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ON THE CYCLE STRUCTURE OF THE PRODUCT OF RANDOM MAXIMAL CYCLES

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Abstract. The subject of this paper is the cycle structure of the random permutation σ of $[N]$, which is the product of k independent random cycles of maximal length N. We use the character-based Fourier transform to study the counts of cycles of σ by length and also the distribution of the elements of the subset $[\ell]$ among the cycles of σ.

1. Introduction

Enumeration of permutations of a set $[N] = \{1, 2, \ldots, N\}$ according to the numbers of cycles of various lengths has a long and glorious history. The plentiful results are not infrequently cast in the probabilistic light, if the assumption is made that a permutation is chosen *uniformly at random* among all $N!$ permutations. The techniques vary widely, from bijective methods to multivariate generating functions to functional limit theorems, allowing to find solutions, exact or asymptotic, of rather delicate, enumerative-probabilistic, problems. More recently there has been a growing interest in the probabilities regarding distribution of the elements of a subset $S \subseteq [N]$ among the cycles of the random permutation. For instance, we can determine the probability that each of the entries in S will be in a different cycle, or that all entries of S will be in the same cycle, or that each cycle of p will contain at least one entry of S. See [21] for results of this kind.

The classic, and more recent, problems become much more difficult if, instead of a uniformly random permutation, we consider a random permutation which is a *product* of random *maximal* cycles. That is, our sample space is now that of all ordered k-tuples $(p_1, p_2, \ldots, p_k)$, where all p_i are maximal cycles of length N. One can investigate the random permutation $\sigma := p_1 \cdots p_k$ under the assumption that $p_1, \ldots, p_k$ are maximal cycles, chosen uniformly at random, and independently of each other, from all $(N-1)!$ such cycles. The problem to determine the probability that the permutation σ has prescribed counts of cycles of all lengths is equivalent to a special case of the problem to count all factorizations of σ belonging to a given conjugacy class into the product of k permutations each belonging to its own conjugacy class. There is a long, glorious history of research in this classic

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area, starting with Hurwitz [16] who was motivated by deep connections between factorizations and topological equivalence classes of functions on associated Riemann surfaces; see [18] for the history of the subject, and a relatively recent paper by Poulalhon and Schaeffer [25]. Still when counting the products σ that do not fill a single conjugacy class, or few such classes, that usage of the existing results on factorization is quite problematic.

1.1. **Motivation and recent results.** Among the sources of our inspiration are Zagier's classic formula for the distribution of the number of cycles in σ for $k = 2$ (see [33]), and the more recent results by Stanley [29] and by Bernardi et al. [3], again for $k = 2$. For the problem in [33], the admissible products of cycles form a union of too many conjugacy classes for invoking the factorization results. In [3], a formula is proved for the probability that σ, the product of two maximal cycles, separates the *given disjoint* subsets of $[N]$, i.e., no two of those subsets are represented in the same cycle of σ. For this problem the labels of elements in $[N]$ are at the core of counting, and the set of admissible products is a union of *subsets of* conjugacy classes.

Beside their intrinsic interest, solutions of the mentioned problems may lead to surprising applications. In [5], Bóna and Flynn used a result of Stanley [29] concerning the special case $S = \{1, 2\}$ and $k = 2$ to prove an exact formula for the average number of block interchanges needed to sort a permutation, a problem motivated by genome sorting. Equally interesting are the methods that can be used, as they come from a wide array of areas in mathematics, such as character theory, multivariate Gaussian integration, bijective combinatorics and the summation techniques for hypergeometric sums.

1.2. **Overview: methods and results.** In 1986 Harer and Zagier [15] discovered a remarkable formula for the bivariate generating function of the number of cycles in the product of a maximal cycle and a random, fixed-point free, involution of $[2n]$, thus solving a difficult problem of enumerating the chord diagrams by the genus of an associated surface. The proof was based on evaluation of multidimensional Gaussian integrals. Soon after, Jackson [17], and later Zagier [33], found alternative proofs that used characters of the symmetric group S_{2n}. Recently, the second author found a different, character-based proof in [24]. Its core consists of computing and *marginally* inverting the Fourier transform of the underlying probability measure on S_{2n}. In the present paper, we use the techniques in [24], see also an earlier paper by Chmutov and Pittel [7], to investigate the product of k maximal cycles in S_N. In order to make the discussion reasonably self-contained, we will introduce the necessary definitions and facts from [24] in Section 2.

We begin Section 3 with Lemma 3.1 that states an explicit formula for the probability distribution of the number of cycles in σ, the product of k random, independent, maximal cycles in S_N. Not surprisingly, the distribution is expressed through the Stirling numbers of first kind. The formulas for the probability that σ is the identity permutation and for the probability that σ is a maximal cycle obtained by Stanley in [28] are the special cases of the identity in this lemma. Our analysis also delivers a well-known formula found by Zagier for the case $k = 2$. See Corollary 3.5 for this special case; see the Appendix by Zagier in [18] for the original result of Zagier. In Corollary 3.6, we also obtain a bivariate generating function for the distribution of the number of cycles for the product of three cycles. We conclude this section with a relatively compact integral formula for the probability that the product of two cycles belongs to a given conjugacy class.

Then, in Section 4, we turn to the following general question. Let $p_A(N, \ell; k)$ be the probability that the number of elements of $[\ell] = \{1, 2, \dots, \ell\}$ in each cycle of σ comes from the set $A \subseteq \mathbb{Z}_{\geq 0}$. What can we say about $p_A(N, \ell; k)$?

To this end, for a general A, we first enumerate the admissible permutations by the cycle counts and then evaluate the sum of character values over all admissible permutations for irreducible representations labeled by one-hook Young diagrams. Then we consider the special case when $A = \mathbb{Z}_{>0}$, i.e., when each cycle of σ contains at least one element of $[\ell]$. Using the inverse Fourier transform, we find an alternating sum expression for this probability with $N - \ell + 1$ binomial-type summands. This result is proved in Theorem 4.2. For $k = 2$, this sum reduces to two notably simpler expressions, that can be efficiently computed for moderate ℓ and moderate $N - \ell$ respectively.

Next we investigate the case of $A = \{0, \ell\}$, that is, when all elements of ℓ are in the same cycle of σ. This computation is longer than its counterpart in the previous case, and it leads to a general formula for $p_A(N, \ell; k)$, given in Theorem 4.5, that is analogous to that for $A = \mathbb{Z}_{>0}$. Again, if $k = 2$, then the formula shrinks to a pair of computationally efficient sums for moderate ℓ and moderate $N - \ell$ respectively. For $\ell = 2$ and $\ell = 3$, we recover the results obtained by Stanley [29].

Having experimented with *Maple*, we feel confident that the residual sums for $k = 2$ in either of the two cases do not have a more compact presentation.

After this, in Section 5, we turn to our most technical problem. We consider disjoint subsets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t$ of $[N]$ such that $|\mathcal{S}_j| = \ell_j$; define $\ell = \sum_j \ell_j$. Let $p(N, \vec{\ell}; k)$ denote the probability that no cycle of σ contains elements from more than one $\mathcal{S}_j$, a property to which we refer by saying that σ *separates* the sets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_t$. Bernardi et al. [3] found a striking formula for $p(N, \vec{\ell}; 2)$ that involved an alternating sum of $\ell - t + 1$ terms. Remarkably, the factor $\prod_j \ell_j!$ aside, the rest of the formula depends on ℓ and t only. In Lemma 5.1, we show that the separation probability continues to have this latter property for all $k \geq 2$, and find an alternating sum formula with $N - \ell + t + 1$ terms for this probability, which is computationally efficient if t and $N - \ell$ are both bounded as N grows. Then, for $k = 2$, we are able to simplify this formula to one that is close in appearance, but is significantly different from the formula in [3]. This formula is given in Theorem 5.5, and it still contains a sum of $\ell - t + 1$ summands, but the signs are no longer alternating.

Finally, in Section 6, we consider the following question. Let us say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if no two elements of $[\ell]$ are neighbors, *and* each element of $[\ell]$ has a neighbor from $[N] \setminus [\ell]$. Then, for a general $k \geq 2$, we find a *two-term* formula for the probability that σ blocks the elements of $[\ell]$. This formula is proved in Theorem 6.1.

While on occasion our proofs deliver the already known results, we hope that the employed techniques can be used for a broader variety of problems on cyclic structure of the products of random permutations.

In conclusion we should mention the recent paper [11] by Féray and Rattan. Among other results, it was proved in [11] that the number of ways to write a given permutation σ as the product of an n-cycle and a permutation $[(n-a) - \text{cycle}] \cup \rho$, depends only on the product of signs of σ and ρ and the first $(a-1)$ multiplicities of cycle lengths in σ, thus generalizing a theorem by Boccara [4]. Féray and Rattan also found a recurrence equation for separation probabilities for the product of n-cycle and $(n-a)$ cycle. Their methods differ significantly from ours.

2. Preliminaries

A key observation is that the set of all maximal cycles forms a *conjugacy class* in the symmetric group S_N, a class with particularly simple character values. We mention that permutations generated by a given conjugacy class were studied for instance by Diaconis and Shahshahani [8], and, from a more algebraic point of view, by Larsen and Shalev [19]. Some of that work resulted in asymptotic formulas. However, the research in the present paper is in the line of work of proving *exact* formulas in combinatorial enumeration, like the mentioned results in [3, 27, 28, 29].

Let us start with the Fourier inversion formula for a general probability measure P on S_N:

$$P(s) = \frac{1}{N!} \sum_{\lambda \vdash N} f^{\lambda} \operatorname{tr}\big(\rho^{\lambda}(s^{-1}) \hat{P}(\rho^{\lambda})\big), \quad s \in S_N. \tag{1}$$

(See Diaconis and Shahshahani [8] and Diaconis [9] for a lucid discussion and applications of the Fourier transform for asymptotic analysis of the random walks on the symmetric group S_N and other simple groups.) Here λ is a generic partition of the integer N, ρ^{λ} is the irreducible representation of S_N associated with λ, $f^{\lambda} = \dim(\rho^{\lambda})$, and $\hat{P}(\rho^{\lambda})$ is the $f^{\lambda} \times f^{\lambda}$ matrix-valued Fourier transform of $P(\cdot)$ evaluated at ρ^{λ}, $\hat{P}(\rho^{\lambda}) = \sum_{s \in S_N} \rho^{\lambda}(s) P(s)$. Let us evaluate the right-hand side of (1) for $P = P_{\sigma}$, the probability measure on S_N induced by $\sigma = \prod_{j=1}^{k} \sigma_j$, where σ_j is uniform on a conjugacy class C_j. Let P_{σ_j} be the probability measure on S_N induced by σ_j, i.e., P_{σ_j} is uniform on C_j. As the σ_j are independent, we have $P_{\sigma}(s) = \sum_{s_1,\dots,s_k} \prod_j P_{\sigma_j}(s_j)$,

$(s_1 \cdots s_k = s)$, that is, P_σ is the convolution of $P_{\sigma_1}, \dots, P_{\sigma_k}$. So, by multiplicativity of the Fourier transform for convolutions, $\hat{P}_\sigma(\rho^\lambda) = \prod_j \hat{P}_{\sigma_j}(\rho^\lambda)$. Since each P_{σ_j} is supported by the single conjugacy class C_j, we have $\hat{P}_{\sigma_j}(\rho^\lambda) = \frac{\chi^\lambda(C_j)}{f^\lambda} I_{f^\lambda}$, where I_{f^λ} is the $f^\lambda \times f^\lambda$ identity matrix, see [9]. So

$$\hat{P}_\sigma(\rho^\lambda) = \prod_{j=1}^{k} \hat{P}_{\sigma_j}(\rho^\lambda) = (f^\lambda)^{-k} \prod_{j=1}^{k} \chi^\lambda(C_j)\, I_{f^\lambda},$$

and (1) becomes

$$\begin{aligned} P_\sigma(s) &= \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \left(\prod_{j=1}^{k} \chi^\lambda(C_j) \right) \operatorname{tr}\big(\rho^\lambda(s^{-1}) I_{f^\lambda}\big) \\ &= \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+1} \chi^\lambda(s) \prod_{j=1}^{k} \chi^\lambda(C_j); \end{aligned} \tag{2}$$

see Stanley [28, Exercise 7.67].

Remark. For the special case $s = \mathrm{id}$, the identity (2) becomes

$$\mathrm{P}_\sigma(\mathrm{id}) = \frac{1}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^{k} \chi^\lambda(C_j).$$

Since the left-hand side is just $\mathcal{N}(C_1, \dots, C_k)$, the number of ways to write the identity permutation as the product of elements of $C_1, \dots, C_k$, divided by $\prod_{j=1}^{k} |C_j|$, we obtain the well-known S_N-version of Frobenius' identity

$$\mathcal{N}(C_1, \dots, C_k) = \frac{\prod_{j=1}^{k} |C_j|}{N!} \sum_\lambda (f^\lambda)^{-k+2} \prod_{j=1}^{k} \chi^\lambda(C_j). \tag{3}$$

We will use (2) for $C_j \equiv \mathcal{C}_N$, where $\mathcal{C}_N$ is the conjugacy class of all maximal cycles. By the Murnaghan–Nakayama rule (see [26, Lemma 4.10.2] or [28, Section 7.17, Equation (7.75)]), $\chi^\lambda(\mathcal{C}_N) = 0$ unless the diagram λ is a single hook λ^*, with one row of length λ_1 and one column of height λ^1, so $\lambda_1 + \lambda^1 = N + 1$. In that case

$$\chi^\lambda(\mathcal{C}_N) = (-1)^{\lambda^1 - 1}. \tag{4}$$

As for f^{λ^*}, the number of Standard Young Tableaux of shape λ^*, by applying the hook length formula (or simply selecting the entries that go in the first column), we obtain

$$f^{\lambda^*} = \frac{N!}{N \prod_{r=1}^{\lambda_1 - 1} r \prod_{s=1}^{\lambda^1 - 1} s} = \binom{N-1}{\lambda_1 - 1}. \tag{5}$$

The equations (2), (4) and (5) imply

$$P_\sigma(s) = \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\chi^{\lambda^*}(s),$$

or more generally: for $\mathcal{A}\subseteq S_N$,

$$(6)\qquad \begin{aligned} P_\sigma(\mathcal{A}) &= \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\chi^{\lambda^*}(\mathcal{A}),\\ \chi^{\lambda^*}(\mathcal{A}) &:= \sum_{s\in\mathcal{A}}\chi^{\lambda^*}(s). \end{aligned}$$

By the Murnaghan–Nakayama rule, given a hook diagram λ^*, the value of $\chi^{\lambda^*}(s)$ depends on s only through $\vec{\nu} = \vec{\nu}(s) := \{\nu_r\}_{r\ge 1}$, where $\nu_r = \nu_r(s)$ is the total number of r-long cycles in the permutation s. It was proved in [24] that

$$(7)\qquad \chi^{\lambda^*}(s) = (-1)^{\lambda^1+\nu}\,[\xi^{\lambda_1}]\,\frac{\xi}{1-\xi}\prod_{r\ge 1}(1-\xi^r)^{\nu_r},$$

$\nu(s) := \sum_r \nu_r(s)$ being the total number of cycles of s. From (7) it follows that

$$(8)\qquad \begin{aligned} \sum_{s:\,\vec{\nu}(s)=\vec{\nu}}\chi^{\lambda^*}(s) &= (-1)^N N!\,\mathcal{A}(N,\nu,\lambda_1),\\ \mathcal{A}(N,\nu,\lambda_1) &:= \binom{N-1}{N-\lambda_1}\sum_{\ell\ge 1}(-1)^{\ell}\,\frac{c(\ell,\nu)}{\ell!}\binom{N-\lambda_1}{N-\ell}, \end{aligned}$$

where $c(\ell,\nu)$ is the total number of permutations of $[\ell]=\{1,2,\dots,\ell\}$ with ν cycles; see the proof of Theorem 2.1 and Equation (2.20) in [24]. The formulas (2), (6)–(8) are the basis of the proofs of a series of exact formulas in the rest of the paper.

Remark 2.1. Some asymptotic approximations for the product of maximal cycles can be obtained directly via the total variation distance bound based on the Fourier inversion formula (1). This powerful approach was pioneered by Diaconis and Shahshahani (see [8], [9]). Observe that the conjugacy class C_N belongs to the alternating group A_N for N odd, and to $A_N^c = S_N\setminus A_N$ for N even. So for N odd, the product of k cycles is in A_N for every k. For N even, the product of k cycles is in A_N if and only if k even. A minor modification of the argument in [7] yields that in each case the total variation distance between the distribution of product of k independent, uniformly random, N-cycles and the uniform distribution on the corresponding range (A_N or A_N^c, depending on parities of N and k) decays as $N^{-k/2}$ for $n\to\infty$. The interested reader may wish to compare this general error estimate with the particular remainder term estimates arising from our exact results and the known/folklore results for a uniformly random permutation from A_N or A_N^c.

3. Distribution of the number of cycles in σ

To stress dependence of σ on k, in this section we will write $\sigma^{(k)}$ instead of σ. Let $\nu(\sigma)$ denote the number of cycles in a permutation σ. We start with the following general formula.

Lemma 3.1. *The identity*

$$\begin{aligned} P(\nu(\sigma^{(k)}) = \nu) = (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \\ \times \sum_{\ell \geq 1} (-1)^\ell \frac{c(\ell,\nu)}{\ell!} \binom{N-\lambda_1}{N-\ell} \end{aligned} \tag{9}$$

holds.

Proof. Combination of (8), (6), and $\lambda^1 + \lambda_1 = N+1$ proves (9). $\square$

Corollary 3.2. *For $k \geq 2$, the identity*

$$P(\sigma^{(k)} = \mathrm{id}) = P(\nu(\sigma) = N) = \frac{1}{N!} \sum_{r=0}^{N-1} (-1)^{kr} \binom{N-1}{r}^{-k+2}, \tag{10}$$

holds.

Proof. Use Formula (9) and the fact that $c(\ell,\nu) = 0$ for $\ell < \nu$. $\square$

Note that Formula (10) appears as Equation (7.181) in [28]. In the special case of $k = 2$ Corollary 3.2 yields

$$\mathrm{P}(\sigma^{(2)} = \mathrm{id}) = \tfrac{1}{(N-1)!}. \tag{11}$$

This is an obvious result, since the inverse of a uniformly random cycle is again a uniformly random cycle.

The special case of $k = 3$ is not so obvious. However, combining (10) and the identity

$$\sum_{r=a}^{n} \frac{(-1)^r}{\binom{n}{r}} = \frac{n+1}{n+2} \left[\frac{(-1)^a}{\binom{n+1}{a}} + (-1)^n \right] \tag{12}$$

(see [30], [28, Equation (7.211)], [31]), we have a non-obvious answer

$$\mathrm{P}(\sigma^{(3)} = \mathrm{id}) = \frac{1 + (-1)^{N-1}}{(N-1)!\,(N+1)}, \tag{13}$$

see [28, Exercise 7.67 (d)].

The remarkable identity (12) followed from the elementary formula for a special case of the beta integral:

$$\binom{n}{r}^{-1} = (n+1) \int_0^1 t^r (1-t)^{n-r}\, dt. \tag{14}$$

Note that for even N, Equation (13) returns zero probability, and that is how it should be, since the product of three even cycles is an odd permutation, and therefore, cannot be the identity. Furthermore, since $\sigma^{(k)} = \sigma^{(k-1)}\sigma_k$, the product $\sigma^{(k)}$ is the identity if and only if $\sigma^{(k-1)} = (\sigma_k)^{-1}$, which is a maximal cycle. As $(\sigma_k)^{-1}$ is uniform on the set of all $(N-1)!$ maximal cycles, *and* independent of $\sigma^{(k-1)}$, we see then that

$$\mathrm{P}(\sigma^{(k-1)} \text{ is a cycle}) = (N-1)!\,\mathrm{P}(\sigma^{(k)} = \mathrm{id}). \tag{15}$$

In the special case of $k = 2$, we rediscover a result that has been proved several times, with different methods.

Corollary 3.3. *We have*

$$P(\sigma^{(2)} \text{ is a cycle}) = \frac{1 + (-1)^{N-1}}{N+1}. \tag{16}$$

Proof. This is immediate from Equations (13) and (15). □

For even N, the statement of Corollary 3.3 is obvious, since the product of two maximal cycles is an even permutation, and, hence, it cannot be an N-cycle for even N. For odd N, the result is equivalent to a well-known, but not at all obvious, fact that there are $\frac{2(N-1)!}{N+1}$ ways to factor a given maximal cycle into a product of two maximal cycles; see for instance [6] and the references therein. In general, Equations (10) and (15) imply the following.

Corollary 3.4. *For all positive integers k, we have*

$$P(\sigma^{(k)} \text{ is a cycle}) = \frac{1}{N}\sum_{r=0}^{N-1}(-1)^{(k+1)r}\binom{N-1}{r}^{-k+1}. \tag{17}$$

Further, it follows from (9) that, for every real number x, we have

$$\begin{aligned}
\mathrm{E}\big[x^{\nu(\sigma^{(k)})}\big] &= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \\
&\qquad \times \sum_{\ell\ge 1} \frac{(-1)^\ell}{\ell!}\binom{N-\lambda_1}{N-\ell} \sum_{\nu\ge 1} x^\nu c(\ell,\nu) \\
&= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \sum_{\ell\ge 1}\binom{N-\lambda_1}{N-\ell}\binom{-x}{\ell} \\
&= (-1)^N \sum_{\lambda_1=1}^{N} (-1)^{k(N-\lambda_1)} \binom{N-1}{N-\lambda_1}^{-k+2} \binom{N-\lambda_1-x}{N} \\
&= (-1)^N \sum_{r=0}^{N-1} (-1)^{kr}\binom{N-1}{r}^{-k+2}\binom{r-x}{N}. \qquad (18)
\end{aligned}$$

For a positive integer x, the non-zero contributions to the sum come from $r < \min\{N, x\}$. So, for instance,

$$\mathrm{E}\big[2^{\nu(\sigma^{(k)})}\big] = N + 1 + \frac{(-1)^k}{(N-1)^{k-2}}, \quad N > 1,$$

$$\mathrm{E}\big[3^{\nu(\sigma^{(k)})}\big] = 2(N+2)_2 - \frac{N+1}{(N-1)^{k-2}} + \binom{N-1}{2}^{-k+2}, \quad N > 2,$$

where we use the notation $(a)_b = a(a-1)\cdots(a-b+1)$, for integers $a \geq b \geq 0$.

For $k = 2$ and $x > N$, Equation (18) implies the following corollary.

Corollary 3.5. *We have*

$$\begin{aligned} E\big[x^{\nu(\sigma^{(2)})}\big] &= (-1)^N \sum_{\lambda_1=1}^{N} \binom{N-\lambda_1-x}{N} \\ &= \sum_{\lambda_1=1}^{N} \binom{\lambda_1+x-1}{N} = \sum_{j=N}^{N+x-1} \binom{j}{N} - \sum_{j=N}^{x-1} \binom{j}{N} \\ &= \binom{N+x}{N+1} - \binom{x}{N+1} = \binom{N+x}{N+1} + (-1)^N \binom{N-x}{N+1}. \end{aligned} \tag{19}$$

Of course, the identity (19) holds for all x. It is equivalent to Zagier's result, (see the Appendix by Zagier in [18]), stating that

$$\mathrm{P}(\nu(\sigma^{(2)}) = \nu) = (1 + (-1)^{N-\nu})\,[x^\nu]\binom{N+x}{N+1}.$$

For $k = 3$, we can prove the following (weighted) analogue of Corollary 3.5.

Corollary 3.6. *We have*

$$\sum_{N\geq 1} \frac{y^N}{N} E\big[x^{\nu(\sigma^{(3)}(N))}\big] = \int_0^1 \frac{(1-y(1-t))^{-x} - (1-y(1-t))^x}{1-yt(1-t)}\,dt; \tag{20}$$

here $\sigma^{(3)}(N)$ is the product of 3 random cycles of length N, and $|x| \leq 1$, $|y| < 1$.

Note that the right-hand side of (20) is an odd function of x. This should be expected, since–regardless of the parity of N–the number of cycles in $\sigma^{(3)}(N)$ is odd. In particular, using

$$(1-y(1-t))^{-x} - (1-y(1-t))^x = 2\sum_{r \text{ odd}} \frac{\log^r(1-y(1-t))^{-1}}{r!}\,x^r,$$

we obtain

$$\sum_{N\geq 1} \frac{y^N}{N} \mathrm{P}(\sigma^{(3)}(N) \text{ has } r \text{ cycles}) = \frac{2}{r!}\int_0^1 \frac{\log^r\big(1-y(1-t)\big)^{-1}}{1-yt(1-t)}\,dt. \tag{21}$$

Here is a probabilistic interpretation of this identity. For $y\in(0,1)$, introduce an integer-valued random variable $\mathcal{N}$ such that

$$\mathrm{P}(\mathcal{N}=N)=\frac{y^N}{N}\log(1-y)^{-1},\quad N\ge 1,$$

i.e., $\mathcal{N}$ has the logarithmic distribution with parameter y. So we (A) choose the random length $\mathcal{N}$ of the permutations, (B) conditioned on $\mathcal{N}$, generate, in order, three independent uniform cycles of length $\mathcal{N}$, and (C) compute their product $\sigma^{(3)}(\mathcal{N})$. Then identity (21) means that, for r odd,

$$\mathrm{P}(\sigma^{(3)}(\mathcal{N})\text{ has }r\text{ cycles})=\frac{2}{r!\log(1-y)^{-1}}\int_0^1\frac{\log^r\bigl(1-y(1-t)\bigr)^{-1}}{1-yt(1-t)}\,dt.$$

Proof of Corollary 3.6. Since both sides of (20) are analytic for $|y|<1$, it suffices to prove the identity for $|y|\le 1/3$. From (18), (14) and

$$(-1)^N\binom{r-x}{N}=[z^N](1-z)^{r-x},$$

we obtain

$$\begin{aligned}N^{-1}\mathrm{E}\bigl[x^{\nu(\sigma^{(3)}(N))}\bigr]&=[z^N]\sum_{r=0}^{N-1}(1-z)^{r-x}\int_0^1(1-t)^{N-1-r}t^r\,dt\\&=[z^N](1-z)^{-x}\int_0^1(1-t)^{N-1}\sum_{r=0}^{N-1}\left(-\frac{(1-z)t}{1-t}\right)^r dt\\&=[z^N](1-z)^{-x}\int_0^1\frac{(1-t)^N+(-1)^{N+1}\bigl((1-z)t\bigr)^N}{1-tz}\,dt.\end{aligned}$$

Next

$$y^N(1-t)^N[z^N]\,\frac{(1-z)^{-x}}{1-tz}=[z^N]\,\frac{\bigl(1-(1-t)yz\bigr)^{-x}}{1-t(1-t)yz};$$

so

$$\begin{aligned}\int_0^1\sum_{N\ge1}y^N(1-t)^N[z^N]\,\frac{(1-z)^{-x}}{1-tz}\,dt&=\int_0^1\sum_{N\ge1}[z^N]\,\frac{\bigl(1-(1-t)yz\bigr)^{-x}}{1-t(1-t)yz}\,dt\\&=\int_0^1\frac{\bigl(1-(1-t)y\bigr)^{-x}}{1-t(1-t)y}\,dt-1.\end{aligned}\tag{22}$$

Further, by the Cauchy integral formula,

$$\begin{aligned}y^N[z^N]\frac{(1-z)^{-x}}{1-tz}\bigl((1-z)t\bigr)^N&=\frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z^{N+1}(1-tz)}\bigl(y(1-z)t\bigr)^N\,dz\\&=\frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\left(\frac{y(1-z)t}{z}\right)^N\,dz.\end{aligned}$$

On the circle $|z| = 2/3$, we have $\left|\frac{y(1-z)t}{z}\right| \le 5|y|/2 \le 5/6$; so summing over $N \ge 1$,

$$\begin{aligned}\sum_{N\ge 1}(-1)^{N+1}y^N[z^N]&\frac{(1-z)^{-x}}{1-tz}\big((1-z)t\big)^N\\ &= \frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\frac{\frac{y(1-z)t}{z}}{1+\frac{y(1-z)t}{z}}\,dz\\ &= \frac{1}{2\pi i}\oint_{|z|=2/3}\frac{(1-z)^{-x}}{z(1-tz)}\cdot\frac{y(1-z)t}{z+y(1-z)t}\,dz.\end{aligned}$$

For $t > 0$, in the circle $|z| \le 2/3$ the integrand has two poles, both simple, at $z = 0$ and $z = -\frac{yt}{1-yt}$, with respective residues equal to 1 and $-\frac{(1-yt)^x}{1-yt(1-t)}$. Thus

$$\sum_{N\ge 1}(-1)^{N+1}y^N[z^N]\frac{(1-z)^{-x}}{1-tz}\big((1-z)t\big)^N = 1-\frac{(1-yt)^x}{1-yt(1-t)}.$$

Integrating for $t \in [0,1]$ and adding to (22), we obtain

$$\sum_{N\ge 1}\frac{y^N}{N}\mathrm{E}\big[x^{\nu(\sigma^{(3)}(N))}\big] = \int_0^1\frac{(1-y(1-t))^{-x}-(1-yt)^x}{1-yt(1-t)}\,dt,$$

which is equivalent to (20), as $t(1-t)$ is symmetric with respect to $t = 1/2$. As a partial check, the integral on the right-hand side equals $\log(1-y)^{-1}$ if $x = 1$. □

Our final result in this section is a relatively compact integral formula for $P_n(\boldsymbol{\nu})$, the probability that $\sigma^{(2)}$ has ν_ℓ cycles of length ℓ, where $1 \le \ell \le n$, for *arbitrary* $\boldsymbol{\nu}$, i.e., satisfying the only constraint $\sum_\ell \ell\nu_\ell = N$. Since $\sigma^{(2)}$ is even, $P_n(\boldsymbol{\nu}) = 0$ if $\sum_{\ell\text{ even}}\nu_\ell$ is odd.

Theorem 3.7. *We have*

$$P_n(\boldsymbol{\nu}) = \frac{N}{\prod_\ell \ell^{\nu_\ell}\nu_\ell!}\int_0^1\prod_{\ell\ge 1}\big[t^\ell+(-1)^{\ell+1}(1-t)^\ell\big]^{\nu_\ell}\,dt.$$

Proof. First, the number of permutations s with cycle structure $\boldsymbol{\nu}$ is equal to $N!/\prod_\ell \ell^{\nu_\ell}\nu_\ell!$. Furthermore, for every such permutation s, Formula (7) implies that $\lambda^1+\lambda_1 = N+1$ and $\nu = \sum_\ell \nu_\ell$. Setting $r = N-\lambda_1$, and choosing a positive ρ, we obtain

$$\chi^{\lambda^*}(s) = (-1)^{N+r}\frac{1}{2\pi i}\oint_{|\xi|=\rho}\frac{1}{\xi^{r+1}(1-\xi)}\prod_{\ell\ge 1}(\xi^\ell-1)^{\nu_\ell}\,d\xi.$$

Here the circular contour is traversed counter-clockwise, and ρ is arbitrary, as the integrand is singular at $\xi = 0$ only. Substituting $\xi = 1/\eta$, we have

$$\chi^{\lambda^*}(s) = (-1)^{N+r}\frac{1}{2\pi i}\oint\limits_{|\eta|=1/\rho} \frac{1}{\eta^{N-r}(\eta-1)}\prod_{\ell\ge 1}(1-\eta^{\ell})^{\nu_\ell}\,d\eta,$$

with the contour traversed counter-clockwise again. Substituting this formula in Equation (6), and using (14), we have

$$\begin{aligned}
P_{\sigma^{(2)}}(s) &= \frac{(-1)^N}{N!}\sum_{r=0}^{N-1}\binom{N-1}{r}^{-1}\chi^{\lambda^*}(s)\\
&= \frac{(-1)^N N}{N!}\sum_{r=0}^{N-1}\chi^{\lambda^*}(s)\int_0^1 t^r(1-t)^{N-1-r}\,dt\\
&= \frac{(-1)^N}{(N-1)!}\,\frac{1}{2\pi i}\oint\limits_{|\eta|=1/\rho}\left(\int_0^1(1-t)^{N-1}\sum_{r=0}^{N-1}\left(-\frac{t\eta}{1-t}\right)^r dt\right)\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta\\
&= \frac{(-1)^N}{(N-1)!}\int_0^1\left(\frac{1}{2\pi i}\oint\limits_{|\eta|=1/\rho}\frac{(1-t)^N-(-t\eta)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta\right)dt.
\end{aligned}$$

Pick $\varepsilon\in(0,1)$ and consider $t\le 1-\varepsilon$. Choose $\rho>(1-\varepsilon)/\varepsilon$. For this ρ, the internal integrand has two singular points, $\eta=0$ and $\eta=-(1-t)/t$, respectively within and outside of the integration contour. Crucially,

$$\frac{-(-t\eta)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)} = \frac{-(-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta-1}$$

has no singularity at $\eta=0$, and for $t>0$, we have

$$\frac{(1-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)} = O(|\eta|^{-2}),\quad |\eta|\to\infty,$$

as $\sum_\ell \ell\nu_\ell = N$. So, by the residue theorem, the internal integral equals

$$\begin{aligned}
\frac{1}{2\pi i}\oint\limits_{|\eta|=1/\rho}&\frac{(1-t)^N}{1-t+t\eta}\cdot\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\,d\eta\\
&= -t^{-1}\,(1-t)^N\cdot\left.\frac{\prod_\ell(1-\eta^\ell)^{\nu_\ell}}{\eta^N(\eta-1)}\right|_{\eta=-\frac{1-t}{t}}\\
&= (-1)^N\prod_{\ell\ge 1}\left[t^\ell+(-1)^{\ell+1}(1-t)^\ell\right]^{\nu_\ell},
\end{aligned}$$

for all $0 < t \le 1-\varepsilon$. Letting $\varepsilon \to 0$, we obtain

$$P_{\sigma^{(2)}}(s) = \frac{1}{(N-1)!}\int_0^1 \prod_{\ell\ge 1}\big[t^\ell + (-1)^{\ell+1}(1-t)^\ell\big]^{\nu_\ell}\,dt.$$

Multiplying the result by $N!/\prod_\ell \ell^{\nu_\ell}\nu_\ell!$ we complete the proof. $\square$

Corollary 3.8. *Let $P_{N,r}$ denote the probability that all cycles of $\sigma^{(2)}$ are of the same length $r\ge 2$, i.e., $N\equiv 0$ (mod r) and $\nu_r = N/r$. (So $P_{N,r}=0$ if r is even and $N\not\equiv 0 \pmod{2r}$.) Then*

$$\begin{aligned} P_{N,r} &= \frac{N}{r^{N/r}(N/r)!}\int_0^1 \big[t^r + (-1)^{r+1}(1-t)^r\big]^{N/r}\,dt \\ &= \frac{N}{(N+1)r^{N/r}(N/r)!}\sum_{\substack{0\le j\le N\\ j\equiv 0 \pmod r}} (-1)^{j(r+1)/r}\frac{\binom{N/r}{j/r}}{\binom{N}{j}}. \end{aligned} \tag{23}$$

In particular,

$$P_{N,2} = \frac{\big(1+(-1)^{N/2}\big)N/2}{2^{N/2}(N/2+1)!},\qquad P_{N,3} = \frac{N}{(N/3)!\,(12)^{N/3}}\sum_{j=0}^{N/3}\binom{N/3}{j}\frac{3^j}{2j+1}, \tag{24}$$

$$P_{N,4} = \frac{\big(1+(-1)^{N/4}\big)N/2}{8^{N/4}(N/4)!}\sum_{j=0}^{N/4}\binom{N/4}{j}\frac{1}{N/4+2j+1}. \tag{25}$$

Proof. The second identity in (23) follows from the binomial formula for the integrand $\big[t^r+(-1)^{r+1}(1-t)^r\big]^{N/r}$, and summation of the resulting beta integrals. Furthermore, denoting $u=2t-1$, we have

$$\big[t^r+(-1)^{r+1}(1-t)^r\big]^{N/r} = \begin{cases} u^{N/2}, & r=2,\\ \left(\frac{3u^2+1}{4}\right)^{N/3}, & r=3,\\ \left(\frac{u^3+u}{2}\right)^{N/4}, & r=4.\end{cases}$$

Switching to $u=2t-1$ in the integral in (23), using the binomial formula and integrating, we obtain Formulas (24)–(25). Using the same device for $r>4$ would have necessitated the multinomial formula, leading to multifold sums, progressively more complex than the formula in the second line of (23). $\square$

Corollary 3.9. *For all positive integers N, we have*

$$P(\sigma^{(2)} \text{ is an involution}) = N\sum_{\substack{\nu_1+2\nu_2=N\\ \nu_2 \text{ even}}}\frac{1}{\nu_1!\,2^{\nu_2}(\nu_2+1)!}.$$

The identities equivalent to (24) were proved in [10] by using the sum-type formulas for the total number of ways to represent a maximal cycle as a product of a maximal cycle and a permutation from a given conjugacy class, see [13], [27], [14]. The sequence $(N-1)!\,P_{N,2}$ is listed by Sloane as A035319, and known as the counts of certain rooted maps, see [32]. The sequence $(N-1)!\,P_{N,3}$ is listed in Sloane as A178217.

4. Probability that the occupancy numbers of the cycles of σ by the elements of $[\ell]$ belong to a given set

In the section title and elsewhere below σ is $\sigma^{(k)}$, the product of k random maximal cycles. Let $A \subseteq \mathbb{Z}_{\ge 0}$ be given. Recall that $p_A(N,\ell;k)$ is the probability that the number of elements of $[\ell]$ in each cycle of σ belongs to the set A. Let $\mathcal{A}$ denote the set of all permutations s with this property, so that $p_A(N,\ell;k) = P_\sigma(s \in \mathcal{A})$.

The examples include: (1) $A_1 = \mathbb{Z}_{>0}$; each cycle must contain at least one element of $[\ell]$; (2) $A_2 = \{0,\ell\}$; one of the cycles of σ contains the whole set $[\ell]$; (3) $A_3 = \{0,1\}$; each element of $[\ell]$ belongs to a distinct cycle of σ.

The problem of finding an explicit formula for $p_A(N,\ell;k)$ was solved by Stanley [29] in the special cases of $k=2$, $\ell=2$ and $A=\{0,2\}$ or $A=\{0,1\}$. Recently, Bernardi et al. [3] solved the case $k=2$, $A=A_3$ for $\ell\ge 2$. In fact they solved a general problem of separation probability for t disjoint sets $\mathcal{S}_1,\dots,\mathcal{S}_t$. We will present an alternative solution for this problem.

To evaluate $p_A(N,\ell;k)$, consider first $Q_A(\vec{\nu},\ell)$, the total number of permutations s of $[N]$, with $\vec{\nu}(s) = \{\nu_r(s)\} = \{\nu_r\} = \vec{\nu}$, such that the number of elements of $[\ell]$ in every cycle is an element of A. The reason we need $Q_A(\vec{\nu},\ell)$ is that the key formula (7) expresses $\chi^{\lambda *}(s)$ through the cycle counts $\nu_r(s)$, $r\ge 1$.

Theorem 4.1. *For all $\ell \ge 2$, we have*

$$Q_A(\vec{\nu},\ell) = (N-\ell)!\,\ell!\,[w^\ell]\prod_r \frac{1}{\nu_r!}\left(\frac{\sum_{a\in A}\binom{r}{a}w^a}{r}\right)^{\nu_r}. \tag{26}$$

Proof. To evaluate $Q_A(\vec{\nu},\ell)$, it suffices to evaluate its counterpart $\mathcal{Q}_A(\vec{\nu},\ell)$ that corresponds to the case when cycles of the same length are ordered, because

$$Q_A(\vec{\nu},\ell) = \frac{\mathcal{Q}_A(\vec{\nu},\ell)}{\prod_r \nu_r!}.$$

Introduce the integers $a_{r,j}\ge 0$, $b_{r,j}\ge 0$ that stand for the generic numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ in the j-th cycle of length r, ($j\in[\nu_r]$). Let $\mathbf{a}$ (respectively $\mathbf{b}$) denote the sequence of the numbers $a_{r,j}$ (respectively $b_{r,j}$), in increasing order of the indices r, and (for each r) in increasing order of the indices j. For $\mathbf{a}$, $\mathbf{b}$ to be admissible we must have

$$a_{r,j}+b_{r,j}=r, \tag{27}$$

$$a_{r,j}\in A, \tag{28}$$

$$\sum_{r,j} a_{r,j}=\ell, \quad \sum_{r,j} b_{r,j}=N-\ell. \tag{29}$$

Given $\mathbf{a}$ and $\mathbf{b}$, the number of ways to partition $[\ell]$ into the subsets of cardinality $a_{r,j}$ and to partition $[N]\setminus[\ell]$ into the subsets of cardinality $b_{r,j}$ is

$$\frac{\ell!}{\prod_{r,j} a_{r,j}!}\cdot\frac{(N-\ell)!}{\prod_{r,j} b_{r,j}!}.$$

(Here the subsets form two ordered sequences, each in increasing order of r, and (given r) in increasing order of $j\in[\nu_r]$.) Given two such set partitions of $[\ell]$ and $N\setminus[\ell]$, the number of ways to form a cycle from the (r,j)-th set belonging to the first partition and the (r,j)-th set belonging to the second partition is $(a_{r,j}+b_{r,j}-1)!=(r-1)!$. Therefore

(30)

$$\begin{aligned}\mathcal{Q}_A(\vec{\nu},\ell) &= (N-\ell)!\,\ell! \sum_{\substack{\mathbf{a},\mathbf{b}\text{ meet}\\ (27),(28),(29)}} \prod_r ((r-1)!)^{\nu_r} \prod_{j\in[\nu_r]} \frac{1}{a_{r,j}!\,b_{r,j}!}\\ &= (N-\ell)!\,\ell!\,[w^\ell]\prod_r \frac{1}{r^{\nu_r}} \prod_{j\in[\nu_r]} \sum_{a_{r,j}\in A} \binom{r}{a_{r,j}} w^{a_{r,j}}\\ &= (N-\ell)!\,\ell!\ [w^\ell]\prod_r \left(\frac{\sum_{a\in A}\binom{r}{a}w^a}{r}\right)^{\nu_r}. \qquad \square\end{aligned}$$

Let $\mathcal{A}(\vec{\nu}):=\{s\in\mathcal{A}:\vec{\nu}(s)=\vec{\nu}\}$. Then, using (7), $\nu=\sum_r \nu_r$ and (26), we conclude that

$$\begin{aligned}&\chi^{\lambda^*}(\mathcal{A}(\vec{\nu})) = (-1)^{\lambda^1+\nu} Q_A(\vec{\nu},\ell)\cdot[\xi^{\lambda_1}]\frac{\xi}{1-\xi}\prod_{r\ge 1}(1-\xi^r)^{\nu_r}\\ &= (-1)^{\lambda^1}(N-\ell)!\,\ell!\cdot[\xi^{\lambda_1}w^\ell]\ \frac{\xi}{1-\xi}\prod_r \frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}.\end{aligned}$$

This identity implies

$$\begin{aligned}\chi^{\lambda^*}(\mathcal{A}) &= (-1)^{\lambda^1}(N-\ell)!\,\ell!\\ &\times[\xi^{\lambda_1}w^\ell]\ \frac{\xi}{1-\xi} \sum_{\substack{\vec{\nu}:\\ 1\nu_1+2\nu_2+\cdots=N}} \prod_r \frac{1}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}.\end{aligned} \tag{31}$$

The expression in the second line of (31) equals

$$
\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\sum_{\vec{\nu}\geq\mathbf{0}}\prod_r \frac{(x^r)^{\nu_r}}{\nu_r!}\left(-\frac{(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}\\
&\qquad=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\sum_{\nu_r\geq 0}\frac{1}{\nu_r!}\left(-\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)^{\nu_r}\\
&\qquad=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\prod_r\exp\left(-\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right)\\
&\qquad=[\xi^{\lambda_1} w^\ell x^N]\frac{\xi}{1-\xi}\exp\left(-\sum_{r\geq 1}\frac{x^r(1-\xi^r)\left(\sum_{a\in A}\binom{r}{a}w^a\right)}{r}\right). \qquad (32)
\end{aligned}
$$

4.1. **Probability that each cycle of σ contains at least one element of $[\ell]$.** In this case $A = A_1 = \mathbb{Z}_{>0}$. Therefore

$$\sum_{a\in A}\binom{r}{a}w^a = (1+w)^r - 1.$$

In this section, we prove the following result and discuss some of its special cases.

Theorem 4.2. *For all positive integers ℓ and k, we have*

$$p_{A_1}(N,\ell;k) = \binom{N}{\ell}^{-1}\sum_{\lambda_1=\ell}^{N}(-1)^{(k-1)(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+1}\binom{\lambda_1-1}{\ell-1}. \qquad (33)$$

Proof. Using (31), (32) and $\sum_{j\geq 1} z^j/j = -\log(1-z)$, $|z|<1$, we obtain

$$
\begin{aligned}
\chi^{\lambda^*}(\mathcal{A}) &= (-1)^{\lambda^1}(N-\ell)!\,\ell!\\
&\quad\times[\xi^{\lambda_1} w^\ell x^N]\,\frac{\xi}{1-\xi}\,\frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)}. \qquad (34)
\end{aligned}
$$

Let us simplify this formula. Write

$$
\begin{aligned}
[w^\ell]\frac{1-x(1+w)}{1-\xi x(1+w)} &= \frac{\xi-1}{\xi}[w^\ell]\frac{1}{1-\xi x(1+w)}\\
&= \frac{1-\xi}{\xi^2 x}\,[w^\ell]\left(w-\frac{1-\xi x}{\xi x}\right)^{-1}\\
&= \frac{1-\xi}{\xi^2 x}\binom{-1}{\ell}(-1)^{-1-\ell}\left(\frac{1-\xi x}{\xi x}\right)^{-1-\ell}\\
&= -\frac{1-\xi}{\xi^2 x}\left(\frac{\xi x}{1-\xi x}\right)^{1+\ell}.
\end{aligned}
$$

Therefore

$$[\xi^{\lambda_1} w^\ell x^N]\ \frac{\xi}{1-\xi}\ \frac{\big(1-x(1+w)\big)(1-\xi x)}{\big(1-\xi x(1+w)\big)(1-x)} = -[\xi^{\lambda_1} x^N]\,(1-x)^{-1}\left(\frac{\xi x}{1-\xi x}\right)^\ell$$
$$= -[x^N]\,x^{\lambda_1}(1-x)^{-1}\cdot[y^{\lambda_1}]\left(\frac{y}{1-y}\right)^\ell$$
$$= -[y^{\lambda_1-k}](1-y)^{-\ell} = -\binom{\lambda_1-1}{\lambda_1-\ell},$$

where $\binom{a}{b}=0$ for $b<0$. So (34) becomes

$$\chi^{\lambda^*}(\mathcal{A}) = (-1)^{\lambda^1-1}(N-\ell)!\,\ell!\binom{\lambda_1-1}{\lambda_1-\ell}. \tag{35}$$

Combining (35) and (6) we conclude that

$$\begin{aligned} p_{A_1}(N,\ell;k) &= \frac{1}{N!}\sum_{\lambda^*}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\chi^{\lambda^*}(\mathcal{A}) \\ &= \binom{N}{\ell}^{-1}\sum_{\lambda_1=\ell}^{N}(-1)^{(k-1)(N-\lambda_1)}\binom{N-1}{N-\lambda_1}^{-k+1}\binom{\lambda_1-1}{\ell-1}, \end{aligned} \tag{36}$$

which was to be proved. □

Note that as $N\to\infty$, the dominant contribution to the right-hand side in (36) comes from $\lambda_1 = N$, so that $p_{A_1}(N,\ell;k) = \ell/N + O(N^{-2\ell+1})$; the formula is useful for $\ell > 1$. (Of course, $p_{A_1}(N,1;k) = P(\sigma^{(k)}$ is a cycle), see (17).) We remark that ℓ/N is the exact probability that every cycle of the uniformly random permutation of $[N]$ contains at least one element of $[\ell]$; see [21, Section 3, Exercise 6].

Theorem 4.3. *For all positive integers ℓ, we have*

$$\begin{aligned} p_{A_1}(N,\ell;2) &= (-1)^{N-1}N\binom{N}{\ell}^{-1} \\ &\quad\times\left[\frac{(-1)^{\ell-1}}{(N+\ell)\binom{N+\ell-1}{\ell-1}} + \sum_{j=0}^{\ell-1}\frac{(-1)^{N+\ell-j}\binom{N}{j}}{N+\ell-j}\right] \\ &= (-1)^{N+\ell}N\binom{N}{\ell}^{-1}\sum_{i=0}^{N-\ell}(-1)^i\binom{N}{i}\frac{1}{i+\ell}. \end{aligned} \tag{37}$$

Remark. The first (respectively second) identity is computationally efficient when ℓ (respectively $N-\ell$) is moderately valued.

Proof. For the proof we need a certain binomial identity. Introduce

$$S_{n,a,b} = \sum_{r=a+b}^{n}(-1)^r\frac{\binom{r-a}{b}}{\binom{n}{r}}. \tag{38}$$

This function is relevant since (36) is equivalent to

$$p_{A_1}(N,\ell;2) = (-1)^{N-1}\binom{N}{\ell}^{-1} S_{N-1,0,\ell-1}. \tag{39}$$

As we mentioned earlier,

$$S_{n,0,0} = (1+(-1)^n)\,\frac{n+1}{n+2}, \tag{40}$$

(see [30], [28], [31]), and the key element of the proofs was identity (14).

In fact, in [30] the equation (14) was used to derive a sum-type formula, still with $n+1$ terms, for

$$\sum_{r=0}^{n}(-1)^r\,\frac{x^r}{\binom{n}{r}},$$

that yielded (40) via setting $x=1$. We also use (14) but avoid an intermediate sum with $n+1$ terms, and instead differentiate the resulting integral with respect to the parameter x. Here are the details. For $a+b\le n$, define

$$\mathcal{S}_{n,a,b}(x) := \sum_{r=a+b}^{n}(-1)^r\,\frac{x^{r-a}}{\binom{n}{r}}.$$

Observe that $S_{n,a,b} = \frac{1}{b!}\,\frac{d^b\mathcal{S}_{n,a,b}(x)}{dx^b}\Big|_{x=1}$. Now

$$\begin{aligned}
\mathcal{S}_{n,a,b}(x) &= (n+1)\int_0^1\left(\sum_{r=a+b}^{n}(-1)^r x^{r-a}t^r(1-t)^{n-r}\right)dt\\
&= (n+1)(-1)^{a+b}\int_0^1 t^{a+b}(1-t)^{n-a-b}x^b\sum_{r=a+b}^{n}\left(-\frac{xt}{1-t}\right)^{r-a-b}dt\\
&= (n+1)(-1)^{a+b}\int_0^1 \frac{x^b+(-1)^{n-a-b}x^{n-a+1}\left(\frac{t}{1-t}\right)^{n-a-b+1}}{1+\frac{xt}{1-t}}\\
&\quad\times t^{a+b}(1-t)^{n-a-b}\,dt.
\end{aligned} \tag{41}$$

To compute $(d^b/dx^b)\mathcal{S}_{n,a,b}(x)$ at $x=1$, we differentiate b times the right-hand side of (41) with respect to x by carrying the operation inside the integral and then setting $x=1$. So

$$\begin{aligned}
\frac{d^b\mathcal{S}(n,a,x)}{dx^b}\Big|_{x=1} &= (-1)^{a+b}(n+1)\\
\times\int_0^1 \frac{\partial^b}{\partial x^b}\,&\frac{x^b+(-1)^{n-a-b}x^{n-a+1}\left(\frac{t}{1-t}\right)^{n-a-b+1}}{1+\frac{xt}{1-t}}\Bigg|_{x=1} t^{a+b}(1-t)^{n-a-b}\,dt.
\end{aligned} \tag{42}$$

By $(uv)^{(b)} = \sum_j \binom{b}{j} u^{(j)} v^{(b-j)}$, the partial derivative at $x = 1$ is

$$\sum_{j=0}^{b} \binom{b}{j} \left((b)_j + (-1)^{n-a-b} \left(\tfrac{t}{1-t}\right)^{n-a-b+1} (n-a+1)_j \right)$$
$$\times (-1)^{b-j} \frac{(b-j)!}{\left(1+\frac{t}{1-t}\right)^{b-j+1}} \cdot \left(\frac{t}{1-t}\right)^{b-j}$$
$$= \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j}$$
$$\times \left[b!\, t^{b-j}(1-t) + (-1)^{n-a-b}(b-j)!\,(n-a+1)_j \frac{t^{n-a+1-j}}{(1-t)^{n-a-b}} \right]$$
$$= b! \left[(1-t)^{b+1} + \sum_{j=0}^{b} (-1)^{n-a-j} \binom{n-a+1}{j} \frac{t^{n-a+1-j}}{(1-t)^{n-a-b}} \right].$$

Substituting the last expression in (42) and using (14) we obtain

$$(43)\qquad \begin{aligned} S_{n,a,b} = \sum_{r=a+b}^{n} (-1)^r \frac{\binom{r-a}{b}}{\binom{n}{r}} = (n+1) \Bigg[& \frac{(-1)^{a+b}}{(n+2+b)\binom{n+b+1}{a+b}} \\ & + \sum_{j=0}^{b} (-1)^{n+b-j} \binom{n-a+1}{j} \frac{1}{n+2+b-j} \Bigg]. \end{aligned}$$

Using yet another identity from [31], namely

$$\sum_{j=0}^{u} (-1)^j \binom{u}{j} \frac{1}{v+j+1} = \frac{1}{(u+v+1)\binom{u+v}{v}},$$

we transform Equation (43) into

$$(44)\qquad S_{n,a,b} = (-1)^{a+b}(n+1) \sum_{i=0}^{n-a-b} (-1)^i \binom{n-a+1}{i} \frac{1}{i+a+b+1}.$$

So, applying Formulas (43), (44) for $n = N-1$, $a = 0$ and $b = \ell - 1$, and using (39), we immediately obtain the double identity (37). □

Remark. A hard-working reviewer pointed out that just equivalence of (33) and (37) follows from classic transformation formulas in the theory of hypergeometric series, see [12], [2]. However, neither (33) nor (37) were known before.

Example 4.4. Using the first expression in (37), we obtain

$$p_{A_1}(N, 1; 2) = \begin{cases} \frac{2}{N+1}, & \text{if } N \text{ is odd}, \\ 0, & \text{if } N \text{ is even}. \end{cases}$$

This is equivalent to the result already mentioned in Section 3, since $p_{A_1}(N,1;2)$ is indeed equal to the probability that σ is a maximal cycle.

4.2. **Probability that the elements $1,\dots,\ell$ are in the same cycle of σ.** This time $A = A_2 = \{0,\ell\}$, so that

$$\sum_{a\in A_2}\binom{r}{a}w^a = 1+\binom{r}{\ell}w^{\ell}. \tag{45}$$

Our goal in this section is to prove the following theorem and its special case of $k=2$.

Theorem 4.5. *For all integers $\ell \geq 2$, we have*

$$\begin{aligned}p_{A_2}(N,\ell;k) &= \frac{1}{N!}\sum_{\lambda_1=1}^{N}(-1)^{k(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\sum_{s\text{ admissible}}\chi^{\lambda^*}(s)\\ &= \frac{1}{\ell}\binom{N}{\ell}^{-1}\sum_{\lambda_1}(-1)^{(k+1)(\lambda^1-1)}\binom{N-1}{\lambda_1-1}^{-k+1}\\ &\quad\times\left\{1_{\{\lambda_1<N\}}\left[\binom{N-1}{\ell-1}-\binom{N-\lambda_1-1}{\ell-1}\right]+1_{\{\lambda_1=N\}}\binom{N}{\ell}\right\}.\end{aligned}$$

Proof. By and large, the proof is similar to that in the case of A_1, but the computation is more involved. The reader may want to revisit the proofs of Formulas (31) and (32) before proceeding further.

Formula (45) implies

$$Q_{A_2}(\vec{\nu},\ell) = (N-\ell)!\,\ell!\;[w^{\ell}]\prod_r\frac{1}{\nu_r!}\left(\tfrac{1+\binom{r}{\ell}w^{\ell}}{r}\right)^{\nu_r}. \tag{46}$$

So, using (31), (32) and $\nu = \sum_r \nu_r$, we conclude that

$$\begin{aligned}\chi^{\lambda^*}(\mathcal{A}) &= (-1)^{\lambda^1}(N-\ell)!\,\ell!\\ &\quad\times[\xi^{\lambda_1}w^{\ell}]\;\frac{\xi}{1-\xi}\sum_{\substack{\vec{\nu}:\\ 1\nu_1+2\nu_2+\cdots=N}}\prod_r\frac{1}{\nu_r!}\left(-(1-\xi^r)\frac{1+\binom{r}{\ell}w^{\ell}}{r}\right)^{\nu_r}.\end{aligned} \tag{47}$$

Since $\sum_r r\nu_r = N$, using the same intermediate steps as in the proof of (32), we see that the expression in the second line of (47) equals

$$\begin{aligned}&[\xi^{\lambda_1}w^{\ell}x^N]\frac{\xi}{1-\xi}\sum_{\vec{\nu}\geq\mathbf{0}}\prod_r\frac{(x^r)^{\nu_r}}{\nu_r!}\left(-(1-\xi^r)\frac{1+\binom{r}{\ell}w^{\ell}}{r}\right)^{\nu_r}\\ &\qquad= [\xi^{\lambda_1}w^{\ell}x^N]\frac{\xi}{1-\xi}\,\exp\left(-\sum_{r\geq1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^{\ell}}{r}\right).\end{aligned}$$

Here, using the identities

$$\sum_{b\ge a}\binom{b}{a}z^b=\frac{z^a}{(1-z)^{a+1}},\quad \sum_{r>1}\frac{z^r}{r}=-\log(1-z),$$

we obtain

$$\begin{aligned}\sum_{r\ge1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}&\\ &=-\log(1-x)+\log(1-x\xi)+\frac{w^\ell}{\ell}\sum_{r\ge1}\binom{r-1}{\ell-1}\left(x^r-(x\xi)^r\right)\\ &=\log\frac{1-x\xi}{1-x}+\frac{w^\ell}{\ell}\left(\frac{x^\ell}{(1-x)^\ell}-\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\right).\end{aligned}$$

Consequently, we have

$$\begin{aligned}[w^\ell]\exp\left(-\sum_{r\ge1}x^r(1-\xi^r)\frac{1+\binom{r}{\ell}w^\ell}{r}\right)&\\ &=\frac{1}{\ell}\frac{1-x}{1-x\xi}\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right).\end{aligned}$$

Therefore the expression in the second line of (47) is equal to

$$\begin{aligned}\frac{1}{\ell}\,[\xi^{\lambda_1}x^N]\,\frac{\xi}{1-\xi}\cdot\frac{1-x}{1-x\xi}&\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)\\ &=\frac{1}{\ell}\,[\xi^{\lambda_1}x^N]\left(\frac{1}{1-\xi}-\frac{1}{1-x\xi}\right)\left(\frac{(x\xi)^\ell}{(1-x\xi)^\ell}-\frac{x^\ell}{(1-x)^\ell}\right)\\ &=:\frac{1}{\ell}(T_1+T_2+T_3+T_4).\end{aligned}$$

Here,

$$\begin{aligned}T_1&=[\xi^{\lambda_1}x^N]\,\frac{1}{1-\xi}\cdot\frac{(x\xi)^\ell}{(1-x\xi)^\ell}\\ &=[\xi^{\lambda_1}]\,\frac{\xi^N}{1-\xi}\,[y^N]\frac{y^\ell}{(1-y)^\ell}=1_{\{\lambda_1=N\}}\binom{N-1}{\ell-1};\end{aligned}\tag{48}$$

next

$$\begin{aligned}T_2&=-[\xi^{\lambda_1}x^N]\,\frac{1}{1-\xi}\cdot\frac{x^\ell}{(1-x)^\ell}\\ &=-[x^{N-\ell}]\,\frac{1}{(1-x)^\ell}=-\binom{N-1}{\ell-1};\end{aligned}\tag{49}$$

next

$$
\begin{aligned}
T_3 &= -[\xi^{\lambda_1} x^N] \, \frac{(x\xi)^\ell}{(1-x\xi)^{\ell+1}} \\
&= -1_{\{\lambda_1 = N\}} \, [y^{N-\ell}] \, \frac{1}{(1-y)^{\ell+1}} = -1_{\{\lambda_1=N\}} \binom{N}{\ell};
\end{aligned} \tag{50}
$$

and finally

$$
\begin{aligned}
T_4 &= [\xi^{\lambda_1} x^N] \, \frac{1}{1-x\xi} \, \frac{x^\ell}{(1-x)^\ell} \\
&= [x^N] \, \frac{x^{\lambda_1+\ell}}{(1-x)^\ell} = [x^{N-\lambda_1-\ell}] \, \frac{1}{(1-x)^\ell} \\
&= 1_{\{\lambda_1 < N\}} \binom{N-\lambda_1-1}{\ell-1}.
\end{aligned} \tag{51}
$$

It follows from (48), (49), (50) and (51) that

$$
\begin{aligned}
&\frac{1}{\ell}(T_1+T_2+T_3+T_4) \\
&\qquad = -\frac{1}{\ell}\left\{ 1_{\{\lambda_1<N\}} \left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1} \right] + 1_{\{\lambda_1=N\}} \binom{N}{\ell} \right\}.
\end{aligned}
$$

So (47) becomes

$$
\begin{aligned}
\chi^{\lambda^*}(\mathcal{A}) &= (-1)^{\lambda^1-1}(N-\ell)!\,\ell! \\
&\times \frac{1}{\ell}\left\{ 1_{\{\lambda_1<N\}} \left[\binom{N-1}{\ell-1} - \binom{N-\lambda_1-1}{\ell-1} \right] + 1_{\{\lambda_1=N\}} \binom{N}{\ell} \right\}.
\end{aligned} \tag{52}
$$

Combining (52) and (6), we obtain the statement that was to be proved. □

Corollary 4.6. *For all integers $\ell \geq 2$, we have*

$$
\begin{aligned}
p_{A_2}(N,\ell;2) &= \frac{1}{\ell} - \frac{1}{N(N+1)} \\
&\quad + \frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}} \left[\binom{N+\ell}{\ell}^{-1} + N \sum_{j=0}^{\ell-2} (-1)^{N-j} \binom{N-1}{j} \frac{1}{N+\ell-j} \right] \\
&= \frac{1}{\ell} - \frac{1}{N(N+1)} \\
&\quad + (-1)^{\ell+1} \binom{N-1}{\ell-1}^{-1} \sum_{i=0}^{N-\ell} (-1)^i \binom{N-1}{i} \frac{1}{i+\ell+1}.
\end{aligned} \tag{53}
$$

Proof. By Theorem 4.5, for $k=2$ we have: with $r := N - \lambda_1$,

$$\begin{aligned} p_{A_2}(N,\ell;2) &= \frac{1}{\ell} + \frac{1}{\ell}\binom{N}{\ell}^{-1} \sum_{r=1}^{N-1} (-1)^r \\ &\qquad \times \binom{N-1}{r}^{-1} \left[\binom{N-1}{\ell-1} - \binom{r-1}{\ell-1} \right] \\ &= \frac{1}{\ell} + \frac{1}{\ell}\binom{N}{\ell}^{-1} \left[\binom{N-1}{\ell-1} S_{N-1,1,0} - S_{N-1,1,\ell-1} \right]. \end{aligned} \tag{54}$$

According to (43) and (44), we have

$$\begin{aligned} S_{N-1,1,0} &= S_{N-1,0,0} - 1 = \left[1 + (-1)^{N-1}\right] \frac{N}{N+1} - 1, \\ S_{N-1,1,\ell-1} &= (-1)^{\ell} \left[\binom{N+\ell}{\ell}^{-1} + N \sum_{j=0}^{\ell-1} (-1)^{N-j} \binom{N-1}{j} \frac{1}{N+\ell-j} \right] \\ &= (-1)^{\ell} N \sum_{i=0}^{N-1-\ell} (-1)^i \binom{N-1}{i} \frac{1}{i+\ell+1}. \end{aligned}$$

Substituting the expression for $S_{N-1,1,0}$ and the first expression for $S_{N-1,1,\ell-1}$ in (54), we obtain after simple algebra

$$\begin{aligned} p_{A_2}(N,\ell;2) &= \frac{1}{\ell} + \left[\frac{1+(-1)^{N-1}}{N+1} - \frac{1}{N} \right] \\ &\quad + \frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}} \left[\binom{N+\ell}{\ell}^{-1} + N \sum_{j=0}^{\ell-1} (-1)^{N-j} \binom{N-1}{j} \frac{1}{N+\ell-j} \right] \\ &= \frac{1}{\ell} - \frac{1}{N(N+1)} \\ &\quad + \frac{(-1)^{\ell+1}}{\ell\binom{N}{\ell}} \left[\binom{N+\ell}{\ell}^{-1} + N \sum_{j=0}^{\ell-2} (-1)^{N-j} \binom{N-1}{j} \frac{1}{N+\ell-j} \right]. \end{aligned} \tag{55}$$

(The term $(-1)^{N-1}/(N+1)$ and the $(\ell-1)$-th term in the sum over j cancelled each other.) Using the second expression for $S_{N-1,1,\ell-1}$, we obtain the alternative formula

$$\begin{aligned} p_{A_2}(N,\ell;2) &= \frac{1}{\ell} - \frac{1}{N(N+1)} \\ &\quad + (-1)^{\ell+1} \binom{N-1}{\ell-1}^{-1} \sum_{i=0}^{N-\ell} (-1)^i \binom{N-1}{i} \frac{1}{i+\ell+1}. \end{aligned} \tag{56}$$

□

The equivalent formulas (55) and (56) are computationally efficient for moderate ℓ and moderate $N-\ell$, respectively. In particular, plugging $\ell = 2, 3$ into (55) and simplifying, we recover Stanley's results [29].

5. The probability that σ separates the disjoint sets $\mathcal{S}_1, \dots, \mathcal{S}_t$

Let $\mathcal{S}_1, \dots, \mathcal{S}_t$ be disjoint subsets of $[N]$. Let $\ell_j = |\mathcal{S}_j|$, $1 \le j \le t$, $\ell = \sum_j \ell_j$. Introduce $p(N, \vec{\ell}; k)$, the probability that the permutation σ separates the sets $\mathcal{S}_1, \dots, \mathcal{S}_t$, meaning that no cycle of σ contains a pair of elements from two distinct sets $\mathcal{S}_i$ and $\mathcal{S}_j$. Bernardi et al. [3] were able to derive a striking formula for $p(N, \vec{\ell}; 2)$:

(57)

$$p(N, \vec{\ell}; 2) = \frac{(N-\ell)! \prod_j \ell_j!}{(N+t)(N-1)!} \left[\frac{(-1)^{N+\ell}\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}} + \sum_{j=0}^{\ell-t} \frac{(-1)^j \binom{\ell-t}{j}\binom{N+j+1}{\ell}}{\binom{N+t+j}{j}} \right],$$

which is a sum of $\ell - t + 2$ terms. Remarkably, $\prod_j \ell_j!$ aside, the rest of this expression does not depend on the individual ℓ_j. Equation (57) is very efficient for values of ℓ, t relatively small compared to N.

In this section, we apply first our approach to obtain a formula for $p(N, \vec{\ell}; k)$ when $k \ge 2$.

Theorem 5.1. *Introduce*

$$K(N, \ell, t; r) = \left[\xi^{r-\ell+t}\eta^{N-\ell}\right] \left(\frac{1-\xi}{1-\eta}\right)^{t-1} (1-\xi\eta)^{-\ell-1},$$

and define $\alpha_k(N,t) = t-1$ if k is odd, and $\alpha_k(N,t) = N+t$ if k is even. Then

(58)

$$p(N, \vec{\ell}; k) = \frac{(-1)^{\alpha_k(N,t)} \prod_j \ell_j!}{(N)_\ell} \sum_{r=\ell-t}^{N-1} (-1)^{(k+1)r} \binom{N-1}{r}^{-k+1} K(N, \ell, t; r),$$

i.e., $p(N, \vec{\ell}; k)$ is $\prod_j \ell_j!$ times a factor dependent on ℓ and t. Consequently, the probability that the cycles of the product of k random cycles of length N partition the set of cardinality ℓ into t subsets is given by

$$p(N, \ell, t; k) = \frac{(-1)^{\alpha_k(N,t)}\binom{\ell-1}{t-1}}{t!\,(N)_\ell} \sum_{r=\ell-t}^{N-1} (-1)^{(k+1)r} \binom{N-1}{r}^{-k+1} K(N, \ell, t; r).$$

Formula (58) is computationally efficient for $\ell - t$ close to N.

Proof. Let $\mathcal{A}(\vec{\ell})$ denote the set of all permutations that separate the sets $\mathcal{S}_1, \dots, \mathcal{S}_t$, and let $\mathcal{A}(\vec{\nu}, \vec{\ell}) = \{s \in \mathcal{A}(\vec{\ell}) : \vec{\nu}(s) = \vec{\nu}\}$, $Q(\vec{\nu}, \vec{\ell}) = |\mathcal{A}(\vec{\nu}, \vec{\ell})|$. Each cycle of such a permutation either does not contain any element of $\bigcup_j \mathcal{S}_j$, or contains some of the elements of exactly one set $\mathcal{S}_j$. Since

$\left|[N] \setminus \bigcup_j \mathcal{S}_j\right| = N - \ell$, denoting $\prod_j w_j^{\ell_j} = \vec{w}^{\,\vec{\ell}}$, analogously to (30) we have

(59)
$$\begin{aligned}\frac{Q(\vec{\nu}, \vec{\ell})}{(N-\ell)!\prod_j \ell_j!} &= [y^{N-\ell}\vec{w}^{\,\vec{\ell}}]\left[\prod_r \frac{1}{\nu_r!}\left(\frac{y^r + \sum_{j=1}^t \sum_{a>0}\binom{r}{a} w_j^a y^{r-a}}{r}\right)^{\nu_r}\right]\\ &= [y^{N-\ell}\vec{w}^{\,\vec{\ell}}]\left[\prod_r \frac{1}{\nu_r!}\left(\frac{-(t-1)y^r + \sum_{j=1}^t (w_j+y)^r}{r}\right)^{\nu_r}\right].\end{aligned}$$

(The interested reader may wish to check this identity by following the steps of the proof of (30).) Using (59), we obtain

$$\begin{aligned}\chi^{\lambda^*}(\mathcal{A}(\vec{\ell})) = (-1)^{\lambda^1}(N-\ell)! \prod_j \ell_j!\\ \times [\xi^{\lambda_1} y^{N-\ell}\vec{w}^{\,\vec{\ell}}]\,\frac{\xi}{1-\xi}\sum_{\vec{\nu}}\prod_r \frac{1}{\nu_r!}\\ \times\left(-\frac{(1-\xi^r)\big(-(t-1)y^r + \sum_j (w_j+y)^r\big)}{r}\right)^{\nu_r},\end{aligned} \tag{60}$$

the sum being for $\vec{\nu} \geq \mathbf{0}$ with $\sum_r r\nu_r = N$. So the expression in the second line of (60) equals

$$\begin{aligned}&[\xi^{\lambda_1} x^N y^{N-\ell}\vec{w}^{\,\vec{\ell}}]\frac{\xi}{1-\xi}\sum_{\vec{\nu}\geq\mathbf{0}}\prod_r \frac{(x^r)^{\nu_r}}{\nu_r!}\\ &\qquad\times\left(-\frac{(1-\xi^r)\big[-(t-1)y^r + \sum_j (w_j+y)^r\big]}{r}\right)^{\nu_r}\\ &= [\xi^{\lambda_1} x^N y^{N-\ell}\vec{w}^{\,\vec{\ell}}]\frac{\xi}{1-\xi}\\ &\qquad\times\prod_r \exp\left(-\frac{x^r(1-\xi^r)\big[-(t-1)y^r + \sum_{j=1}^t (w_j+y)^r\big]}{r}\right).\end{aligned}$$

Using $\prod_{r\geq 1}\exp(-\eta^r/r) = 1-\eta$, ($|\eta| < 1$), we transform the last product into

$$\left(\frac{1-\xi xy}{1-xy}\right)^{t-1}\prod_{j=1}^t \frac{1-x(w_j+y)}{1-\xi x(w_j+y)}.$$

Further

$$[\vec{w}^{\,\vec{\ell}}]\prod_{j=1}^{t}\frac{1-x(w_j+y)}{1-\xi x(w_j+y)}=\left(\frac{1-xy}{1-\xi xy}\right)^t\cdot\prod_{j=1}^{t}[w_j^{\ell_j}]\frac{1-\frac{xw_j}{1-xy}}{1-\frac{\xi xw_j}{1-\xi xy}}$$
$$=\left(\frac{1-xy}{1-\xi xy}\right)^t\prod_{j=1}^{t}\left[\left(\frac{\xi x}{1-\xi xy}\right)^{\ell_j}-\frac{x}{1-xy}\left(\frac{\xi x}{1-\xi xy}\right)^{\ell_j-1}\right]$$
$$=\left(\frac{1-xy}{1-\xi xy}\right)^t\left(\frac{\xi x}{1-\xi xy}\right)^{\ell-t}\left(\frac{(\xi-1)x}{(1-\xi xy)(1-xy)}\right)^t$$
$$=\frac{(\xi x)^{\ell-t}x^t(\xi-1)^t}{(1-\xi xy)^{\ell+t}}.$$

So we need

$$[\xi^{\lambda_1}x^Ny^{N-\ell}]\frac{\xi}{1-\xi}\left(\frac{1-\xi xy}{1-xy}\right)^{t-1}\frac{(\xi x)^{\ell-t}x^t(\xi-1)^t}{(1-\xi xy)^{\ell+t}}$$
$$=(-1)^t[\xi^{\lambda_1-1-(\ell-t)}x^{N-\ell}y^{N-\ell}](1-\xi)^{t-1}(1-xy)^{-t+1}(1-\xi xy)^{-\ell-1}$$
$$=(-1)^t[\xi^{\lambda_1-1-(\ell-t)}z^{N-\ell}]\left(\frac{1-\xi}{1-z}\right)^{t-1}(1-\xi z)^{-\ell-1}$$
$$=(-1)^tK(N,\ell,t;r),\quad r:=\lambda_1-1-(\ell-t).$$

Here

$$\begin{aligned}K(N,\ell,t;r)&=[\xi^{r-\ell+t}z^{N-\ell}]\left(\frac{1-\xi}{1-z}\right)^{t-1}(1-\xi z)^{-\ell-1}\\&=\sum_j(-1)^{r-\Delta-j}\binom{\ell+j}{j}\binom{t-1}{r-\Delta-j}\binom{N-\Delta-j-2}{t-2},\end{aligned}\tag{61}$$

where we set $\Delta=\ell-t$. Obviously $K(N,\ell,t;r)=0$ for $r<\ell-t$, and less obviously for $r\ge N$. Indeed,

$$\begin{aligned}&[z^{N-\ell}]\,(1-\xi)^{t-1}(1-z)^{-t+1}(1-\xi z)^{-\ell+1}\\&\qquad=\sum_{j\le N-\ell}(-1)^{N-\ell-j}\binom{-t+1}{N-\ell-j}[z^j](1-\xi)^{t-1}(1-\xi z)^{-\ell-1},\end{aligned}\tag{62}$$

and the $[z^j]$-factor is a polynomial of ξ of degree $t-1+j\le t-1+N-\ell<r-\ell+t$ if $r\ge N$.

So Equation (60) becomes

$$\chi^{\lambda^*}(\mathcal{A}(\vec{\ell}))=(-1)^{\lambda^1+t}(N-\ell)!\left(\prod_{j=1}^{t}\ell_j!\right)K(N,\ell,t;r).$$

Combining this with Equation (6), and $\lambda_1+\lambda^1=N+1$, we obtain the statement that was to be proved. □

Notice that

$$K(N,N,t;r) = [\xi^{r-N+t}](1-\xi)^{t-1} = (-1)^{r-N+t}\binom{t-1}{r-N+t}.$$

Let $\ell = \sum_j \ell_j = N$. Introducing $\beta_k(N) = N-1$ for k odd, $\beta_k(N) = 0$ for k even, we transform (58) into

$$p(N,\vec{\ell};k) = \frac{(-1)^{\beta_k(N)}\prod_j \ell_j!}{N!}\sum_{r=N-t}^{N-1}(-1)^{kr}\binom{N-1}{r}^{-k+1}\binom{t-1}{r-N+t},$$

an alternating sum of t terms. For $t = N$, note that $\ell_i = 1$ for all i, and that $p(N,\vec{\ell};k) = \mathrm{P}(\sigma = \mathrm{id})$. The resulting formula agrees with (10), since for k odd and N even the sum over $r \in [0, N-1]$ is zero.

5.1. **The case of $k = 2$.** From now on we focus on $k = 2$, and general $\vec{\ell}$. We begin with a relatively compact formula that represents $p(N,\vec{\ell};2)$ as a composition of one-dimensional integration operation and coefficient extraction operation applied to a bivariate rational function.

Theorem 5.2. *Recalling the notation $(a)_b = a(a-1)\cdots(a-b+1)$, we have*

$$\begin{aligned} p(N,\vec{\ell};2) = {} & \frac{(-1)^{N+\ell} N \prod_j \ell_j!}{(N)_\ell} \\ & \times [z^{N-\ell}](1-z)^{-t+1}\int_0^1 \frac{(1-u)^{N+1}u^{\ell-t}}{(1-u+zu)^{\ell+1}}\,du. \end{aligned} \tag{63}$$

Proof. For $k = 2$, Equation (58) becomes

$$p(N,\vec{\ell};2) = \frac{(-1)^{N+t}\prod_j \ell_j!}{(N)_\ell}\sum_{r=\ell-t}^{N-1}(-1)^r\binom{N-1}{r}^{-1}K(N,\ell,t;r), \tag{64}$$

where, by Lemma 5.1,

$$K(N,\ell,t;r) = \left[\xi^{r-\ell+t}\eta^{N-\ell}\right]\left(\frac{1-\xi}{1-\eta}\right)^{t-1}(1-\xi\eta)^{-\ell-1}.$$

In (64) we can extend the summation to $r \in [\ell - t, \infty)$, since $K(N, \ell, t; r) = 0$ for $r \geq N$. Using this observation and (14), we first evaluate

(65)
$$\begin{aligned}
&\sum_{r=\ell-t}^{N-1} (-1)^r \binom{N-1}{r}^{-1} [\xi^{r-\ell+t}]\,(1-\xi)^{t-1}(1-\xi z)^{-\ell-1} \\
&\quad = N \sum_{r=\ell-t}^{\infty} (-1)^r [\xi^{r-\ell+t}]\, \frac{(1-\xi)^{t-1}}{(1-\xi z)^{\ell+1}} \int_0^1 u^r (1-u)^{N-1-r}\, du \\
&\quad = N \int_0^1 (1-u)^{N-1} \left(\sum_{r=\ell-t}^{\infty} \left(-\frac{u}{1-u}\right)^r [\xi^{r-\ell+t}] \frac{(1-\xi)^{t-1}}{(1-\xi z)^{\ell+1}} \right) du \\
&\quad = N \int_0^1 (1-u)^{N-1} \left(-\frac{u}{1-u}\right)^{\ell-t} \left(\sum_{r=\ell-t}^{\infty} [\xi^{r-\ell+t}] \frac{(1+\xi\,\frac{u}{1-u})^{t-1}}{(1+\xi z\,\frac{u}{1-u})^{\ell+1}} \right) du \\
&\quad = N \int_0^1 (1-u)^{N-1} \left(-\frac{u}{1-u}\right)^{\ell-t} \left. \frac{(1+\xi\,\frac{u}{1-u})^{t-1}}{(1+\xi z\,\frac{u}{1-u})^{\ell+1}} \right|_{\xi=1} du \\
&\quad = (-1)^{\ell-t} N \int_0^1 \frac{(1-u)^{N+1} u^{\ell-t}}{(1-u+zu)^{\ell+1}}\, du;
\end{aligned}$$

(in the fifth line we used $\sum_{r \geq 0} [\xi^r] f(\xi) = f(1)$ for the series $f(\xi) = \sum_{r \geq 0} a_r \xi^r$). So (64) is transformed into (63). $\square$

Corollary 5.3. *For $\ell = N$, Formula* (63) *yields*

$$p(N, \vec{\ell}; 2) = \frac{N \prod_j \ell_j!}{N!} \int_0^1 u^{N-t}\, du = \frac{\prod_j \ell_j!}{(N-1)!\,(N-t+1)}. \tag{66}$$

To compare, the separation probability for a uniformly random permutation of $[N]$ is $\prod_j \ell_j!/N!$.

For $\ell_1 = \cdots = \ell_{t-1} = 1$, $\ell_t = N - t + 1$, $(2 \leq t \leq N)$, $p(N, \vec{\ell}; 2)$ is the probability that all elements of a given subset of cardinality $t-1$ are fixed points of $\sigma^{(2)}$; the number of such subsets is $\binom{N}{t-1}$. Furthermore the probability that all the elements of $[N]$ are fixed, i.e., $\sigma^{(2)} = \mathrm{id}$, is $\frac{1}{(N-1)!}$, see (11). So using the inclusion-exclusion formula, we obtain:

$$\mathrm{P}(\sigma^{(2)} \text{ is a derangement}) = N \sum_{\tau=0}^{N-1} \frac{(-1)^\tau}{(N-\tau)\tau!} + \frac{(-1)^N}{(N-1)!}.$$

For comparison, the probability that a uniformly random permutation of $[N]$ is a derangement equals $\sum_{\tau=0}^{N} (-1)^\tau \frac{1}{\tau!}$.

More generally, if in (63) we take the factor $(1-z)^{-t+1}$ inside the integral, and perform extraction of the coefficient of $z^{N-\ell}$ from the resulting integrand, then we end up with an alternating-sign sum of the beta integrals and obtain an explicit formula

$$p(N,\vec{\ell};2)=\frac{N\prod_j \ell_j!}{(N)_\ell}\sum_{k\le N-\ell}(-1)^k\frac{\binom{t+k-2}{t-2}\binom{N-k}{\ell}}{(N-t+1)\binom{N-t}{k}}. \tag{67}$$

This expression is computationally efficient for moderate $N-\ell$, but progressively less useful for larger values of $N-\ell$.

5.2. **An alternative formula deduced by the Gosper–Zeilberger algorithm.** In this section, we will show that Equation (63) can be transformed so that extraction of the coefficient of $z^{N-\ell}$ will lead to a sum with $\ell - t + 2$ number of terms, close in appearance to the formula (57) by Bernardi et al.

Clearly it is the outside factor $(1-z)^{-t+1}$ that causes the number of summands in (67) grow indefinitely with N. To get rid of $(1-z)^{-t+1}$, we resort to repeated integration by parts of the integral, (denote it $I(z)$), with each step producing the outside factor $1-z$. However the factor $u^{\ell-t}$ in the integrand of $I(z)$ would have made the integration process unwieldy; so we apply it instead to $K_1(z)$, where

$$K_\nu(z):=\int_0^1\frac{(1-u)^{N+\nu}}{(1-u+zu)^{t+\nu}}\,du,$$

because

$$I(z)=\frac{(-1)^{\ell-t}}{(t+1)^{(\ell-t)}}\,\frac{d^{\ell-t}K_1(z)}{dz^{\ell-t}}. \tag{68}$$

One integration by parts leads to

$$\begin{aligned}K_1(z)&=\frac{1}{N+2}+\frac{(t+1)(1-z)}{N+2}\int_0^1\frac{(1-u)^{N+2}}{(1-u+zu)^{t+2}}\,du\\&=\frac{1}{N+2}+\frac{(t+1)(1-z)}{N+2}K_2(z).\end{aligned}$$

A clear pattern emerges here. Let us denote by $a^{(b)}$ the rising factorial $a(a+1)\cdots(a+b-1)$. After $\ell-1$ integrations by parts, we get

$$K_1(z)=\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}}{(N+2)^{(j)}}(1-z)^{j-1}+\frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\,(1-z)^{\ell-1}K_\ell(z).$$

So, using (68) and

$$\frac{d^{\ell-t}\big[(1-z)^{\ell-1}K_\ell\big]}{dz^{\ell-t}}=\sum_{\mu=0}^{\ell-t}(-1)^\mu\binom{\ell-t}{\mu}(\ell-1)_\mu(1-z)^{\ell-1-\mu}\frac{d^{\ell-t-\mu}K_\ell}{dz^{\ell-t-\mu}},$$

we obtain

$$(1-z)^{-t+1}I(z)\frac{(t+1)^{\ell-t}}{(-1)^{\ell-t}} = (-1)^{\ell-t}\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(N+2)^{(j)}}(1-z)^{j-\ell}$$
$$+\frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\sum_{\mu=0}^{\ell-t}(-1)^{\mu}\binom{\ell-t}{\mu}(\ell-1)_{\mu}(1-z)^{\ell-t-\mu}\,\frac{d^{\ell-t-\mu}K_{\ell}(z)}{dz^{\ell-t-\mu}}.$$

It remains to extract the coefficient of $[z^{N-\ell}]$ in the right-hand side expression. First,

$$[z^{N-\ell}](1-z)^{j-\ell} = (-1)^{N-\ell}\binom{j-\ell}{N-\ell}.$$

Next, for every $r\ge 0$,

$$[z^r]\frac{d^{\ell-t-\mu}K_{\ell}}{dz^{\ell-t-\mu}} = (-1)^{\ell-t-\mu}(t+\ell)^{(\ell-t-\mu)}[z^r]\int_0^1\frac{(1-u)^{N+\ell}u^{\ell-t-\mu}}{(1-u+zu)^{2\ell-\mu}}\,du$$
$$= (-1)^{\ell-t-\mu}(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}\int_0^1(1-u)^{N-\ell+\mu-r}u^{\ell-t-\mu+r}\,du$$
$$= (-1)^{\ell-t-\mu}\frac{(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}}{(N-t+1)\binom{N-t}{\ell-t-\mu+r}}.$$

So

(69)
$$[z^{N-\ell}]\left\{(1-z)^{\ell-t-\mu}\,\frac{d^{\ell-t-\mu}\,K_{\ell}}{dz^{\ell-t-\mu}}\right\}$$
$$=\sum_{k\le \ell-t-\mu}\left\{[z^k](1-z)^{\ell-t-\mu}\right\}\left\{[z^{N-\ell-k}]\,\frac{d^{\ell-t-\mu}\,K_{\ell}}{dz^{\ell-t-\mu}}\right\}$$
$$=\sum_{k\le \ell-t-\mu}(-1)^k\binom{\ell-t-\mu}{k}\,(-1)^{\ell-t-\mu}\left.\frac{(t+\ell)^{(\ell-t-\mu)}\binom{-2\ell+\mu}{r}}{(N-t+1)\binom{N-t}{\ell-t-\mu+r}}\right|_{r=N-\ell-k}.$$

Collecting the pieces,

$$[z^{N-\ell}](1-z)^{-t+1}I(z)\frac{(t+1)^{\ell-t}}{(-1)^{\ell-t}}$$
$$=(-1)^{N-t}\sum_{j=1}^{\ell-1}\frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(N+2)^{(j)}}\binom{j-\ell}{N-\ell}$$
$$+(-1)^{\ell-t}\frac{(t+1)^{(\ell-1)}}{(N+2)^{(\ell-1)}}\sum_{\mu=0}^{\ell-t}\binom{\ell-t}{\mu}(\ell-1)_{\mu}(t+\ell)^{(\ell-t-\mu)}$$
$$\times\sum_{k\le\ell-t-\mu}(-1)^k\binom{\ell-t-\mu}{k}\frac{\binom{-2\ell+\mu}{N-\ell-k}}{(N-t+1)\binom{N-t}{\mu+k}}.$$

So, since

$$\binom{-a}{b} = (-1)^b \binom{a+b-1}{a-1}, \qquad \frac{(t+1)^{(\ell-1)}(t+\ell)^{(\ell-t-\mu)}}{(t+1)^{(\ell-t)}} = \frac{(2\ell-\mu-1)!}{\ell!},$$

Equation (63) becomes

(70)

$$\begin{aligned} p(N,\vec{\ell};2) = {} & \frac{N\prod_j \ell_j!}{(N)_\ell} \\ & \times \left[(-1)^{N+\ell} \sum_{j=1}^{\ell-1} \frac{(t+1)^{(j-1)}(j-1)_{\ell-t}}{(t+1)^{(\ell-t)}(N+2)^{(j)}} \binom{N-j-1}{\ell-j-1} \right. \\ & + \frac{1}{\ell!(N+2)^{(\ell-1)}(N-t+1)} \\ & \left. \times \sum_{\mu=0}^{\ell-t} \binom{\ell-t}{\mu} (\ell-1)_\mu \sum_{\nu=\mu}^{\ell-t} \binom{\ell-t-\mu}{\ell-t-\nu} \frac{(N+\ell-\nu-1)_{2\ell-\mu-1}}{\binom{N-t}{\nu}} \right]; \end{aligned}$$

ν in the bottom sum comes from substitution $\nu = k + \mu$ in (69). Changing the order of summation, the double sum above equals

$$\begin{aligned} \frac{(\ell-t)!}{(N-\ell)!} \sum_{\nu=0}^{\ell-t} \frac{(N+\ell-\nu-1)!}{(\ell-t-\nu)!} \frac{1}{\binom{N-t}{\nu}} \sum_{\mu=0}^{\nu} \binom{\ell-1}{\mu} \binom{N-\ell}{\nu-\mu} \\ = \frac{(\ell-t)!}{(N-\ell)!} \sum_{\nu=0}^{\ell-t} \frac{(N+\ell-\nu-1)!}{(\ell-t-\nu)!} \frac{\binom{N-1}{\nu}}{\binom{N-t}{\nu}}. \end{aligned} \tag{71}$$

Let $\Sigma(N,\ell,t)$ denote the sum over j in (70).

Lemma 5.4. *We have*

$$\Sigma(N,\ell,t) = \frac{(N-1)_{t-2}\,(\ell-t)!}{(t-2)!\,(N+t)^{(\ell-t+1)}}. \tag{72}$$

Proof. We confirmed this conjecture via the powerful Gosper–Zeilberger algorithm, [23, Chapters 5 and 6], [22]. Given $\Delta \ge 0$, introduce a function of $t \ge 2$, defined by

$$S(t) = \sum_{j=1}^{t-1+\Delta} \frac{(t+1)^{(j-1)}(j-1)_\Delta}{(t+1)^{(\Delta)}(N+2)^{(j)}} \binom{N-j-1}{t+\Delta-j-1}.$$

The non-zero summands are those for $j \in [\Delta+1, t-1+\Delta]$. We can extend summation to $j \in [1,\infty)$, since the last binomial is zero for $j \ge t+\Delta$. We need to show that

$$S(t) = S^*(t) := \frac{(N-1)_{t-2}\Delta!}{(t-2)!\,(N+t)^{(\Delta+1)}}. \tag{73}$$

To do so, first we compute

$$\frac{S^*(t)}{S^*(t-1)} = \frac{\beta(t)}{\alpha(t)},$$
$$\alpha(t) := (t-2)(N+t+\Delta), \quad \beta(t) := (N-t+2)(N+t-1).$$

Next, let $F(t,j)$ stand for the j-term in the series $S(t)$. Introduce the "partner" sequence $G(t,j)$ (which again for each t is 0 for all but finitely many j) such that

$$G(t,j) - G(t,j-1) = \alpha(t)F(t,j) - \beta(t)F(t-1,j), \quad j \geq \Delta+1, \tag{74}$$

and $G(t,\Delta) = 0$.

Equation (73) will be proved if we demonstrate that $G(t,j) = 0$ for j large enough.

Such a sequence $G(t,j)$ can be found either by Gosper's algorithm, interpreted by Doron Zeilberger's *Maple* package EKHAD [34], or without Gosper's algorithm as we show below.

Using (74) and the initial condition and $G(t,\Delta) = 0$, we compute by hand and by *Maple* that

$$G(t,\Delta+1) = -\frac{(\Delta+1)!\,(\Delta+2t-2)}{(N+2)^{(\Delta+1)}}\binom{N-\Delta-2}{t-3},$$
$$G(t,\Delta+2) = -\frac{(\Delta+2)!\,(\Delta+2t-2)(t+\Delta+1)}{(N+2)^{(\Delta+2)}}\binom{N-\Delta-3}{t-4},$$
$$G(t,\Delta+3) = -\frac{(\Delta+3)!\,(\Delta+2t-2)(t+\Delta+2)_2}{2(N+2)^{(\Delta+3)}}\binom{N-\Delta-4}{t-5}.$$

The evidence is unmistakable: it must be true that for all $u \geq 1$

$$G(t,\Delta+u) = -\frac{(\Delta+u)!\,(\Delta+2t-2)\binom{t+\Delta+u-1}{u-1}}{(N+2)^{(\Delta+u)}}\binom{N-\Delta-u-1}{t-u-2}. \tag{75}$$

Sure enough, the inductive step based on the recurrence (74) is easily carried out with a guided assistance of *Maple*. It remains to notice that the last binomial coefficient is zero for $u > t-2$. □

Now we are in a position to announce the main result of this section.

Theorem 5.5. *We have the identity*

$$p(N,\vec{\ell};2) = \frac{(N-\ell)!\,\prod_j \ell_j!}{(N-1)!\,(N+t)}\left[(-1)^{N+\ell}\frac{\binom{N-1}{t-2}}{\binom{N+\ell}{\ell-t}} + \frac{(N+t)(N+1)_{\ell+1}}{(N-t+1)(N+\ell)!\,(\ell)_t}\sum_{\nu=0}^{\ell-t}\frac{(N+\ell-\nu-1)!\,(N-1)_\nu}{(\ell-t-\nu)!\,(N-t)_\nu}\right]. \tag{76}$$

Proof. Combining (71) and (72), we transform (70) into (76). □

The outside factor and the first term inside the square brackets in (76) are exactly those in the identity (57) proved by Bernardi et al. [3]. The second inside term, a sum of $\ell-t+1$ terms, times $\frac{(N+t)(N+1)_{\ell+1}}{(N-t+1)(N+\ell)!\,(\ell)_t}$, is quite different in appearance from its sum counterpart in (57). For $\ell-t\le 5$, *Maple* confirms that the rational functions given by the sums in (57) and in (76) are identical; we did not try to prove equality in general.

A reviewer remarked that the equivalence of the two sums can be proved "hypergeometrically: in hypergeometric terms the sum is a balanced ${}_4F_3$-series; one applies one of Whipple's transformation formulas; there is some cancellation; then one applies the Pfaff-Saalschütz summation".

6. Probability that σ blocks the elements of $[\ell]$

We say that the elements of $[\ell]$ are blocked in a permutation s of $[N]$ if in every cycle of s (1) no two elements of $[\ell]$ are neighbors, and (2) each element from $[\ell]$ has a neighbor from $[N]\setminus[\ell]$.

Let $p(N,\ell;k)$ denote the probability of the event that σ blocks the elements of $[\ell]$. In this final section, we are going to prove the following theorem.

Theorem 6.1. *For all positive integers ℓ and k, the formula*

$$p(N,\ell;k)=\frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}}+(-1)^{k+1}\frac{\binom{N-\ell-1}{\ell-1}}{(N-1)^{k-1}\binom{N}{\ell}}. \tag{77}$$

holds.

Proof. Let $\mathcal{A}(\ell)$ denote the set of all permutations that block the elements of $[\ell]$ and let $\mathcal{A}(\vec\nu,\ell)=\{s\in\mathcal{A}(\ell):\vec\nu(s)=\vec\nu\}$, $Q(\vec\nu,\ell)=|\mathcal{A}(\vec\nu,\ell)|$. To evaluate $Q(\vec\nu,\ell)$, introduce the non-negative integers $a_{r,j}$, $b_{r,j}$ that stand for the numbers of elements from $[\ell]$ and $[N]\setminus[\ell]$ in the j-th cycle of length r, ($j\le\nu_r$), and let $\mathbf{a}$ and $\mathbf{b}$ denote the sequences of the numbers $a_{r,j}$ and $b_{r,j}$, in increasing order of the indices r, and then in increasing order of the indices j. Then

$$\begin{gathered}a_{r,j}+b_{r,j}=r,\\ b_{r,j}>0,\\ \sum_{r,\,j\le\nu_r}a_{r,j}=\ell,\qquad\sum_{r,\,j\le\nu_r}b_{r,j}=N-\ell.\end{gathered} \tag{78}$$

For $a_{r,j}>0$, the number of admissible cycles with parameters $a_{r,j}$, $b_{r,j}$ is

$$c(a_{r,j},b_{r,j}):=(a_{r,j}-1)!\,b_{r,j}!\binom{b_{r,j}-1}{a_{r,j}-1}=(b_{r,j}-1)!\,a_{r,j}!\binom{b_{r,j}}{a_{r,j}}. \tag{79}$$

The last expression works for $a_{r,j}=0$ as well.

Indeed $(a_{r,j}-1)!$ is the total number of directed cycles formed by $a_{r,j}$ elements from $[\ell]$; $b_{r,j}!$ is the total number of ways to linearly order $b_{r,j}$ elements from $[N]\setminus\ell$, and $\binom{b_{r,j}-1}{a_{r,j}-1}$ is the total number of ways to break any such $b_{r,j}$-long sequence into $a_{r,j}$ blocks of positive lengths to be fitted between $a_{r,j}$ cyclically arranged elements from $[\ell]$, starting with the smallest element among them and moving in the cycle's direction, say.

Therefore

$$
\begin{aligned}
Q(\vec{\nu},\ell) &= (N-\ell)!\,\ell! \sum_{\mathbf{a},\,\mathbf{b}\text{ meet (78)}} \prod_{r\ge 1} \frac{1}{\nu_r!} \prod_{j\le \nu_r} \frac{c(a_{r,j},b_{r,j})}{a_{r,j}!\,b_{r,j}!} \\
&= (N-\ell)!\,\ell!\,[w^\ell] \prod_{r\ge 1} \frac{1}{\nu_r!} \left(\sum_{b>0,\,a+b=r} \frac{1}{b}\binom{b}{a} w^a \right)^{\nu_r}.
\end{aligned} \tag{80}
$$

Having found $Q(\vec{\nu},\ell)$, we turn to $p(N,\ell,k)$, the probability that σ blocks the elements of $[\ell]$. Using (7), the equality $\nu=\sum_r \nu_r$, and (80), we obtain

$$
\begin{aligned}
\chi^{\lambda^*}(\mathcal{A}(\vec{\nu},\ell)) &= (-1)^{\lambda^1}(N-\ell)!\,\ell! \\
&\quad\times [\xi^{\lambda_1} w^\ell]\, \frac{\xi}{1-\xi} \prod_r \frac{1}{\nu_r!} \left[-(1-\xi^r) \left(\sum_{\substack{b>0,\\ a+b=r}} \frac{1}{b}\binom{b}{a} w^a \right) \right]^{\nu_r}.
\end{aligned}
$$

This identity implies

$$
\begin{aligned}
\chi^{\lambda^*}(\mathcal{A}(\ell)) &= (-1)^{\lambda^1}(N-\ell)!\,\ell! \\
\times [\xi^{\lambda_1} w^\ell]\, \frac{\xi}{1-\xi} &\sum_{\substack{\vec{\nu}:\\ 1\nu_1+2\nu_2+\cdots=N}} \prod_r \frac{1}{\nu_r!} \left[-(1-\xi^r) \left(\sum_{\substack{b>0,\\ a+b=r}} \frac{1}{b}\binom{b}{a} w^a \right) \right]^{\nu_r}.
\end{aligned} \tag{81}
$$

The expression in the second line of (81) equals

$$
\begin{aligned}
&[\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \sum_{\vec{\nu}\ge \mathbf{0}} \prod_r \frac{(x^r)^{\nu_r}}{\nu_r!} \left[-(1-\xi^r) \left(\sum_{b>0,\,a+b=r} \frac{1}{b}\binom{b}{a} w^a \right) \right]^{\nu_r} \\
&\quad= [\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \prod_r \sum_{\nu_r\ge 0} \frac{1}{\nu_r!} \left[-x^r(1-\xi^r) \left(\sum_{b>0,\,a+b=r} \frac{1}{b}\binom{b}{a} w^a \right) \right]^{\nu_r} \\
&\quad= [\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \prod_r \exp\left[-x^r(1-\xi^r) \left(\sum_{b>0,\,a+b=r} \frac{1}{b}\binom{b}{a} w^a \right) \right] \\
&\quad= [\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi} \exp\left[-\sum_{r\ge 1} [x^r-(x\xi)^r] \left(\sum_{b>0,\,a+b=r} \frac{1}{b}\binom{b}{a} w^a \right) \right].
\end{aligned} \tag{82}
$$

Since

$$\begin{aligned}\sum_{r\geq 1} y^r \sum_{b>0,\, a+b=r} \frac{1}{b}\binom{b}{a} w^a &= \sum_{b>0} \frac{y^b}{b} \sum_a \binom{b}{a}(yw)^a \\ &= \sum_{b>0} \frac{y^b}{b}(1+yw)^b = \sum_{b>0} \frac{[y(1+yw)]^b}{b} \\ &= \log \frac{1}{1-y(1+yw)},\end{aligned}$$

the right-hand side in (82) becomes

$$\begin{aligned}&[\xi^{\lambda_1} w^\ell x^N] \frac{\xi}{1-\xi}\frac{1-x(1+xw)}{1-x\xi(1+x\xi w)} \\ &\quad= [\xi^{\lambda_1} x^N]\, \frac{\xi(1-x)}{(1-\xi)(1-x\xi)}\, [w^\ell] \frac{1-\frac{x^2}{1-x}w}{1-\frac{(x\xi)^2}{1-x\xi}w} \\ &\quad= [\xi^{\lambda_1} x^N]\, \frac{\xi(1-x)}{(1-\xi)(1-x\xi)} \left[\left(\frac{(x\xi)^2}{1-x\xi}\right)^\ell - \frac{x^2}{1-x}\left(\frac{(x\xi)^2}{1-x\xi}\right)^{\ell-1}\right] \\ &\quad= [\xi^{\lambda_1} x^N]\, \frac{\xi}{1-x\xi}\left(\frac{(x\xi)^2}{1-x\xi)}\right)^{\ell-1} \frac{x^2}{1-x\xi}\,(x\xi-1-\xi) \\ &\quad= -[\xi^{\lambda_1} x^N]\, \left(\frac{x^{2\ell}\xi^{2\ell-1}}{(1-x\xi)^\ell} + \frac{x^{2\ell}\xi^{2\ell}}{(1-x\xi)^{\ell+1}}\right) \\ &\quad= -[\xi^{\lambda_1-2\ell+1} x^{N-2\ell}](1-x\xi)^{-\ell} - [\xi^{\lambda_1-2\ell} x^{N-2\ell}](1-x\xi)^{-\ell-1} \\ &\quad= -\binom{N-\ell-1}{\ell-1} 1_{\{\lambda_1=N-1\}} - \binom{N-\ell}{\ell} 1_{\{\lambda_1=N\}}.\end{aligned}$$

So (81) simplifies, greatly, to

$$\begin{aligned}\chi^{\lambda^*}(\mathcal{A}(\ell)) &= (-1)^{\lambda^1-1}(N-\ell)!\,\ell! \\ &\qquad \times \left[\binom{N-\ell-1}{\ell-1} 1_{\{\lambda_1=N-1\}} + \binom{N-\ell}{\ell} 1_{\{\lambda_1=N\}}\right].\end{aligned} \tag{83}$$

The rest is easy. By (6),

$$p(N,\ell;k) = \frac{1}{N!}\sum_{\lambda^*} (-1)^{k(\lambda^1-1)} \binom{N-1}{\lambda_1-1}^{-k+1} \chi^{\lambda^*}(\mathcal{A}(\ell)). \tag{84}$$

Combining this with (83) we conclude that

$$p(N,\ell;k) = \frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}} + (-1)^{k+1} \frac{\binom{N-\ell-1}{\ell-1}}{(N-1)^{k-1}\binom{N}{\ell}}.$$

As a partial check,

$$p(N,\ell;1) = \frac{\binom{N-\ell}{\ell}}{\binom{N}{\ell}} + \frac{\binom{N-\ell-1}{\ell-1}}{\binom{N}{\ell}}$$
$$= \frac{(N-\ell-1)!\,(n-\ell)!}{(N-1)!\,(N-2\ell)!} = \frac{c(\ell, N-\ell)}{(N-1)!},$$

see (78) and (79). □

Remark. Equation (77) shows that $\lim_{k\to\infty} p(N,\ell;k) = \binom{N-\ell}{\ell}/\binom{N}{\ell}$, the probability that the *uniformly* random permutation blocks $[\ell]$.

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