(\mu_\pi) + I_{crs}(\mu_\pi)\right) + O(n)\right)$$

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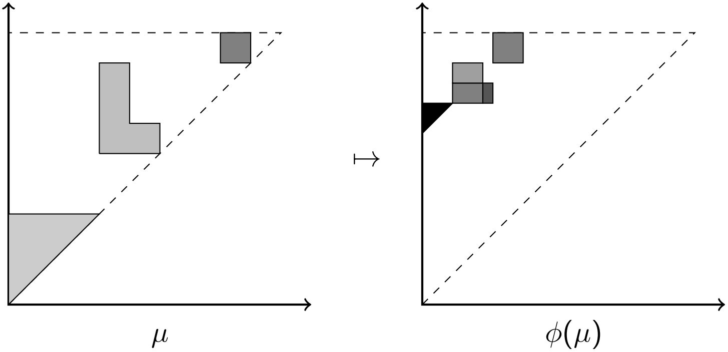
$$\mathcal{H}(\mu) := \frac{1}{2} - 2I_{\dim}(\mu) + I_{crs}(\mu)$$

Playing with $I_{\dim}(\mu) = \int_\Delta (y - x)\, d\mu$



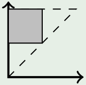
Playing with crossings

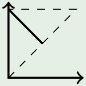
$$I_{crs}(\mu_\pi) = \int_{\Delta^2} \mathbb{1}[x_1 < x_2 < y_1 < y_2] \, d\mu_\pi(x_1, y_1) \, d\mu_\pi(x_2, y_2)$$

Playing with crossings

$$I_{crs}(\mu_\pi) = \int_{\Delta^2} \mathbb{1}[x_1 < x_2 < y_1 < y_2]\, d\mu_\pi(x_1, y_1)\, d\mu_\pi(x_2, y_2)$$

Proposition

If μ has mass in  and μ has uniform marginals then

$I_{crs}(\mu) = 0 \Leftrightarrow \mu =$  $= \Omega$

Summary

μ has uniform marginals

$\mathcal{H}(\mu) = 0 \Leftrightarrow$

Summary

$$\mathcal{H}(\mu) = 0 \Leftrightarrow \begin{array}{l} \mu \text{ has uniform marginals} \\ \mu \text{ inside the top left square} \end{array}$$

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Summary

$$\mathcal{H}(\mu) = 0 \Leftrightarrow \begin{array}{l} \mu \text{ has uniform marginals} \\ \mu \text{ inside the top left square} \\ I_{crs}(\mu) = 0 \end{array} \Leftrightarrow \mu = \Omega$$

The last step (technical) is to prove that $\mu_{\pi^{(n)}} \to \Omega$ iff $\mathcal{H}(\mu_{\pi^{(n)}}) \to \mathcal{H}(\Omega) = 0$.

SUPERTHANK YOU