Orthogonal polynomials and Smith normal form

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(with Dennis Stanton)

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q-Catalan

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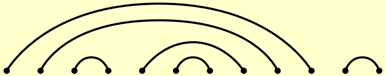

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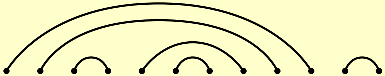

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Theorem (Bessenrodt–Stanley 2015) SSNF of

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exists over $\mathbb{Z}[q]$ and equals

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$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad P_0(x) = 1,\ P_{-1}(x) = 0,\ \text{and } \lambda_n \neq 0$$

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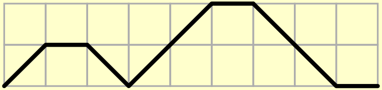

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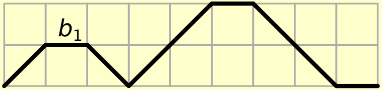

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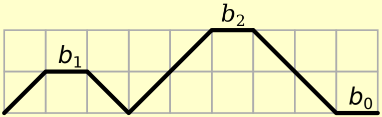

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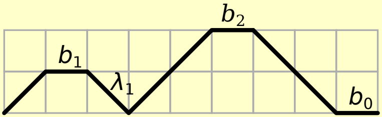

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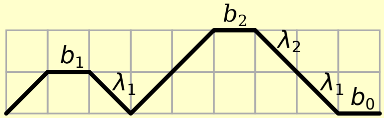

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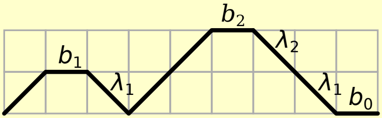

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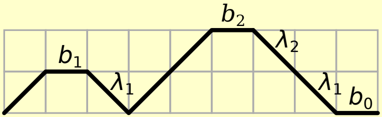

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Theorem (Miller–Stanton)

SSNF of $(\mu_{i+j}) = \mathrm{diag}(\,1\,,\ \lambda_1\,,\ \lambda_1\lambda_2\,,\ \ldots\,,\ \lambda_1\lambda_2\cdots\lambda_n\,)$ over $\mathbb{Z}[b_0, b_1, \ldots, \lambda_1, \lambda_2, \ldots]$

Some examples: moments that are q-

Catalan Motzkin Bell (2) Matchings Perfect matchings $n!!$ $n!$ . . .

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Theorem (Simion–Stanton) $C_n(q) = \mu_n$ for

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Theorem (Médicis–Stanton–White 1995) $B_n(a,q) = \mu_n$ for

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$$B_n(a,q) := \sum_{k=0}^{n} S_q(n,k) a^k = \sum_{\pi \in \Pi_n} a^{\text{block}(\pi)} q^{\text{rs}(\pi)}$$

Theorem (Médicis–Stanton–White 1995) $B_n(a,q) = \mu_n$ for

$$C_{n+1}^a(x;q) = (x - aq^n - [n]_q) C_n^a(x;q) - aq^{n-1}[n]_q C_{n-1}^a(x;q)$$

Corollary (Miller–Stanton)

$$\text{SSNF of } \left(B_{i+j}(a,q) \right) = \text{diag}\left(a^i q^{\binom{i}{2}} [i]!_q \right) \text{ over } \mathbb{Z}[a,q]$$

Some examples: moments that are q-

Catalan Motzkin **Bell (2)** Matchings Perfect matchings $n!!$ $n!$ ...

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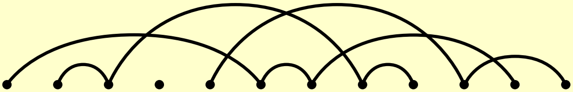

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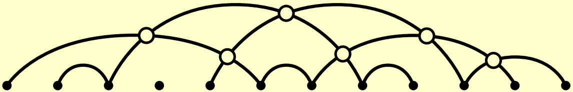

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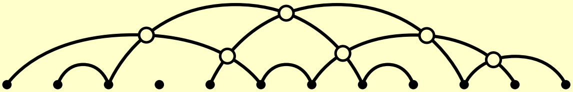

$$\tilde{B}_n(a,q) = \sum_{\pi \in \Pi_n} a^{\text{block}(\pi)} q^{\text{crossing}(\pi)}$$

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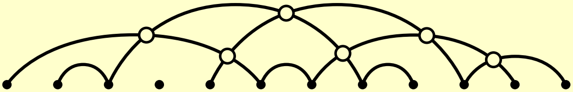

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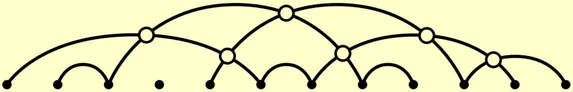

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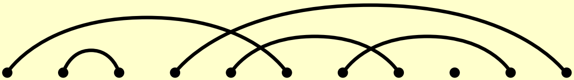

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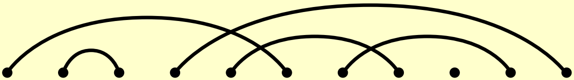

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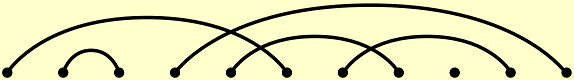

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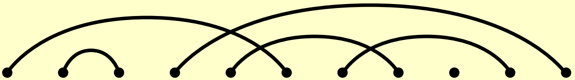

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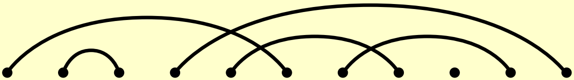

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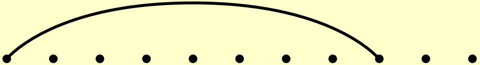

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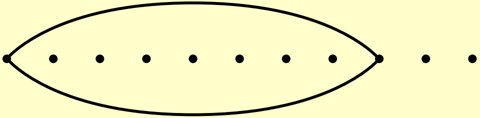

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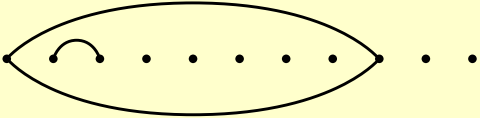

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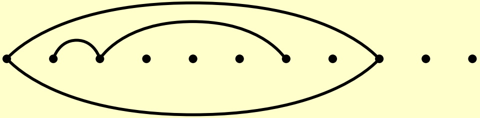

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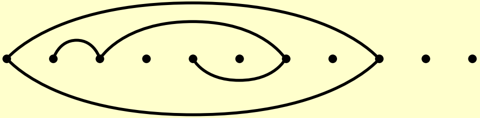

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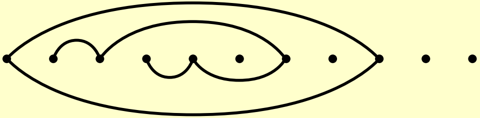

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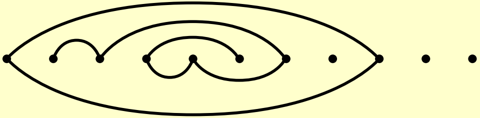

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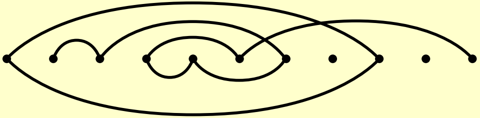

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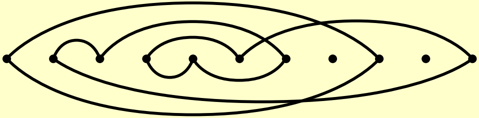

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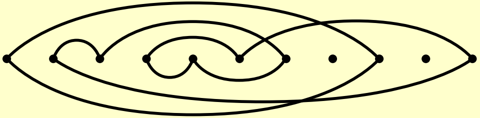

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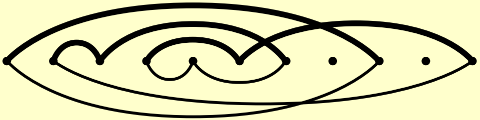

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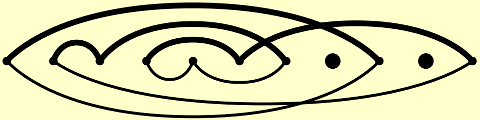

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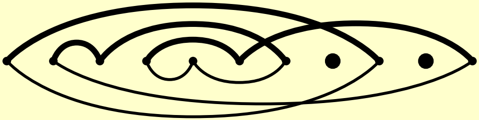

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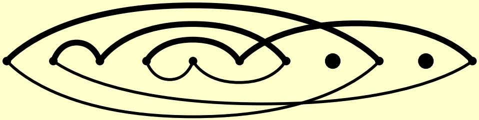

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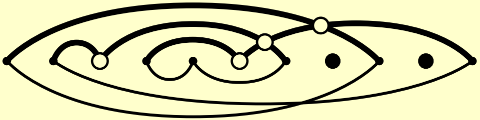

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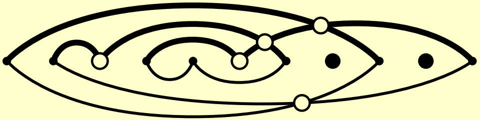

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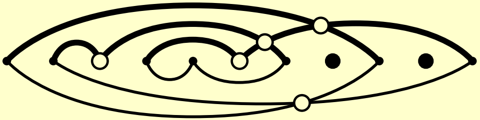

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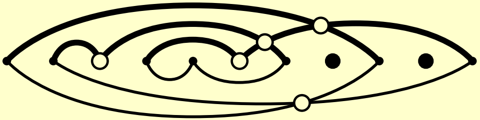

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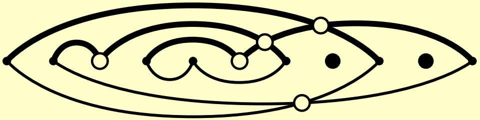

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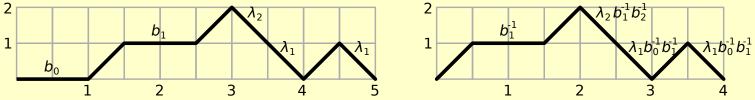

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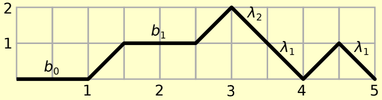

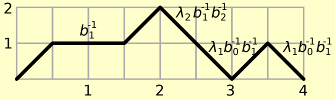

Theorem (Miller–Stanton)

$$\text{SSNF of } (\mu_{i-j})_{i,j=0}^{n} = \text{diag}\left(1\,,\ -\frac{\lambda_1}{b_1}\,,\ \frac{\lambda_1\lambda_2}{b_1 b_2}\,,\ \ldots\,,\ \pm\frac{\lambda_1\lambda_2\ldots\lambda_n}{b_1 b_2\ldots b_n}\right)$$

over $\mathbb{Z}[b_0, b_0^{-1}, b_1, b_1^{-1}, \ldots, \lambda_1, \lambda_2, \ldots]$

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Does $\mathbf{G} := (\langle \mathbf{x}, \mathbf{y} \rangle)_{\mathbf{x},\mathbf{y} \in \mathcal{O}}$ have a SSNF over $\mathbb{K}$?

Gram matrix of lattice

Theorem (Miller–Stanton) $\left(q^{\text{block}(A\vee B)}\right)_{A,B\in\Pi_n}$ has $\mathbb{Z}[q]$-SSNF

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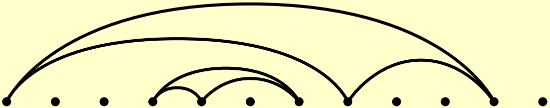

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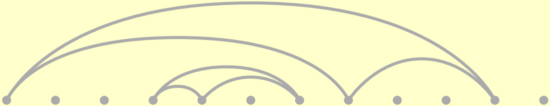

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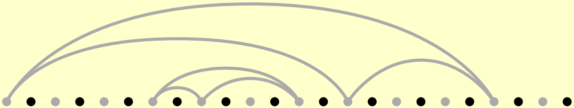

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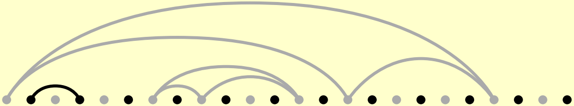

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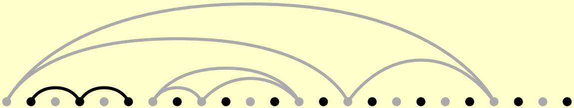

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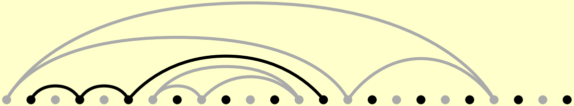

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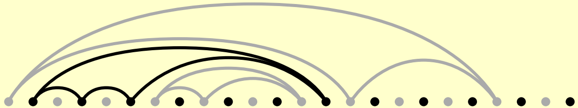

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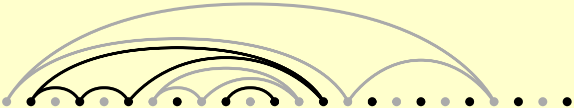

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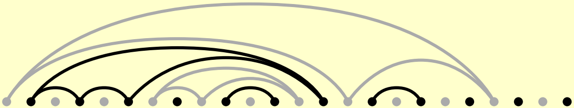

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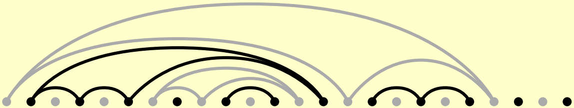

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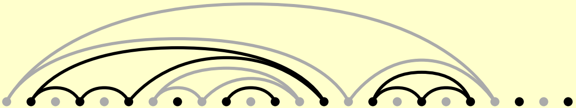

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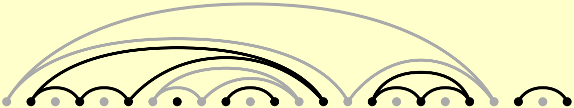

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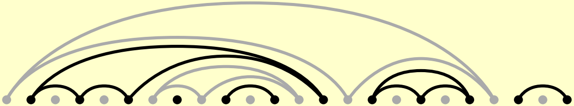

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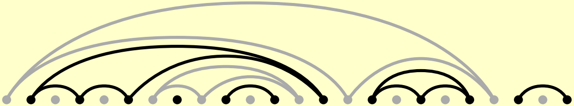

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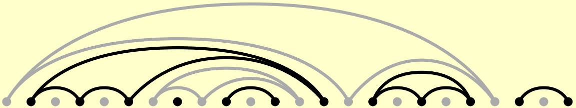

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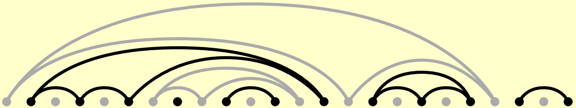

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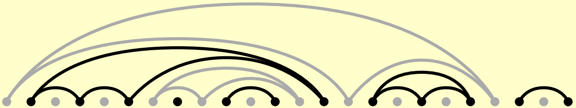

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General conjecture

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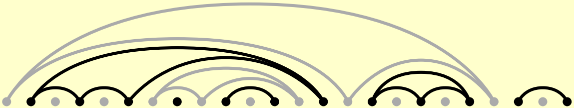

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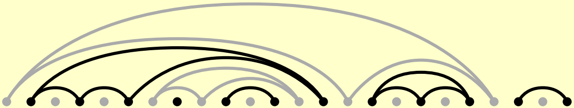

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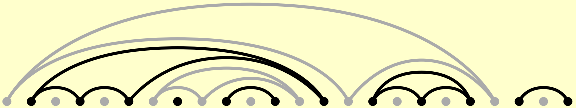

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Bilinear form on noncrossing perfect matchings of $[2n]$

$\langle \quad , \quad \rangle$

Bilinear form on noncrossing perfect matchings of $[2n]$

$\langle \mathbf{m}, \quad \rangle$

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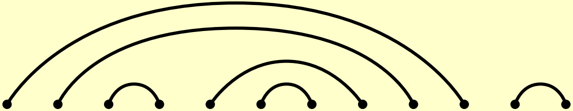

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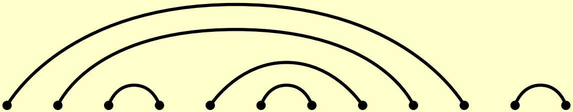

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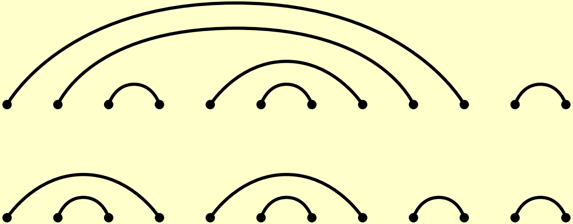

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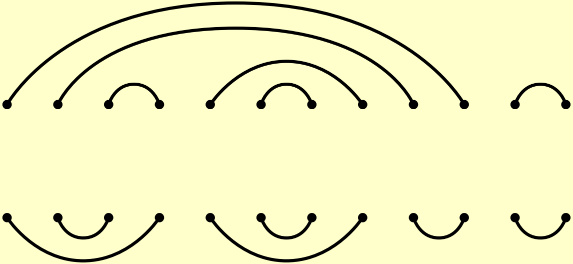

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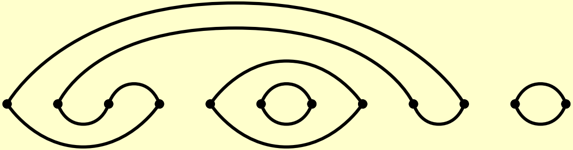

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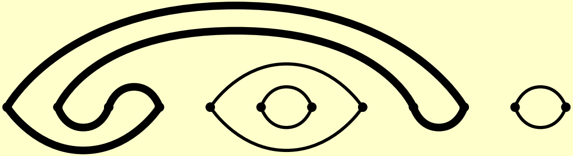

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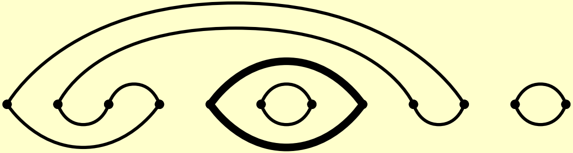

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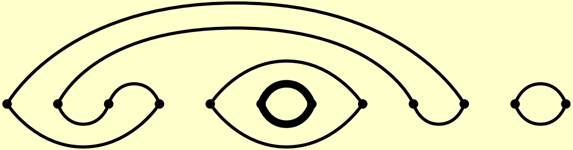

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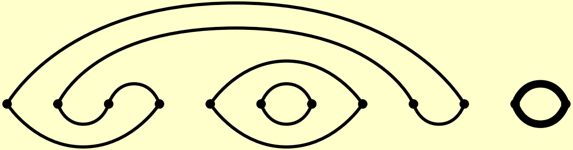

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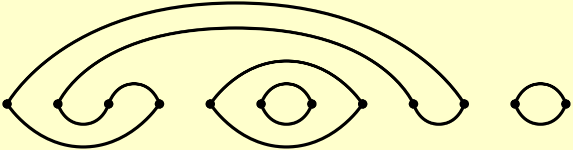

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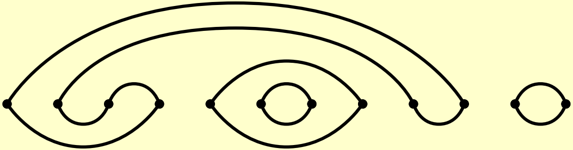

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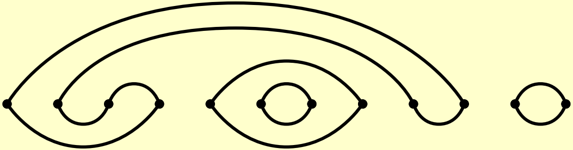

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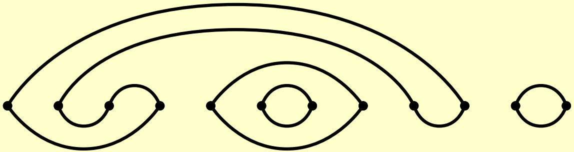

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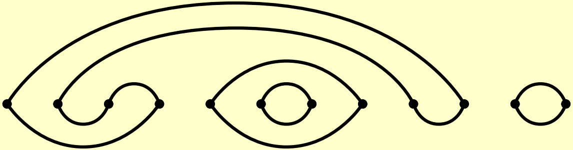

$\mathbf{n = 3}$

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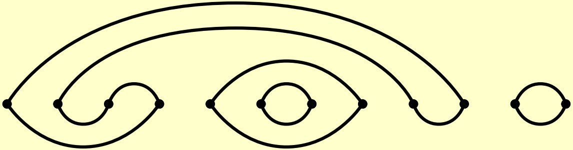

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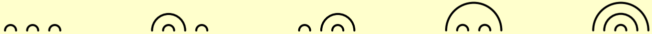

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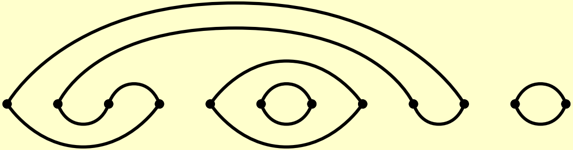

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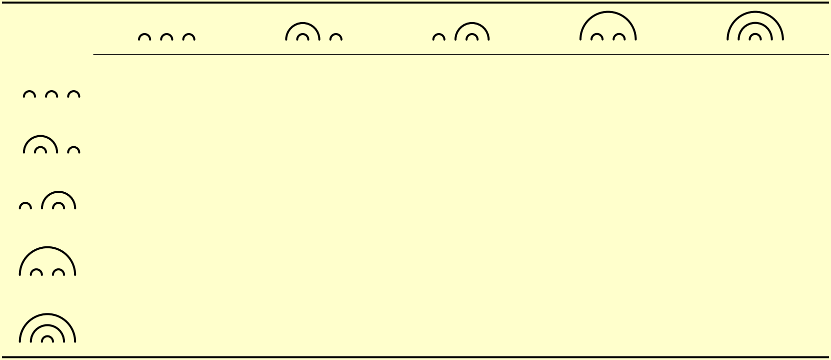

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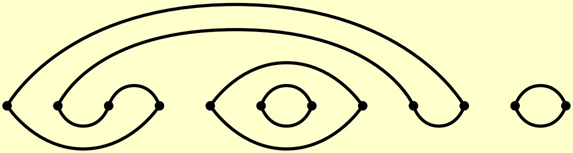

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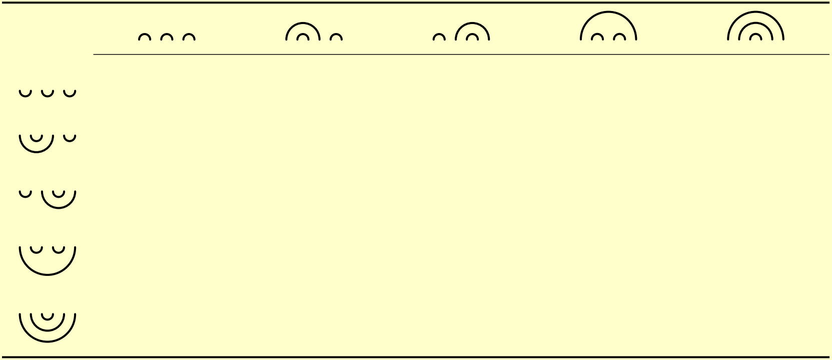

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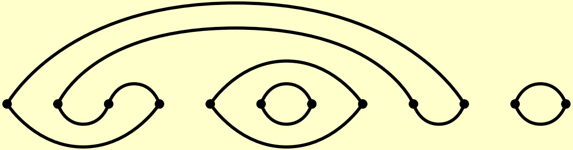

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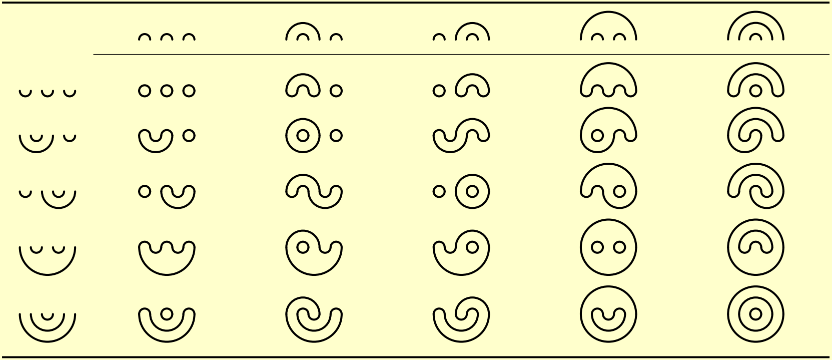

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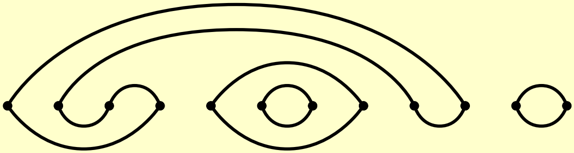
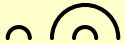


$\mathbf{n = 3}$

					
	q^3	q^2	q^2	q	q^2
	q^2	q^3	q	q^2	q
	q^2	q	q^3	q^2	q
	q	q^2	q^2	q^3	q^2
	q^2	q	q	q^2	q^3

Bilinear form on noncrossing perfect matchings of $[2n]$

$$\begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}$$

Bilinear form on noncrossing perfect matchings of $[2n]$

$$M_n = \begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}$$

Bilinear form on noncrossing perfect matchings of $[2n]$

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Bilinear form on noncrossing perfect matchings of $[2n]$

$\mathbf{n=3}$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & q & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & q \\ -q & 1 & 1 & -q & q^2-1 \end{pmatrix} \begin{pmatrix} q^3 & q^2 & q^2 & q & q^2 \\ q^2 & q^3 & q & q^2 & q \\ q^2 & q & q^3 & q^2 & q \\ q & q^2 & q^2 & q^3 & q^2 \\ q^2 & q & q & q^2 & q^3 \end{pmatrix}$$

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Bilinear form on noncrossing perfect matchings of $[2n]$

$\mathbf{n = 3}$

$$\begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q(q-1)(q+1) & 0 & 0 & 0 \\ 0 & 0 & q(q-1)(q+1) & 0 & 0 \\ 0 & 0 & 0 & q(q-1)(q+1) & 0 \\ 0 & 0 & 0 & 0 & q(q-1)(q+1)(q^2-2) \end{pmatrix}$$

Bilinear form on noncrossing perfect matchings of $[2n]$

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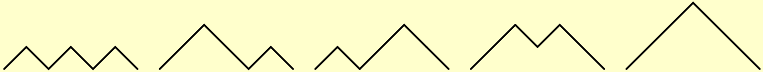

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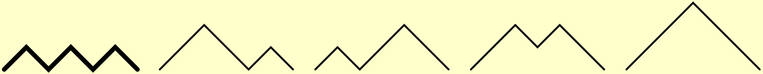

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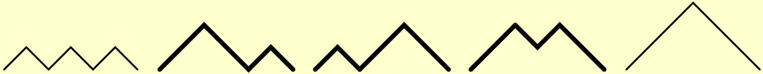

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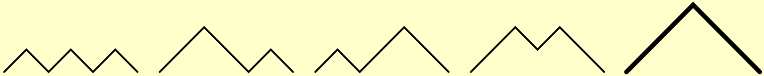

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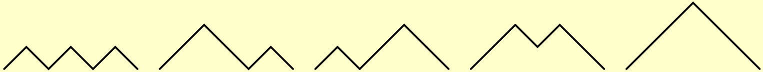

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Bilinear form on noncrossing perfect matchings of $[2n]$

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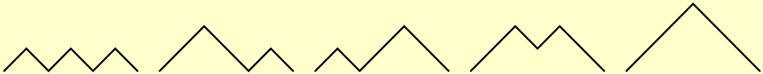
$$\begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q(q-1)(q+1) & 0 & 0 & 0 \\ 0 & 0 & q(q-1)(q+1) & 0 & 0 \\ 0 & 0 & 0 & q(q-1)(q+1) & 0 \\ 0 & 0 & 0 & 0 & q(q-1)(q+1)(q^2-2) \end{pmatrix}$$



Conjecture (Miller–Stanton) M_n has a SSNF over $\mathbb{Z}[q]$. In fact

- SSNF of $M_n = \mathrm{diag}(\ p_1(q)\,\mathrm{I}_{h_1}\ ,\ p_2(q)\,\mathrm{I}_{h_2}\ ,\ \ldots\ ,\ p_n(q)\,\mathrm{I}_{h_n}\ )$
- $h_m = \#\{$ Dyck paths of length $2n$ and height $m\ \}$

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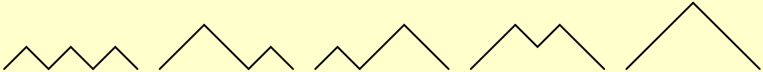

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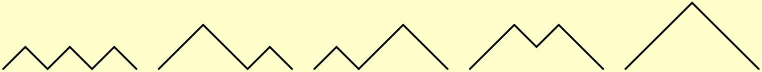

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