On (rational) Shi tableaux

Robin Sulzgruber

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Setting the stage

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Definition Let V be a Euclidean vector space, $\alpha \in V$ a non-zero vector and $k \in \mathbb{Z}$. Define the affine hyperplane

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The Weyl group of Φ is the group generated by the reflections $s_{\alpha,0}$ for $\alpha \in \Phi^+$.

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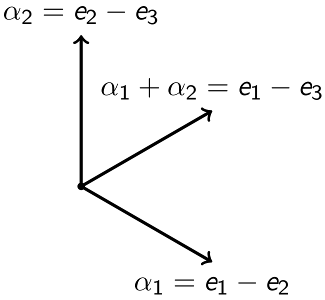

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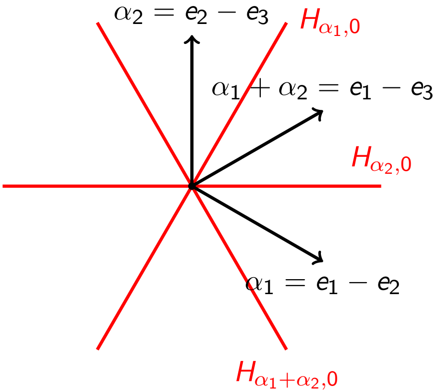

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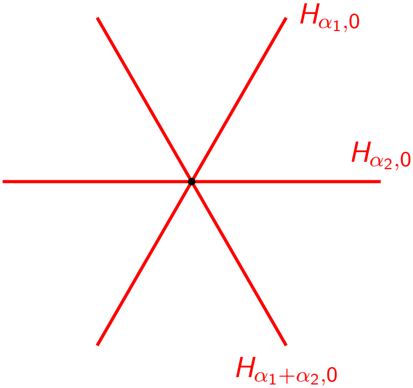

The reflections $s_{\alpha_1,0}, s_{\alpha_2,0}$ generate the symmetric group $\mathfrak{S}_3$.

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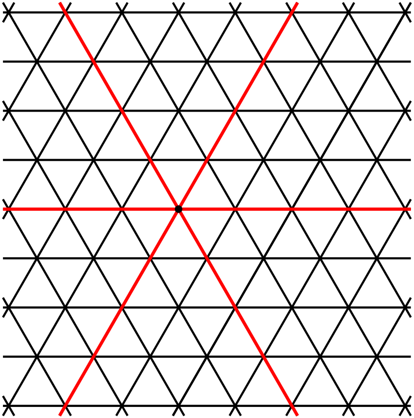

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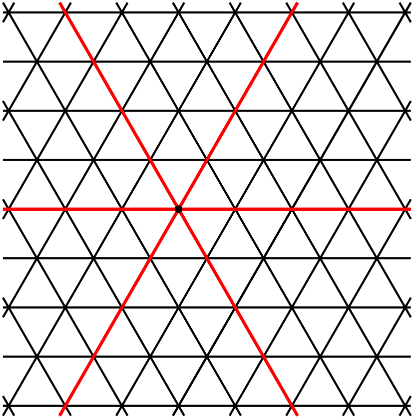

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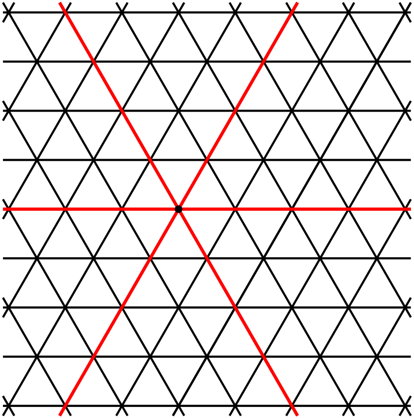
The affine Weyl group $\widetilde{W}$ is the group generated by all reflections in the hyperplanes of Aff. It acts simply transitively on the set of alcoves.



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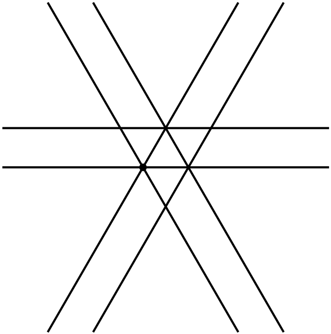

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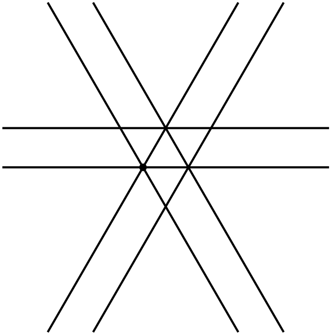
Theorem (Shi 1987, 1997) The Shi arrangement has $(h+1)^r$ regions and

$$\frac{1}{|W|} \prod_{i=1}^{r} (d_i + h)$$

dominant regions.



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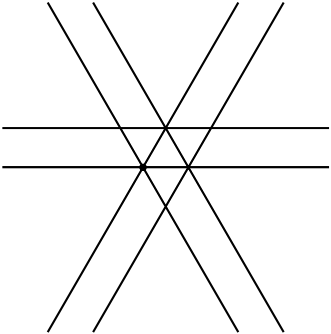

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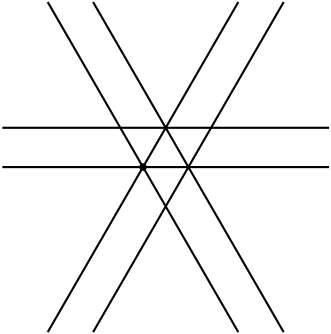

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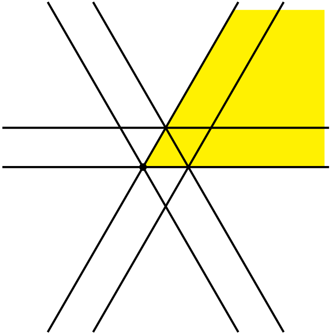

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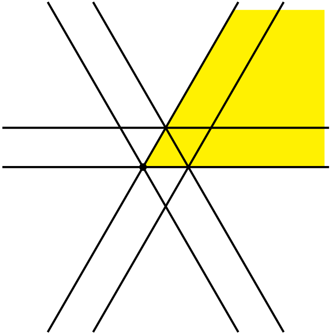

$$(n+1)^{n-1} = 4^2 = 16$$

$$\frac{1}{n+1}\binom{2n}{n} = \frac{6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 5$$

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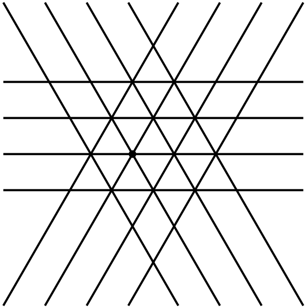

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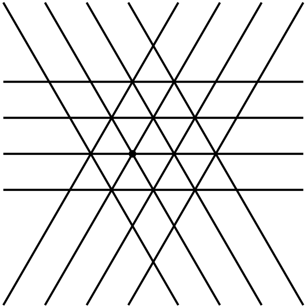

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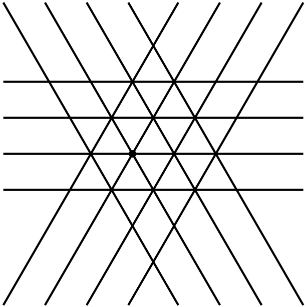

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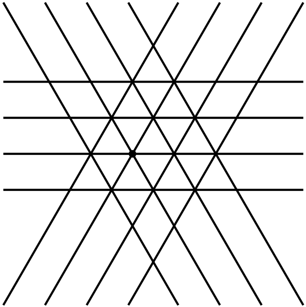

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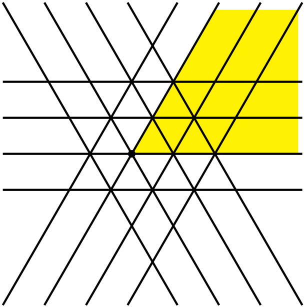

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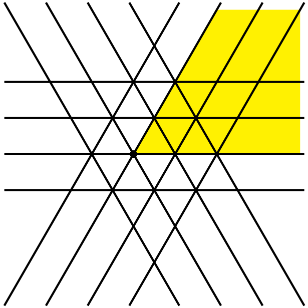

$$(mn+1)^{n-1} = 49$$

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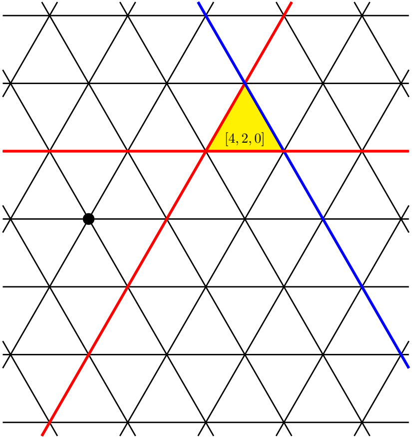

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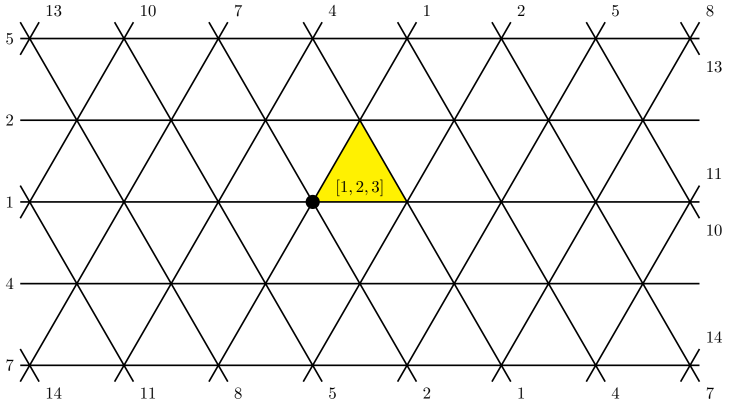

Shi alcoves

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Theorem (Shi 1987, Athanasiadis 2005, Thiel 2015) The regions of the m-Shi arrangement are in bijection with alcoves whose floors have height less than $mh + 1$.

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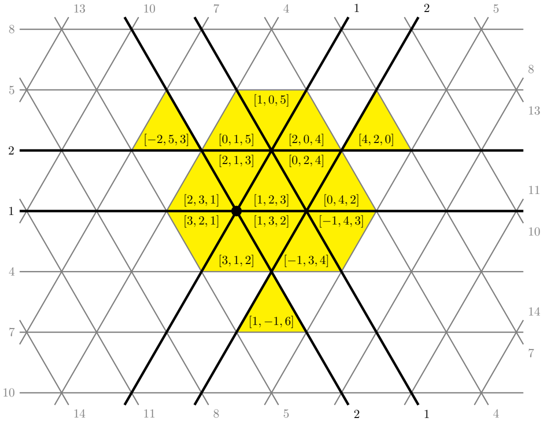

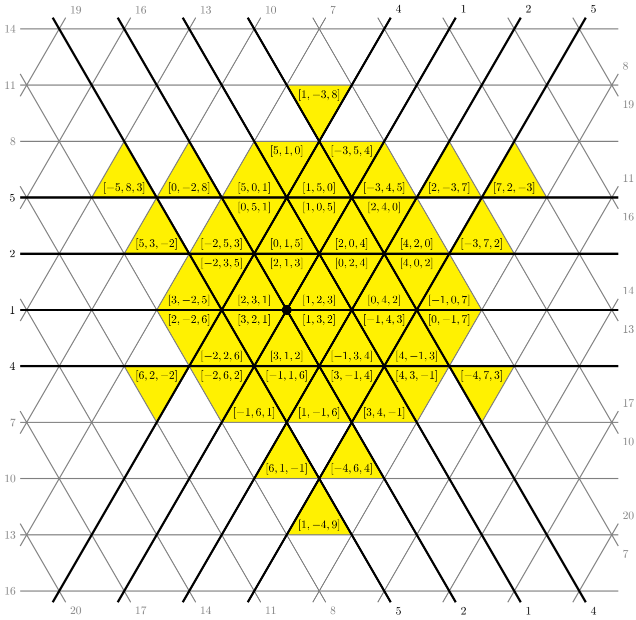

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Inverse Shi alcoves

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Theorem (Fishel, Vazirani 2010) The regions of the m-Shi arrangement are in bijection with the alcoves inside the simplex bounded by the hyperplanes of height $mh + 1$.

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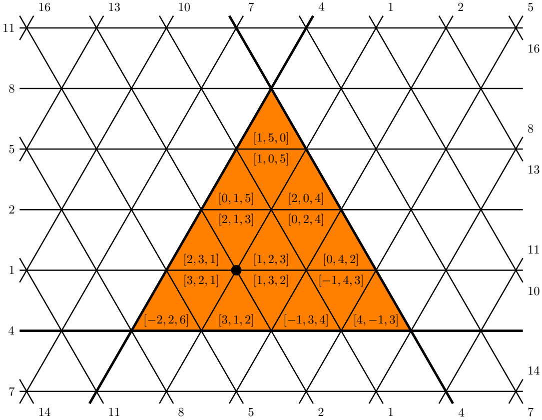

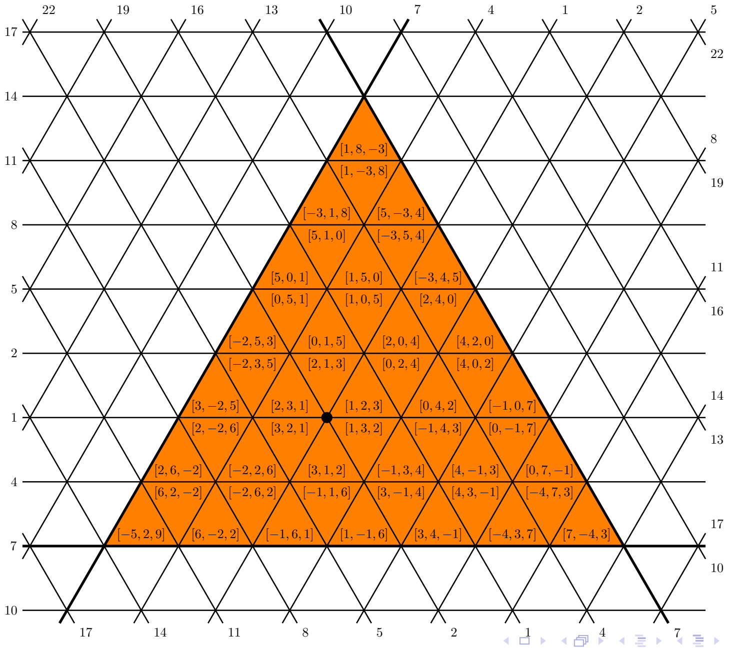
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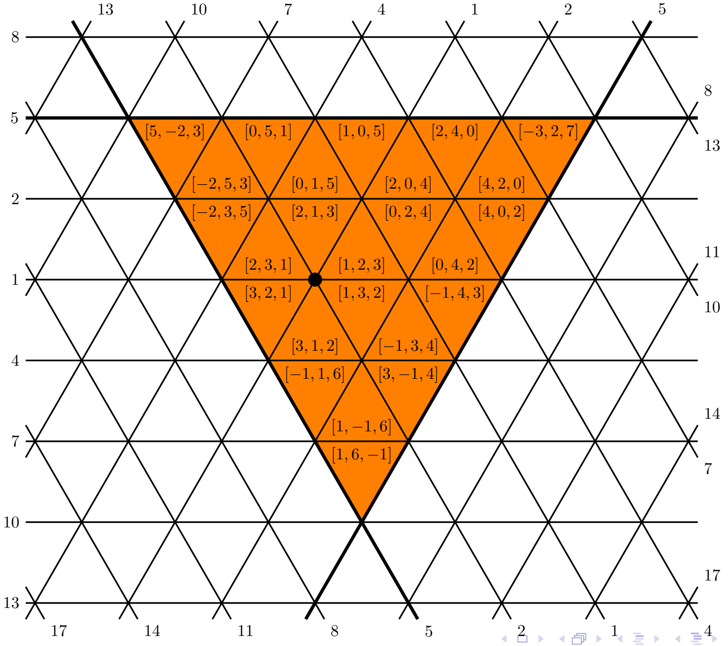
Theorem (Thiel 2015) The number of p-stable alcoves equals p^r. The number of dominant p-stable alcoves equals

$$\frac{1}{|W|}\prod_{i=1}^{r}(p+e_i).$$



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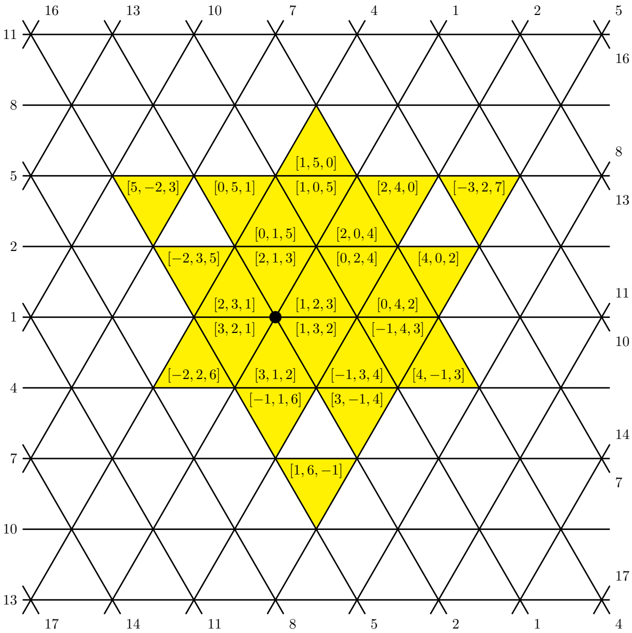

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Definition (Fishel, Tzanaki, Vazirani 2011) Let $w(A_\circ)$ be a dominant Shi alcove and $\alpha \in \Phi^+$. Define $t^{mh+1}(\alpha, w)$ as the number of Shi hyperplanes of the form $H_{\alpha,k}$ that separate $w(A_\circ)$ and $A_\circ$.

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The Shi tableau of w is the collection of the numbers $t^{mh+1}(\alpha, w)$ for $\alpha \in \Phi^+$.

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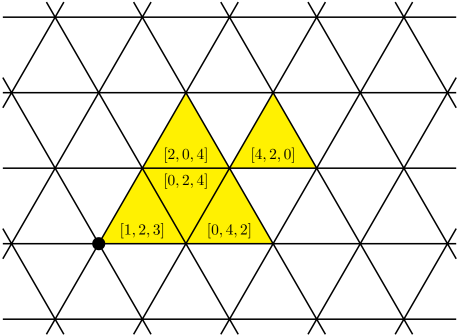

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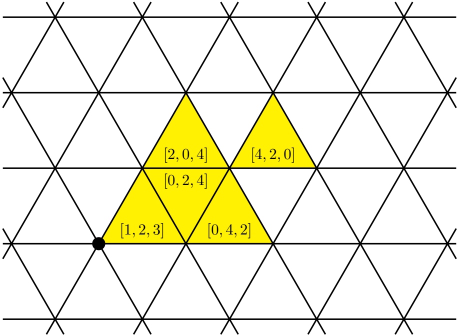

$w = [4, 2, 0]$

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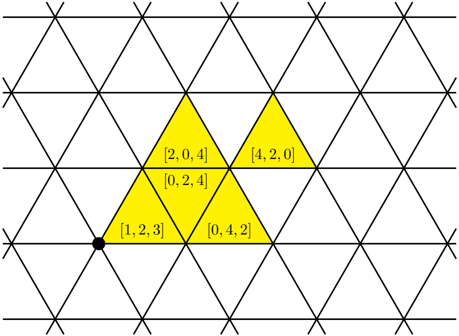

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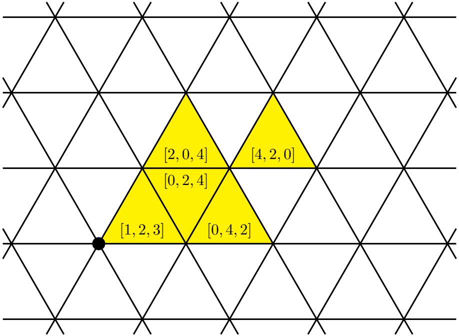

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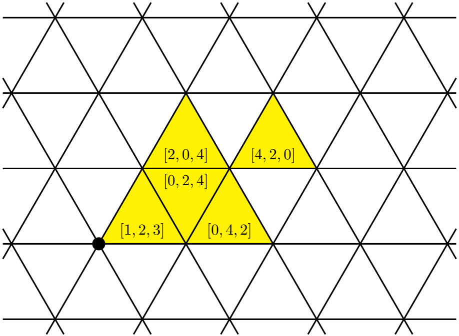

$$w = [4, 2, 0]$$

$$t^4(\alpha_1, w) = 1$$
$$t^4(\alpha_2, w) = 1$$
$$t^4(\alpha_1 + \alpha_2, w) = 1$$

Rational Shi tableaux

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Definition Let $w(A_\circ)$ be dominant and p-stable and $\alpha \in \Phi^+$. Define $t^p(\alpha, w)$ as the number of hyperplanes of the form $H_{\alpha,k}$ with height less than p that separate $w(A_\circ)$ and $A_\circ$.

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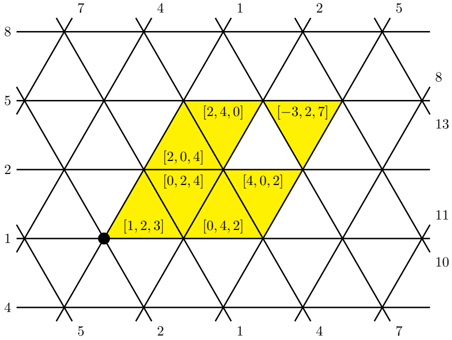

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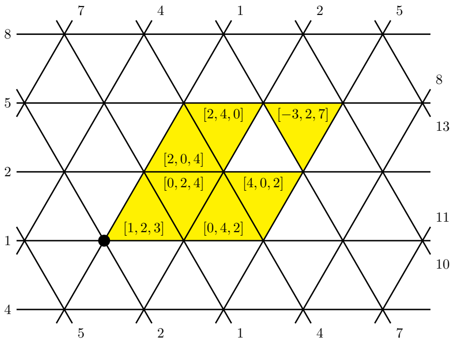

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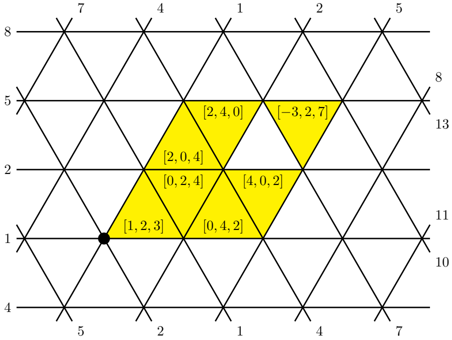

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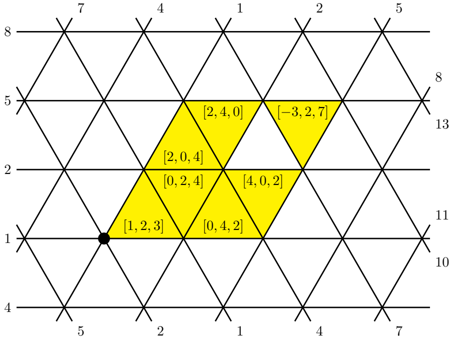

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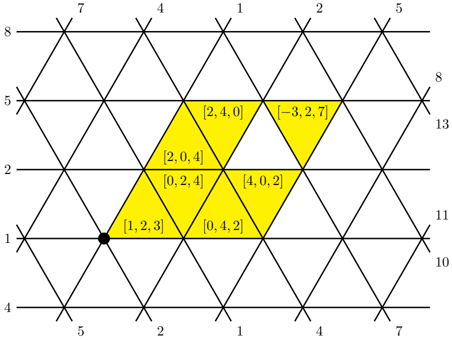

$$w = [-3, 2, 7]$$

$$t^5(\alpha_1, w) = 1$$
$$t^5(\alpha_2, w) = 1$$
$$t^5(\alpha_1 + \alpha_2, w) = 2$$

The Main Conjecture

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Conjecture Every dominant p-stable element $w \in \widetilde{W}$ is uniquely determined by its rational Shi tableau.

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Open Problem Characterise the set of rational Shi tableaux.

Inverting the rational Shi tableau in type A_{n-1}

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Example Consider the affine permutation of type A_4

$$w = [7, -1, 11, 3, -5].$$

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Example Consider the affine permutation of type A_4

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Then the alcove of w^{-1} is contained in the simplex bounded by the hyperplanes of height $p = 8$.

The Shi tableau of w is given by

α_{15} 2 α_{25} 1 α_{35} 2 α_{45} 1

α_{14} 1 α_{24} 2 α_{34} 0

α_{13} 2 α_{23} 1

α_{12} 0

To Dyck paths via row-sums and column-sums

α_{15} 2 α_{25} 1 α_{35} 2 α_{45} 1

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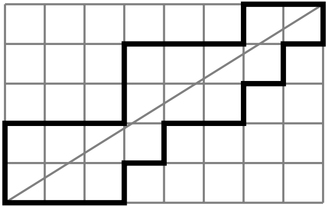
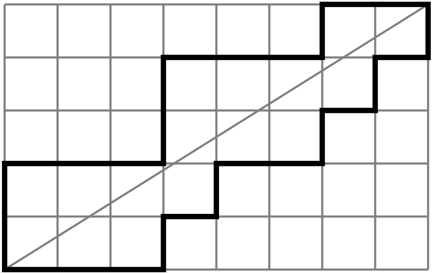


To long cycles (Ceballos, Denton, Hanusa 2016)



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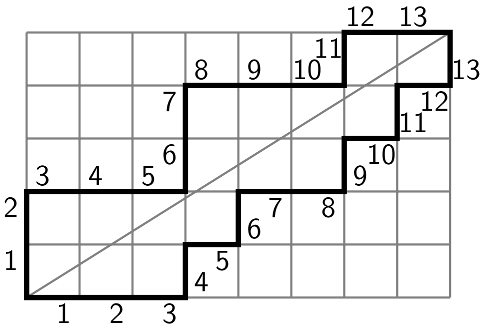

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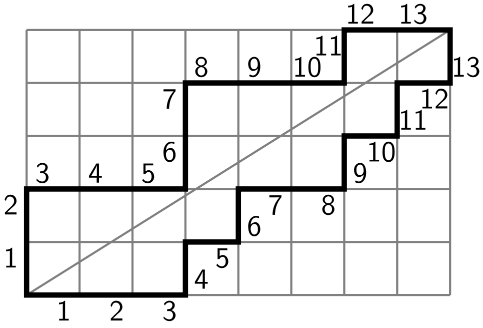

$$(4, 2, 6, 9, 7, 11, 13, 12, 10, 8, 5, 3, 1)$$

Back to Dyck paths (Ceballos, Denton, Hanusa 2016)

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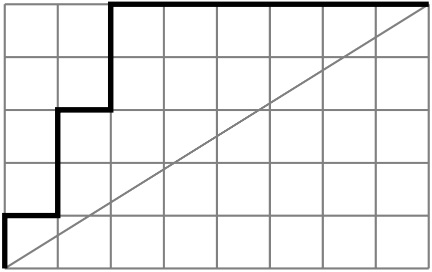

To n and p flush abaci (Anderson 2002)

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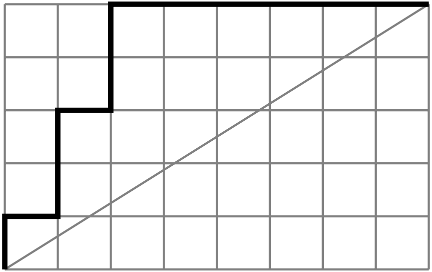

To n and p flush abaci (Anderson 2002)

40	35	30	25	20	15	10	5	0
32	27	22	17	12	7	2	-3	-8
24	19	14	9	4	-1	-6	-11	-16
16	11	6	1	-4	-9	-14	-19	-24
8	3	-2	-7	-12	-17	-22	-27	-32
0	-5	-10	-15	-20	-25	-30	-35	-40

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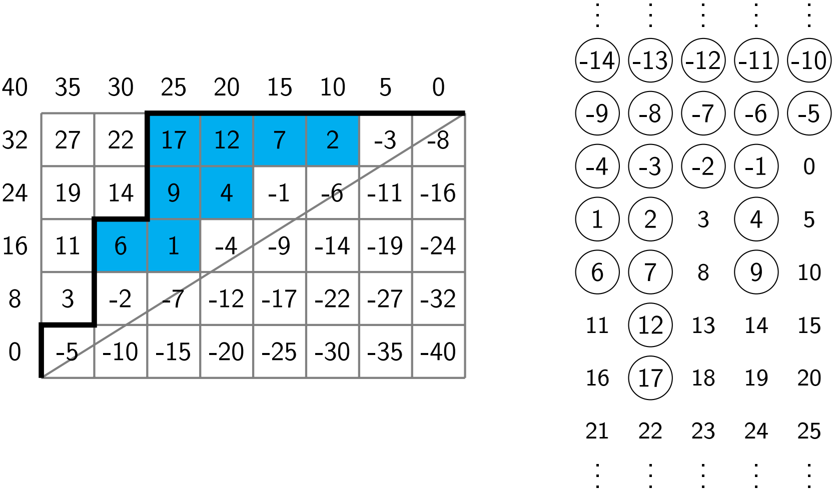

Shift back to affine permutations (Lascoux 2001)

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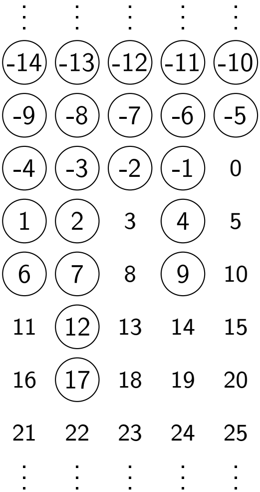

Shift back to affine permutations (Lascoux 2001)

⋮	⋮	⋮	⋮	⋮
-14	**-13**	**-12**	**-11**	**-10**
-9	**-8**	**-7**	**-6**	**-5**
-4	**-3**	**-2**	**-1**	0
1	**2**	3	**4**	5
6	**7**	8	**9**	10
11	**12**	13	14	15
16	**17**	18	19	20
21	22	23	24	25
⋮	⋮	⋮	⋮	⋮

⋮	⋮	⋮	⋮	⋮
-19	**-18**	**-17**	**-16**	**-15**
-14	**-13**	**-12**	**-11**	**-10**
-9	**-8**	-7	**-6**	**-5**
-4	**-3**	-2	**-1**	**0**
1	**2**	3	4	**5**
6	7	8	9	**10**
11	12	13	14	15
16	17	18	19	20
⋮	⋮	⋮	⋮	⋮

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1	**2**	3	**4**	5
6	**7**	8	**9**	10
11	**12**	13	14	15
16	**17**	18	19	20
21	22	23	24	25
⋮	⋮	⋮	⋮	⋮

⋮	⋮	⋮	⋮	⋮
-19	**-18**	**-17**	**-16**	**-15**
-14	**-13**	**-12**	**-11**	**-10**
-9	**-8**	-7	**-6**	**-5**
-4	**-3**	-2	**-1**	**0**
1	**2**	3	4	**5**
6	7	8	9	**10**
11	12	13	14	15
16	17	18	19	20
⋮	⋮	⋮	⋮	⋮

$$w^{-1} = [-7, -4, 4, 7, 15]$$

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⋮	⋮	⋮	⋮	⋮
-14	**-13**	**-12**	**-11**	**-10**
-9	**-8**	**-7**	**-6**	**-5**
-4	**-3**	**-2**	**-1**	0
1	**2**	3	**4**	5
6	**7**	8	**9**	10
11	**12**	13	14	15
16	**17**	18	19	20
21	22	23	24	25
⋮	⋮	⋮	⋮	⋮

⋮	⋮	⋮	⋮	⋮
-19	**-18**	**-17**	**-16**	**-15**
-14	**-13**	**-12**	**-11**	**-10**
-9	**-8**	-7	**-6**	**-5**
-4	**-3**	-2	**-1**	**0**
1	**2**	3	4	**5**
6	7	8	9	**10**
11	12	13	14	15
16	17	18	19	20
⋮	⋮	⋮	⋮	⋮

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This is the end.

Thank you!

Shi coordinates

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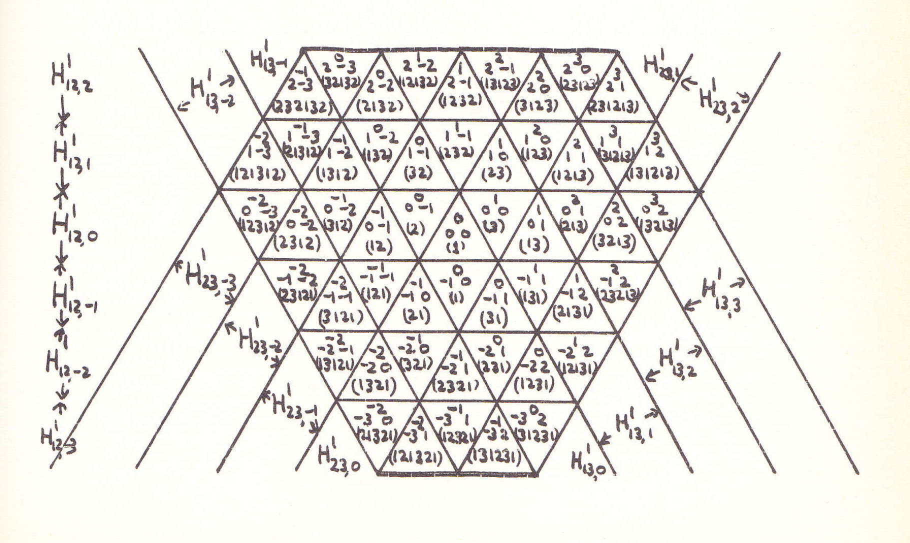

Sign types

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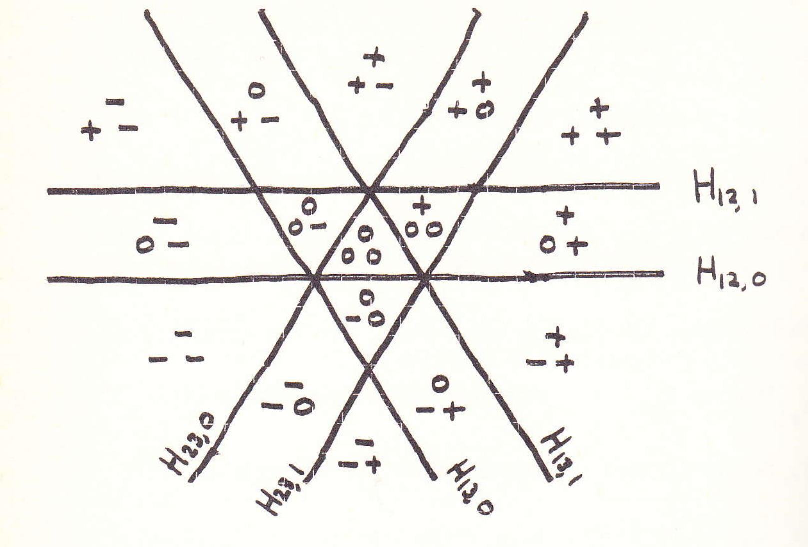

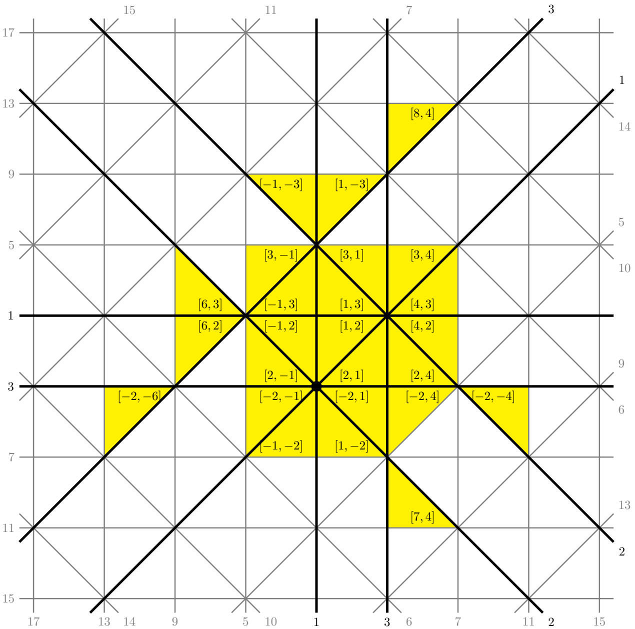
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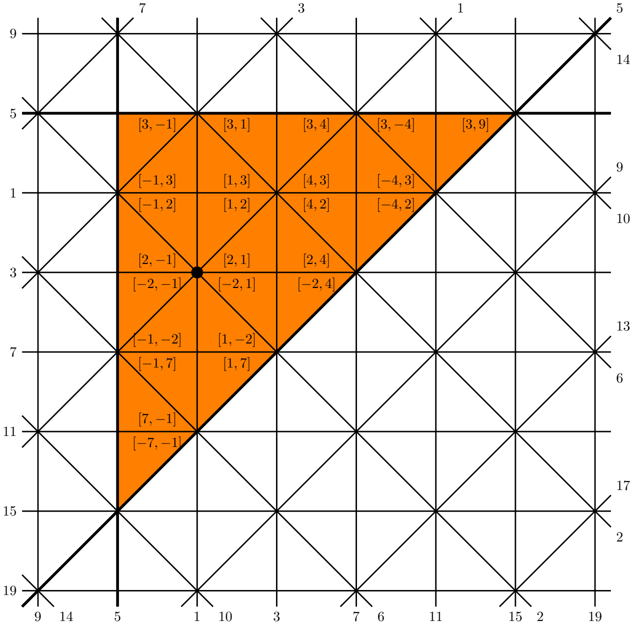
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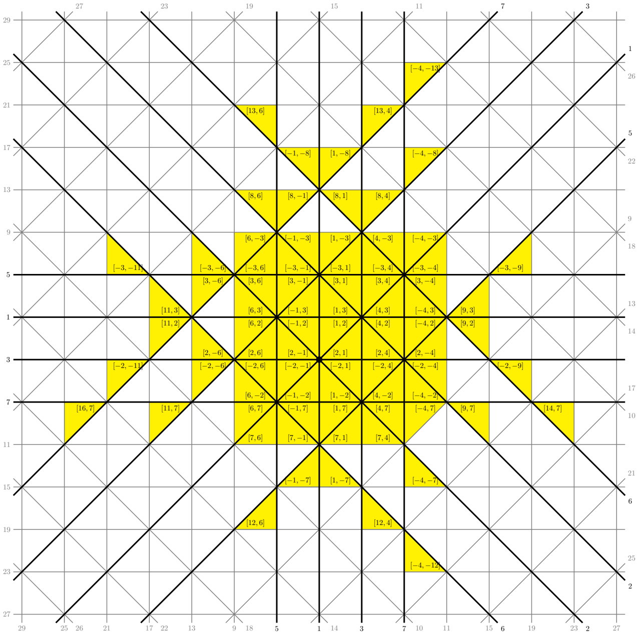

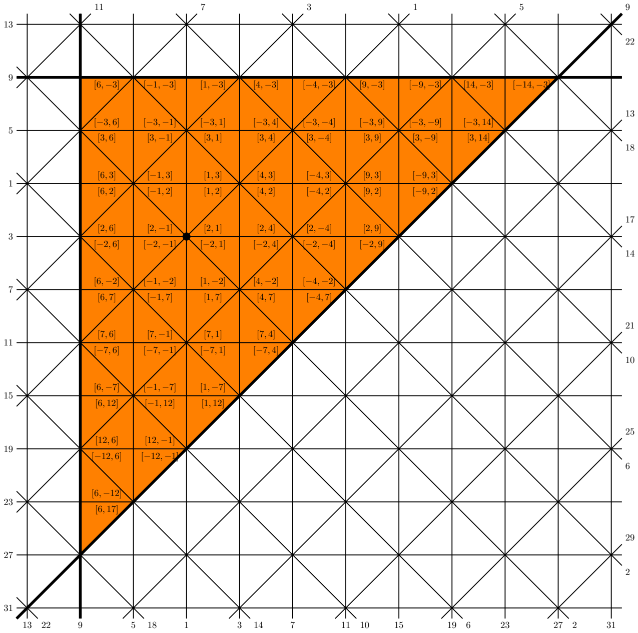

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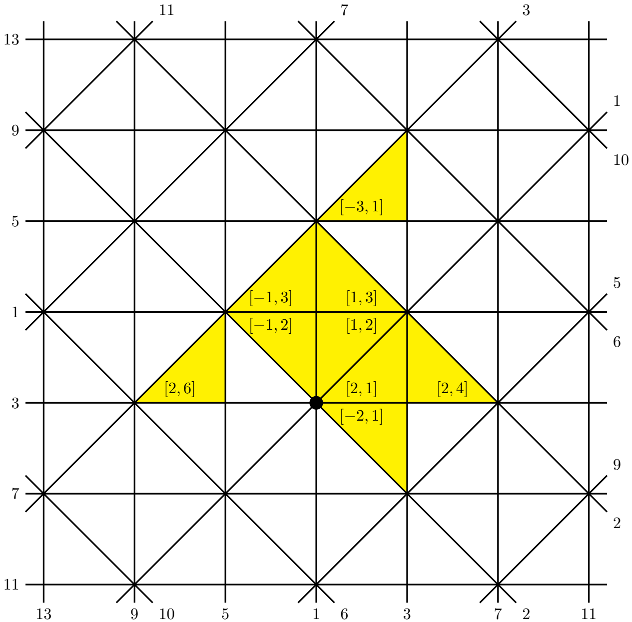
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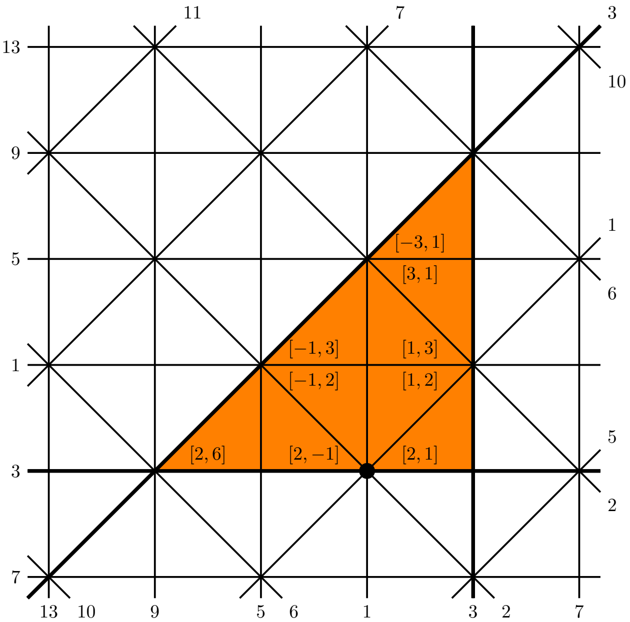
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