A bijection between EW tableaux and permutation tableaux

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joint work with Jason Smith and Einar Steingrímsson

SLC 78, Ottrott

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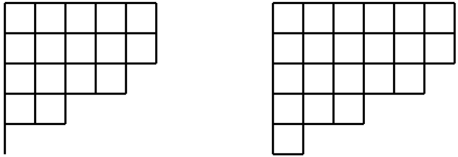
Ferrers diagram

Definition

A Ferrers diagram is a left-aligned collection of cells with a finite number of rows and columns such that the number of cells in each row is weakly decreasing.



(a) F (b) F'

F' is the Ferrers diagram F with an extra column on the left-hand side.

EW-tableaux

Definition (Ehrenborg, van Willigenburg 04)

An EW-*tableau* (EWT) $\mathcal{T}$ is a 0–1 filling of a Ferrers diagram that satisfies the following properties:

1. The top row of $\mathcal{T}$ has a 1 in every cell.
2. Every other row has at least one 0.
3. No four cells of $\mathcal{T}$ that form the corners of a rectangle have 0s in two diagonally opposite corners and 1s in the other two.

The *size* of a EW-tableau is one less than the sum of its number of rows and number of columns.

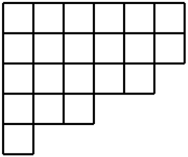
1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	0	
0	1	0			
0					

(c) an EWT

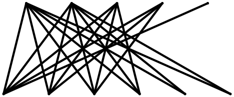
1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	1	
0	1	0			
0					

(d) not an EWT

EWTs and acyclic orientations



F

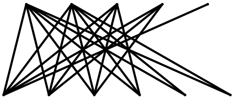


$G(F)$

EWTs and acyclic orientations

1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	0	
0	1	0			
0					

F



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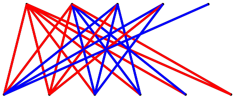
1	1	0	0	0	1
0	1	0	0	0	
0	1	0			
0					

F



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0	1	0	0	0	
0	1	0			
0					

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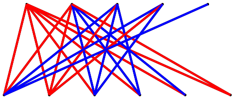
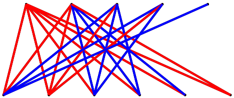


$G(F)$

$(0 = \uparrow = |, 1 = \downarrow = |)$

EWTs and acyclic orientations

1	1	1	1	1	1
1	1	0	0	0	1
0	1	0	0	0	
0	1	0			
0					



F $\qquad\qquad$ $G(F)$

$(0 = \uparrow = |, 1 = \downarrow = |)$

$EWT(F) \leftrightarrow$ {Ac. Or. of G(F) where top-left vertex = unique source}.

Permutation tableaux

Definition (Postnikov 06)

A *permutation tableau* (PT) $\mathcal{T}$ is a 0–1 filling of a Ferrers diagram, some of whose bottom-most rows may be empty, satisfying the following properties:

1. Every column of $\mathcal{T}$ has a 1 in some cell.
2. If a cell has a 1 above it in the same column and a 1 to its left in the same row then it has a 1.

The *size* of a permutation tableau is the sum of its number of rows and number of columns.

0	0	0	1	0
0	1	1	1	1
0	0	1	1	
1	1			

(e) a PT

0	0	0	1	0
0	1	1	1	1
0	0	1	1	
1	0			

(f) not a PT

The main result

Theorem (Ehrenborg, van Willigenburg 04; S., Smith, Steingrímsson ++)

Let F be a Ferrers diagram (possibly with some empty rows). Then the number of PTs of shape F and the number of EWTs of shape F' are the same.

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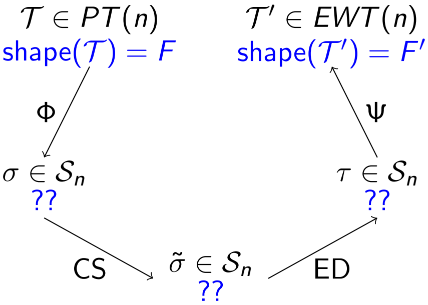
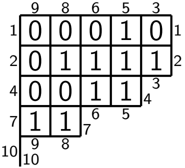
- Formulation in different but equivalent form by EW (04). Proof is recursive.
- A more bijective proof by Josuat-Vergès (10).
- We present a bijection through permutations.

Map of the proof



The map Φ from permutation tableaux to permutations



- Label the rows and columns of the permutation tableau;

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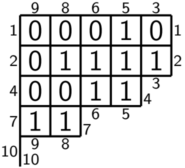

- Label the rows and columns of the permutation tableau;
- For each $1 \leq i \leq n$, construct the path from i to σ_i: enter the row, resp. column, labelled i from the left, resp. top; traverse cells with 0; turn $S \rightarrow E$ or $E \rightarrow S$ when cell is 1; σ_i is the label of the edge through which the path exits on the South-East border;

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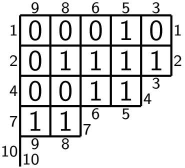

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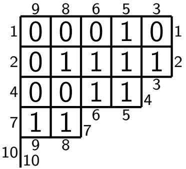
- We get a map $\Phi(\mathcal{T}) := \sigma$

σ_i	3	7	2	6	1	4	9	5	8	10
i	1	2	3	4	5	6	7	8	9	10

.

The map Φ from permutation tableaux to permutations



$\overset{\Phi}{\longmapsto} \sigma :$

σ_i	3	7	2	6	1	4	9	5	8	10
i	1	2	3	4	5	6	7	8	9	10

For $\sigma \in \mathcal{S}_n$, $\mathrm{w\_ex}(\sigma) := \{1 \leq i \leq n;\ i \leq \sigma_i\}$ (weak excedences).

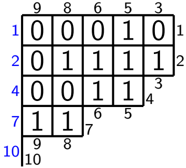
Theorem (Steingrímsson, Williams 07)

Let F be a Ferrers diagram of size n with row labels $RL(F)$, then

$$\Phi : \mathrm{PT}(F) \longrightarrow \{\sigma \in \mathcal{S}_n;\ \mathrm{w\_ex}(\sigma) = RL(F)\}$$

is a bijection.

The map Φ is a bijection



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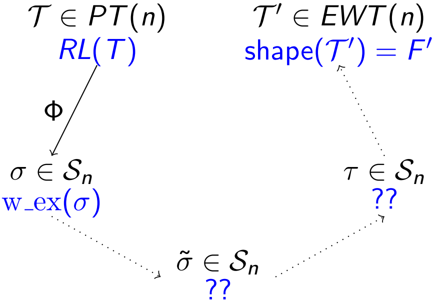
σ_i	3	7	2	6	1	4	9	5	8	10
i	1	2	3	4	5	6	7	8	9	10

$$\Phi : \mathrm{PT}(F) \longrightarrow \{\sigma \in \mathcal{S}_n ; \mathrm{w\_ex}(\sigma) = RL(F)\}$$

- $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is a bijection (can construct σ^{-1}).
- $\mathrm{w\_ex}(\sigma) = RL(F)$.
- Φ is a bijection. We can construct Φ^{-1}.

Progress of the proof



Cyclic Shift CS

For $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, define $\tilde{\sigma} := CS(\sigma) = \sigma_2 \cdots \sigma_n \sigma_1$.

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$$CS : \begin{array}{ccccccccccc} \sigma_i & 3 & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\big\downarrow \text{CS}$$

$$\begin{array}{ccccccccccc} \tilde{\sigma}_i & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 & 3 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

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$$\begin{array}{ccccccccccc} \tilde{\sigma}_i & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 & 3 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

For $\tilde{\sigma} \in \mathcal{S}_n$, $\mathrm{exc}(\tilde{\sigma}) := \{1 \leq i \leq n;\ i < \tilde{\sigma}_i\}$ ((strong) excedences).

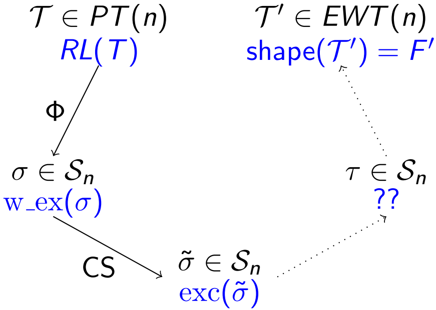
Proposition

For any $2 \leq a_1 < \cdots < a_k$,

$$\begin{array}{rl} CS\ : & \{\sigma \in \mathcal{S}_n;\ \mathrm{w\_ex}(\sigma) = \{1, a_1, \cdots, a_k\}\} \\ \longrightarrow & \{\tilde{\sigma} \in \mathcal{S}_n;\ \mathrm{exc}\,(\tilde{\sigma}) = \{a_1 - 1, \cdots, a_k - 1\}\} \end{array}$$

is a bijection.

Progress of the proof



The map ED : $\tilde{\sigma} \mapsto \tau$

Algorithm: inputs $\tilde{\sigma}$, outputs τ.

0. Initialise $\tau = \emptyset$.
1. $j = \min\{1 \leq i \leq n; i \notin \tau\}$. If $j = +\infty$ return τ. Else $j' = j$ and proceed to 2.
2. Find k s.t. $\tilde{\sigma}_k = j'$. $\tau \leftarrow \tau * k$. If $k \neq j$ then $j' = k$ and repeat 2. Else return to 1.

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Example: $\tilde{\sigma}$:

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

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Example: $\tilde{\sigma}$:

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
i	1	2	3	4	5	6	7	8	9	10

$$\tau = 4$$

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Example: $\tilde{\sigma}$:

$\tilde{\sigma}_i$	7	2	6	1	4	9	5	8	10	3
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$$\tau = 4\ 5$$

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$$\tau = 4\ 5\ 7$$

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$$\tau = 4\ 5\ 7\ 1$$

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$$\tau = 4\ 5\ 7\ 1\ 2$$

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$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

The map ED

$$\tilde{\sigma} : \begin{array}{ccccccccccc} \tilde{\sigma}_i & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 & 3 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

For $\tau \in \mathcal{S}_n$, $\mathrm{des\_bot}(\tau) := \{\tau_i;\ 2 \leq i \leq n \text{ and } \tau_i < \tau_{i-1}\}$.

Theorem (Folklore)

For any $A \subseteq \{1, \cdots, n-1\}$,

$$ED : \{\tilde{\sigma} \in \mathcal{S}_n;\ \mathrm{exc}(\tilde{\sigma}) = A\} \to \{\tau \in \mathcal{S}_n;\ \mathrm{des\_bot}(\tau) = A\}$$

is a bijection.

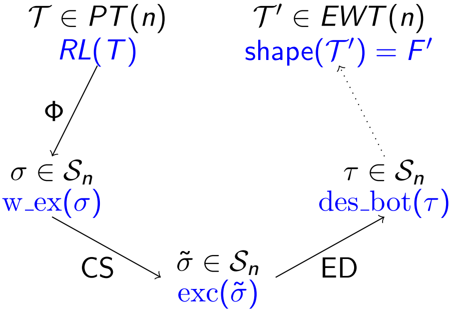
ED is a bijection : proof

$$\tilde{\sigma} : \begin{array}{ccccccccccc} \tilde{\sigma}_i & 7 & 2 & 6 & 1 & 4 & 9 & 5 & 8 & 10 & 3 \\ i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

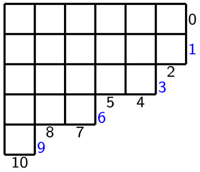
- ED : $\mathcal{S}_n \longrightarrow \mathcal{S}_n$ is a bijection.
- To see that $\mathrm{exc}(\tilde{\sigma}) = \mathrm{des\_bot}(\tau)$, notice that

$$\cdots \begin{array}{c} \tilde{\sigma}_i \\ i \end{array} \cdots \quad \longleftrightarrow \quad \cdots \tilde{\sigma}_i\, i \cdots = \tau$$


$\mathcal{T} \in PT(n)$
$RL(T)$
$\mathcal{T}' \in EWT(n)$
$\text{shape}(\mathcal{T}') = F'$
Φ
$\sigma \in \mathcal{S}_n$
$\text{w\_ex}(\sigma)$
$\tau \in \mathcal{S}_n$
$\text{des\_bot}(\tau)$
CS
$\tilde{\sigma} \in \mathcal{S}_n$
$\text{exc}(\tilde{\sigma})$
ED

$$\tau \ = \ 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

Label edges of a Ferrers diagram F' (with no empty rows), **starting at 0**: rows and columns labelled $0, \cdots, n$ if $n = \text{size}(F')$.



$$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8.$$

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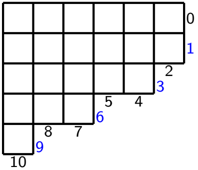

We have $RL'(F') = \mathrm{des\_bot}(\tau)$.

The map Ψ

$\tau =$ 4 5 7 1 2 10 9 6 3 8. Fill the cells of F' as follows.

- Fill the cells of the top row with 1.
- Read τ from left to right. For a letter τ_i:
  - If τ_i is a column, fill remaining cells of that column with 0.
  - If τ_i is a row, fill remaining cells of that row with 1.
- $\mathcal{T}' := \Psi(\tau)$ is the filling of F' we obtain.

The map Ψ

$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8$. Fill the cells of F' as follows.

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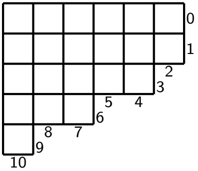

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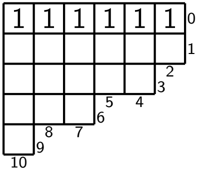

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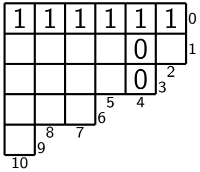

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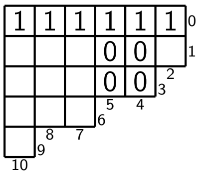

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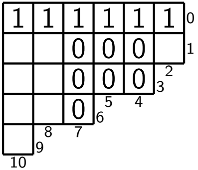

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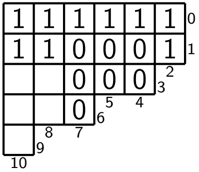

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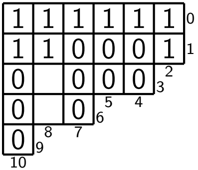

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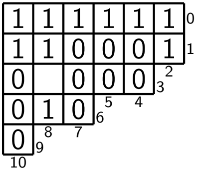

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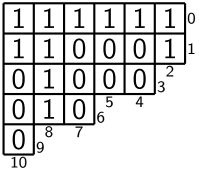

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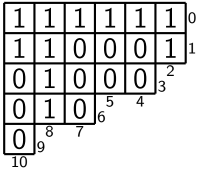
$\tau = 4\ 5\ 7\ 1\ 2\ 10\ 9\ 6\ 3\ 8 \overset{\Psi}{\longmapsto}$ 

Theorem (S., Smith, Steingrímsson ++)

Let $A \subseteq \{1, \cdots, n-1\}$ and F' be the Ferrers diagram such that $RL'(F') = A$. Then:

$$\Psi : \{\tau \in \mathcal{S}_n, \mathrm{des\_bot}(\tau) = A\} \longrightarrow EWT(F')$$

is a bijection.

Proof idea

1. $\Psi(\tau)$ is an EWT by construction.
2. Construct Ψ^{-1}. First erase all entries in the top row. Key lemma: the resulting tableau has at least one all-0 column. Erasing entries as we go, we:

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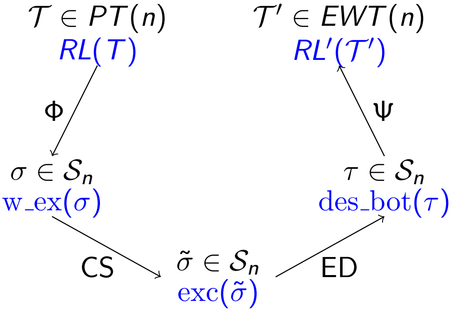
   and iterate.

Map of the proof



Some properties

- A column labelled i of $\mathcal{T}'$ is all-1 iff i is a fixed point of $\tilde{\sigma}$.
- The number of almost-all-0 columns of $\mathcal{T}'$ is the place of first descent in τ.
- The top row of $\mathcal{T}$ is all-0 iff the leftmost column of $\mathcal{T}'$ is its only almost-all-0 column.
- A non top row labelled i of $\mathcal{T}$ is all-0 iff (conditions on row $i-1$ in $\mathcal{T}'$).
- There is a bijection between (0-minimal EWTs of size n with $k+1$ columns) and (the set of binary strings of length $2n-3$ with k 1s and no 11). In particular, the number of 0-minimal EWTs of size n is $\mathrm{Fib}(2n-2)$.
- τ avoids 213 iff (rows of $\mathcal{T}'$ are $0\cdots01\cdots1$ and any leftmost 1 in a row has no 1 directly beneath it).
- $\cdots$

Open questions

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- Other statistics? Number of 1s in $\mathcal{T}$?
- PTs are linked to the PASEP model (Corteel and Williams 07). Link between PASEP and EWTs? In particular, what does cell deletion in $\mathcal{T}$ correspond to in $\mathcal{T}'$?
- Generalisations of PTs and EWTs. Link to Tree-like tableaux (Aval, Boussicault, Nadeau 13). Spanning trees? Sandpile model?

Thank you!