Decomposition of 0-Hecke modules associated to quasisymmetric Schur functions

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Compositions

Let n be a nonnegative integer.

1. A composition of n ($\alpha \vDash n$) is a sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of positive integers which sum up to n. The α_i are called parts.
2. A partition of n ($\lambda \vdash n$) is a composition of n whose parts are weakly decreasing.
3. $\widetilde{\alpha}$ is the partition obtained by sorting the parts of α.

$$\alpha = (1, 4, 3) \vDash 8 \qquad \widetilde{\alpha} = (4, 3, 1) \vdash 8$$

Introduction

	Sym^n	$QSym^n$
Basis	$\{s_\lambda \mid \lambda \vdash n\}$	$\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$
Chara.	$\mathrm{ch}\colon G_0(\mathbb{C}\mathfrak{S}_n) \to Sym^n$ $[\boldsymbol{S}^\lambda] \mapsto s_\lambda$	$\mathrm{Ch}\colon G_0(H_n(0)) \to QSym^n$ $[\boldsymbol{S}_\alpha] \mapsto \mathcal{S}_\alpha$

$$s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha$$

Sym^n homogeneous symmetric functions of degree n

$QSym^n$ homogeneous quasisymmetric functions of degree n

s_λ Schur function

$\boldsymbol{S}^\lambda$ Specht module

$\mathcal{S}_\alpha$ quasisymmetric Schur function

$G_0(A)$ Grothendieck group of $A-$mod

Outline

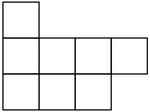
1. Combinatorial background
2. Definition of $\boldsymbol{S}_\alpha$
3. Decomposition of $\boldsymbol{S}_\alpha$

Outline

Composition diagrams

Let $\alpha = (\alpha_1, \ldots, \alpha_l) \vDash n$.

- We obtain the diagram of α by placing α_i boxes in row i beginning with the top row.
- We identify α and its diagram.

$(1, 4, 3) =$ 

Standard composition tableaux

Definition

A standard composition tableau (SCT) of shape $\alpha \vDash n$ is a bijective filling $T\colon \alpha \to [n]$ such that

1. entries increase in first column from top to bottom
2. entries decrease in rows from left to right
3. (triple rule) for all entries $a, b \in T$

$$a < b \text{ and } \begin{array}{|c|c} \hline b & \\ \hline & \multicolumn{1}{|c|}{a} \\ \hline \end{array} \implies \exists c \in T : a < c \text{ and } \begin{array}{|c|c|} \hline b & c \\ \hline & a \\ \hline \end{array}$$

1			
6	5	4	3
8	7	2	

SCT of shape $(1, 4, 3)$

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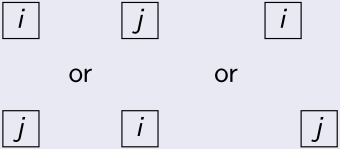
SCT of shape $(1, 4, 3)$

Attacking entries

Let T be an SCT of shape α.

Definition

For $i, j \in T$ i **attacks** j ($i \rightsquigarrow j$) iff $i \neq j$ and



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$$6 \rightsquigarrow 1, 7, 8 \quad 6 \not\rightsquigarrow 2, 3, 4, 5$$

Descents

Let T be an SCT of shape α.

Definition

$D(T) = \{i \in [n-1] \mid i$ weakly left of $i+1\}$ is the descent set of T.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \multicolumn{3}{c}{} \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 & \multicolumn{1}{|c}{} \\ \cline{1-3} \end{array}$$

$$D(T) = \{1, 2, 6\}$$

Outline

The 0-Hecke algebra

Definition

The 0-Hecke algebra $H_n(0)$ is the associative unitary $\mathbb{C}$-algebra generated by $\pi_1, \ldots, \pi_{n-1}$ subject to relations

$$\pi_i^2 = \pi_i$$
$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$$
$$\pi_i \pi_j = \pi_j \pi_i \text{ if } |i-j| \geq 2$$

0-Hecke algebra acts on SCTx

Definition

- $\mathrm{SCT}(\alpha) = \{\text{SCTx of shape } \alpha\}$
- $\boldsymbol{S}_\alpha = \mathrm{span}_{\mathbb{C}}\, \mathrm{SCT}(\alpha)$

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Theorem (Tewari, van Willigenburg 2015)

Let $\alpha \vDash n$. The following action turns $\boldsymbol{S}_\alpha$ into a $H_n(0)$-module. For $T \in SCT(\alpha)$

$$\pi_i T = \begin{cases} T & \text{if } i \notin D(T) \\ 0 & \text{if } i \in D(T) \text{ and } i \rightsquigarrow i+1 \\ s_i T & \text{if } i \in D(T) \text{ and } i \not\rightsquigarrow i+1 \end{cases}$$

where $s_i T$ is the tableau obtained from T by interchanging i and $i+1$.

Example: 0-Hecke action

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \multicolumn{3}{c}{} \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \qquad D(T) = \{1, 2, 6\}$$

$$\pi_i T = \begin{cases} T & \text{if } i = 3, 4, 5, 7 \\ 0 & \text{if } i = 6 \\ s_i T & \text{if } i = 1, 2 \end{cases}$$

$$s_1 T = \begin{array}{|c|c|c|c|} \hline 2 & \multicolumn{3}{c}{} \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \qquad s_2 T = \begin{array}{|c|c|c|c|} \hline 1 & \multicolumn{3}{c}{} \\ \hline 6 & 5 & 4 & 2 \\ \hline 8 & 7 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

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An equivalence relation on SCTx

Let $T_1, T_2 \in \mathsf{SCT}(\alpha)$.

Definition

$T_1 \sim T_2 \iff$ In each column the relative orders of entries in T_1 and T_2 coincide.

1			
6	5	4	3
8	7	2	

$\sim$

4			
6	5	3	1
8	7	2	

$\not\sim$

1			
5	4	3	2
8	7	6	

Note: $\sim$ is equivalence relation on $\mathsf{SCT}(\alpha)$

0-Hecke modules of equivalence classes under $\sim$

Definition

- $\mathcal{E}(\alpha) = {}^{SCT(\alpha)}\!/\!_{\sim}$
- $\boldsymbol{S}_{\alpha,E} = \mathrm{span}_{\mathbb{C}}\, E$ for $E \in \mathcal{E}(\alpha)$

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Proposition (Tewari, van Willigenburg 2015)

- $\pi_i(\boldsymbol{S}_{\alpha,E}) \subseteq \boldsymbol{S}_{\alpha,E}$
- $\boldsymbol{S}_\alpha = \bigoplus_{E\in\mathcal{E}(\alpha)} \boldsymbol{S}_{\alpha,E}$ *as $H_n(0)$-modules.*

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Question

Are the $\boldsymbol{S}_{\alpha,E}$ indecomposable?

A partial order on E

Let $\alpha \vDash n, E \in \mathcal{E}(\alpha),\ T_1, T_2 \in E$.

Definition

$T_1 \preceq T_2 \iff \exists i_1, \ldots, i_r$ s.t. $T_2 = \pi_{i_r} \cdots \pi_{i_1} T_1$.

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$(E, \preceq)$ *is a graded lattice.*

The least element of E is called source tableau of E and denoted by T_0.

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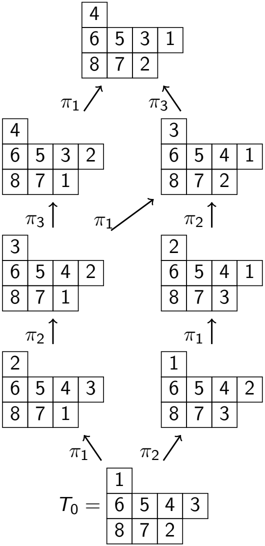
Corollary

$\boldsymbol{S}_{\alpha,E} = H_n(0)\,T_0$.

A graded lattice E



The decomposition of $\boldsymbol{S}_\alpha$

Theorem (K.)

$\mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{C}\,\mathrm{id}$ *for all* $E \in \mathcal{E}(\alpha)$.

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$\boldsymbol{S}_\alpha = \bigoplus_{E \in \mathcal{E}(\alpha)} \boldsymbol{S}_{\alpha,E}$ *is a decomposition into indecomposable* $H_n(0)$*-modules.*

Strategy of the proof

Let $f \in \operatorname{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E})$.

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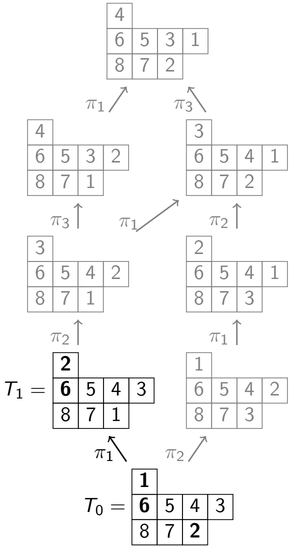
Lemma

$D(T) \not\subseteq D(T_0) \implies a_T = 0.$

$T \in E$ with $D(T) \subseteq D(T_0)$



A special element of $H_n(0)$

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Let $T_0 \neq T \in E$ s.t. $D(T) \subseteq D(T_0)$. Set

$$i = \max\left\{k \in [n] \mid T^{-1}(k) \neq T_0^{-1}(k)\right\}$$
$$j = \min\left\{k \in [n] \mid k > i \text{ and } i \rightsquigarrow k \text{ in } T_0\right\}$$
$$\pi_\sigma = \pi_{j-1} \cdots \pi_{i+1}\pi_i.$$

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Then

1. $\pi_\sigma T_0 = 0$
2. $\pi_\sigma T = s_{j-1} \cdots s_{i+1} s_i T \in E$
3. $\forall T' \in E$ *s.t.* $\delta(T') \leq \delta(T)$:

$$\pi_\sigma T' = \pi_\sigma T \implies T' = T$$

Example for the special element

$$T_0 = \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 2 \\ \cline{1-3}\end{array} \qquad T_1 = \begin{array}{|c|c|c|c|}\cline{1-1} 2 \\ \hline 6 & 5 & 4 & 3 \\ \hline 8 & 7 & 1 \\ \cline{1-3}\end{array}$$

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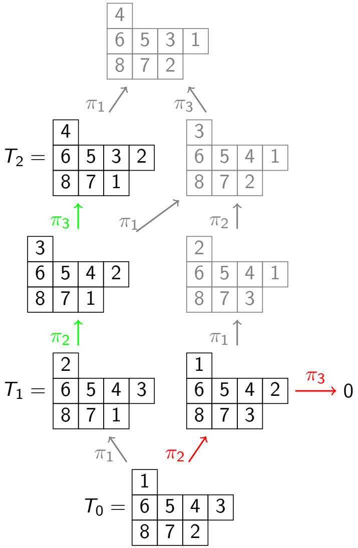
$$j = 4 = \min \{ k \in [n] \mid k > i \text{ and } i \rightsquigarrow k \text{ in } T_0 \}$$

$$\pi_\sigma = \pi_3 \pi_2$$

Application of $\pi_3\pi_2$ on T_0 and T_1



T_1 does not occur in v

Let $f \in \mathrm{End}(\boldsymbol{S}_{\alpha,E})$.

- $v = f(T_0) = a_0 T_0 + a_1 T_1$
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$$\pi_3 \pi_2 v = a_0 \pi_3 \pi_2 T_0 + a_1 \pi_3 \pi_2 T_1 = a_1 T_2.$$

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Hence $a_1 = 0$ and $f = a_0\,\mathrm{id}$. Thus

$$\mathrm{End}_{H_n(0)}(\boldsymbol{S}_{\alpha,E}) = \mathbb{C}\,\mathrm{id}.$$

Thank you!