# Hypergraph polytopes and trialgebras

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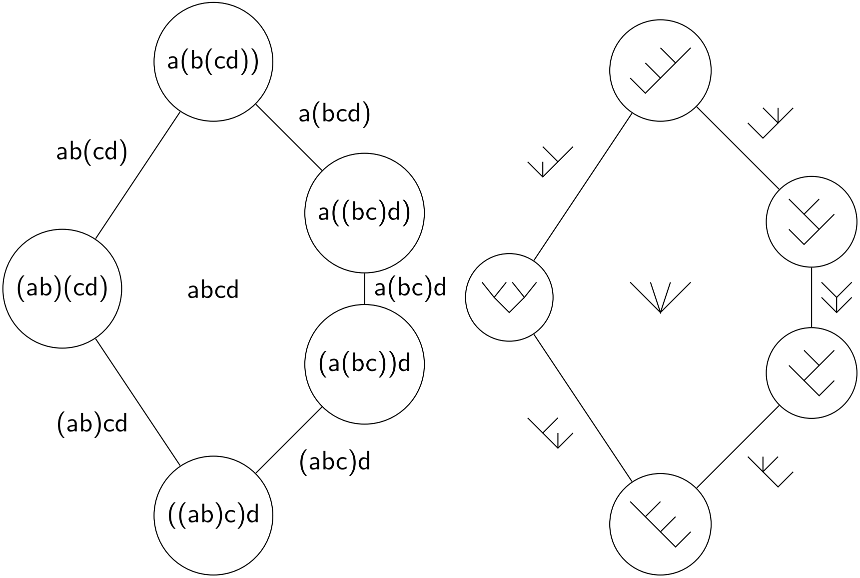
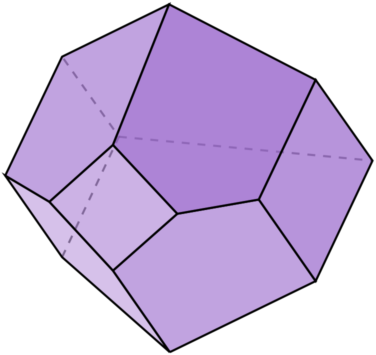
# Outline

# Motivation : Associahedron

# Associahedron and associativity



# Associahedron



with faces of dimension $k$ indexed by parenthesised words ($\leftrightarrow$ planar trees) with $n - k + 1$ parentheses.

# Tridendriform algebras

## Example

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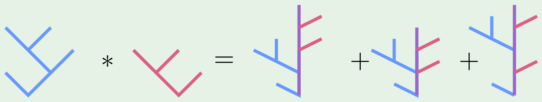
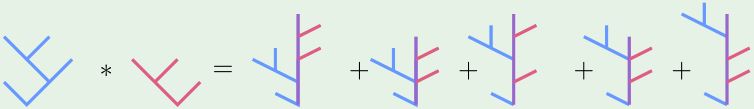
## Example



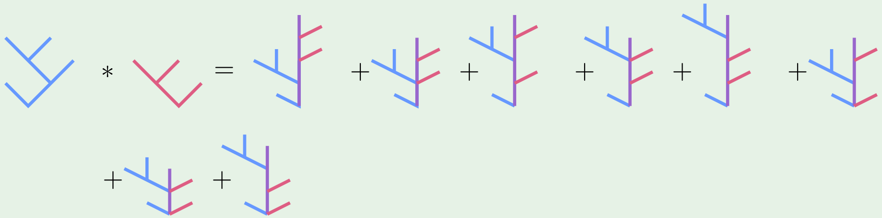
# Tridendriform algebras

## Example



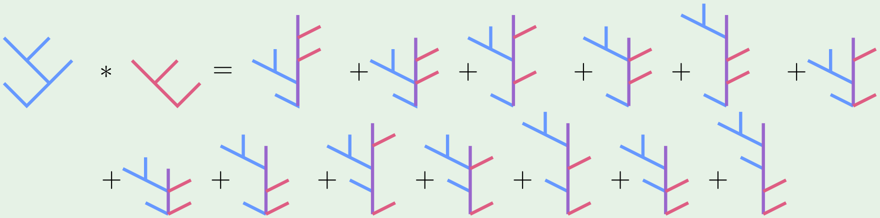
# Tridendriform algebras

## Example



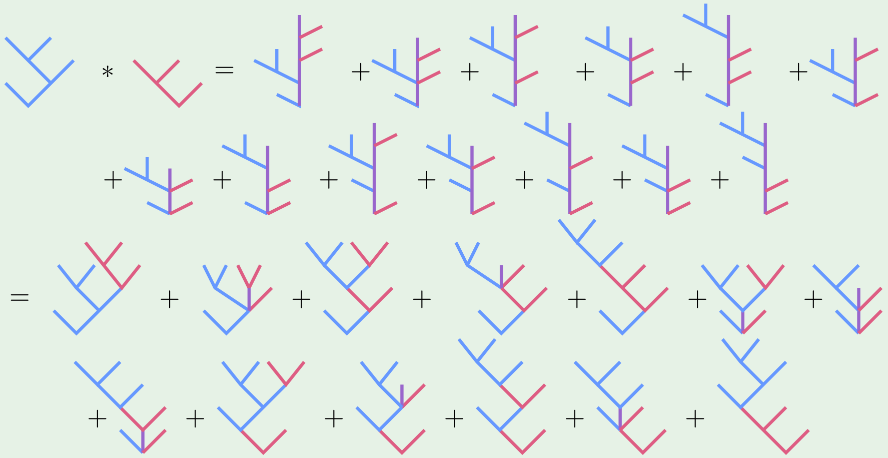
# Tridendriform algebras

## Example



# Tridendriform algebras

## Example



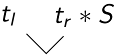
→ Three types of trees (looking at the root)

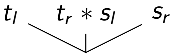
# Recursive definition of tridendriform products

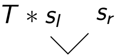
If $T =$ 

and $S =$ 

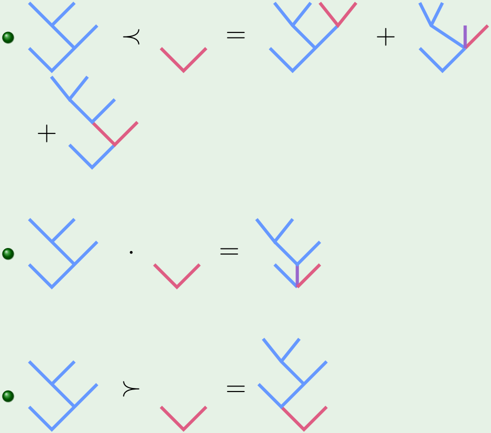
,

$T \prec S =$ 


$T \cdot S =$ 


and $T \succ S =$ 


**Example**



# Free tridendriform algebra

**Definition (Loday, Ronco, 2004 ; Chapoton 2002)**

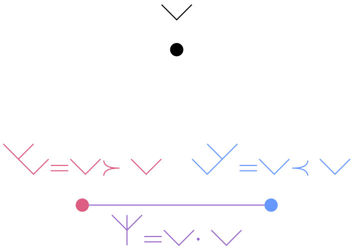
A *tridendriform algebra* is a vector space $A$ endowed with $\prec: A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ: A \otimes A \to A$, satisfying :

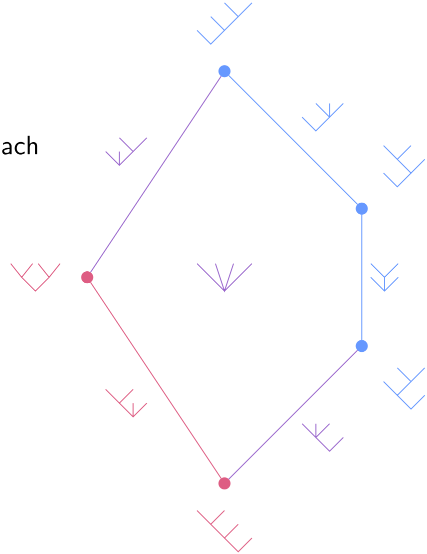
1. $(a \prec b) \prec c = a \prec (b * c)$,
2. $(a * b) \succ c = a \succ (b \succ c)$,
3. $(a \succ b) \prec c = a \succ (b \prec c)$,
4. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
5. $(a \succ b) \cdot c = a \succ (b \cdot c)$,
6. $(a \prec b) \cdot c = a \cdot (b \succ c)$,
7. $(a \cdot b) \prec c = a \cdot (b \prec c)$,

with $* = \prec + \cdot + \succ$

# Link between associahedron and tridendriform algebras

- Faces of the associahedron labelled by planar trees (basis of free tridend.alg.)
- Faces of dimension 0 from $\prec$ and $\succ$. Each use of $\cdot$ increase the dimension of the associated face by one.





# Questions

In literature,

- Labelling of polytopes faces by combinatorial structures
- Existence of polytopes on this structures
- Existence of algebras on this structures

### Question :

Is it possible starting from a family of polytopes to construct an algebra (operad in fact) associated to it?

# Constructs of a hypergraph polytope

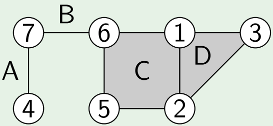
# Hypergraphs

## Definition

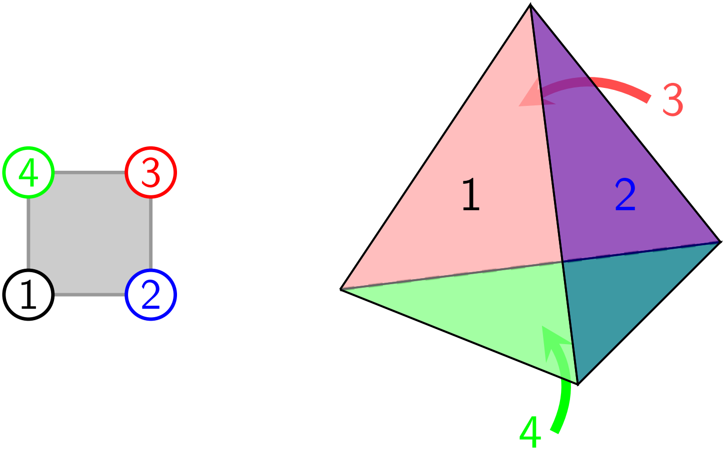
*A hypergraph (with vertex set $V$) is a pair $(V, E)$ where:*

- *$V$ is a finite set, (set of vertices)*
- *$E$ is a subset of $\mathcal{P}(V)$, the powerset of $V$ (set of edges), with $|e| \geq 2$ for every edge $e \in E$.*

## Example of a hypergraph on $[1; 7]$

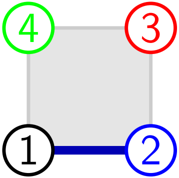


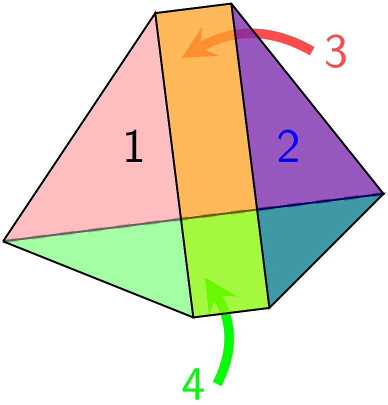
# Hypergraph polytopes [Dosen, Petric] (=Nestohedra [Postnikov])




By default, an edge containing every vertices.

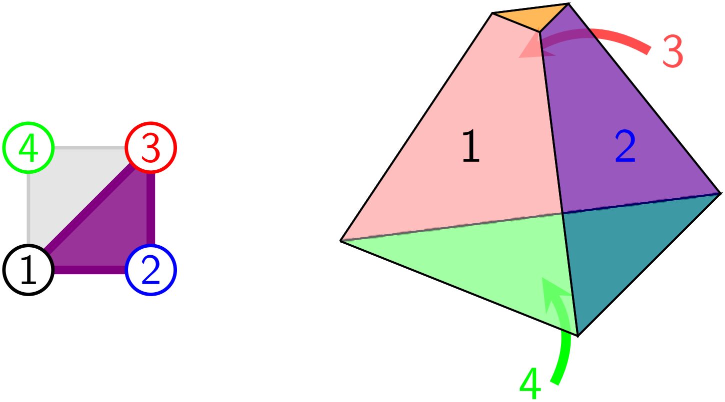
# Hypergraph polytopes (=nestohedra)




Edges $\{a_1, \ldots, a_n\}$ corresponds to truncation of $a_1 \cap \ldots \cap a_n$

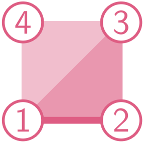
# Hypergraph polytopes

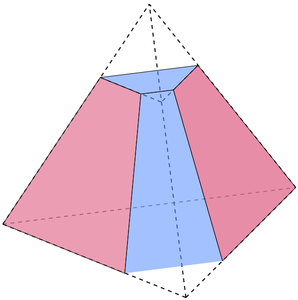



Edges $\{a_1, \ldots, a_n\}$ corresponds to truncation of $a_1 \cap \ldots \cap a_n$

# Hypergraph polytope

Example of the truncature associated with a flag hypergraph :



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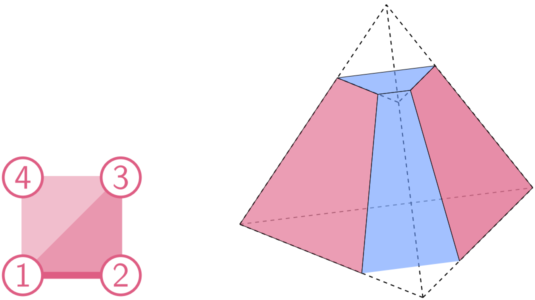


Do you recognize it ?

# Hypergraph polytope

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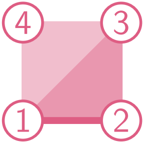
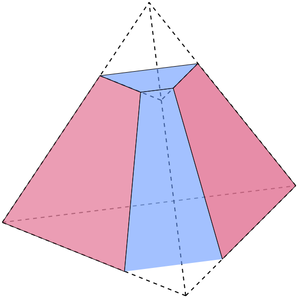


Do you recognize it ?

→ it is the cube !

# Constructs (=tubings, spines)

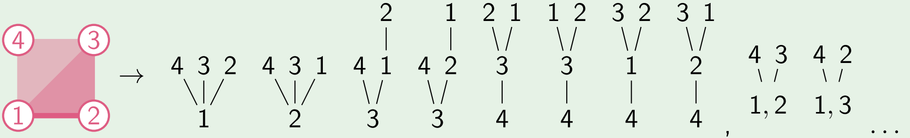
## Construct [Feichtner, Sturmfels ; Dosen, Petric ; Curien, Ivanovic, Obradovic]

A construct of a hypergraph $H$ is defined recursively.For $E \in V(H)$ (set of vertices of $H$),

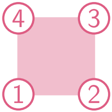
- If $E = V(H)$, the associated construct is a rooted tree with a single node labelled by $E$,
- Otherwise, considering $(T_1, \ldots, T_n)$ constructs of the connected components of $H - E$, we get a construct by adding to this forest of trees a root labelled by $E$.

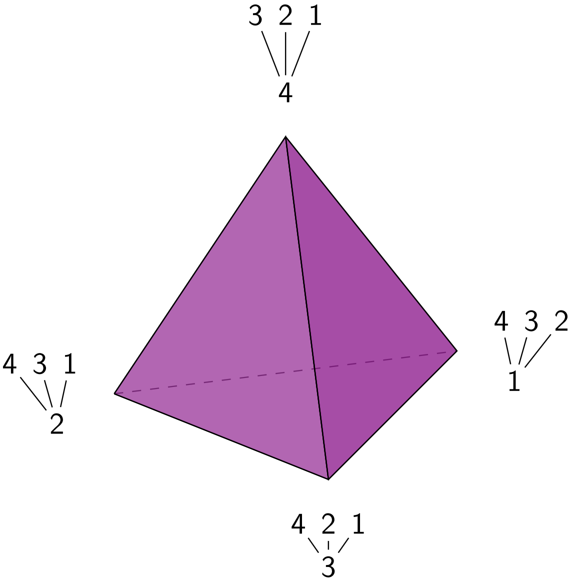
The set of constructs labels faces of the associated polytopes.

## First example:



# Constructs of the simplex



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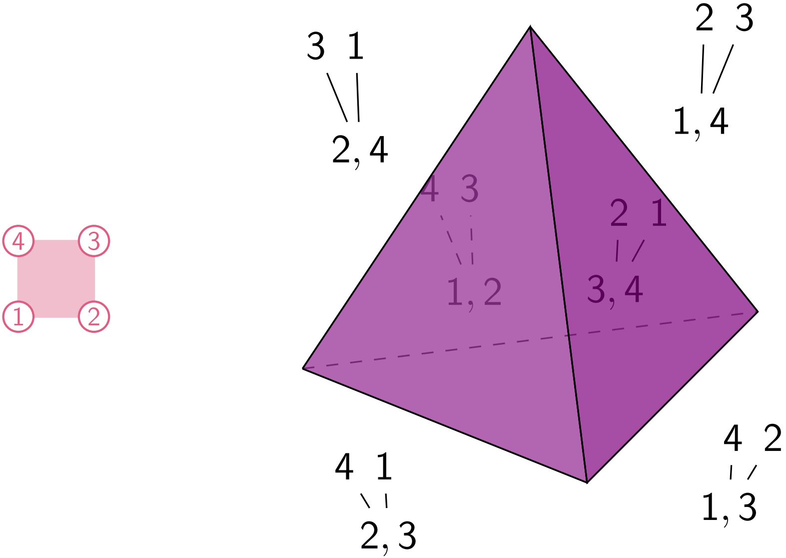

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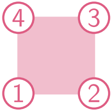


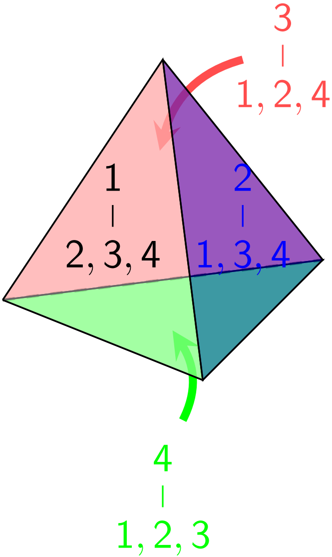

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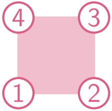
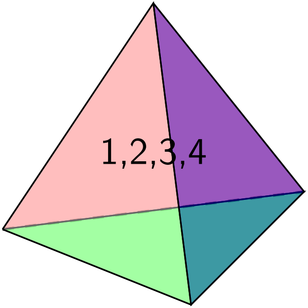
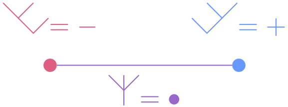


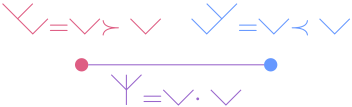
# Combinatorial interpretations of constructs

**Simplex** To a face $\{a_1, \ldots, a_k\}$ of dimension $k$ is associated the multipointed set $(V(H), \{a_1, \ldots, a_k\})$, consisting of vertices of the associated hypergraph pointed in $a_1, \ldots, a_k$

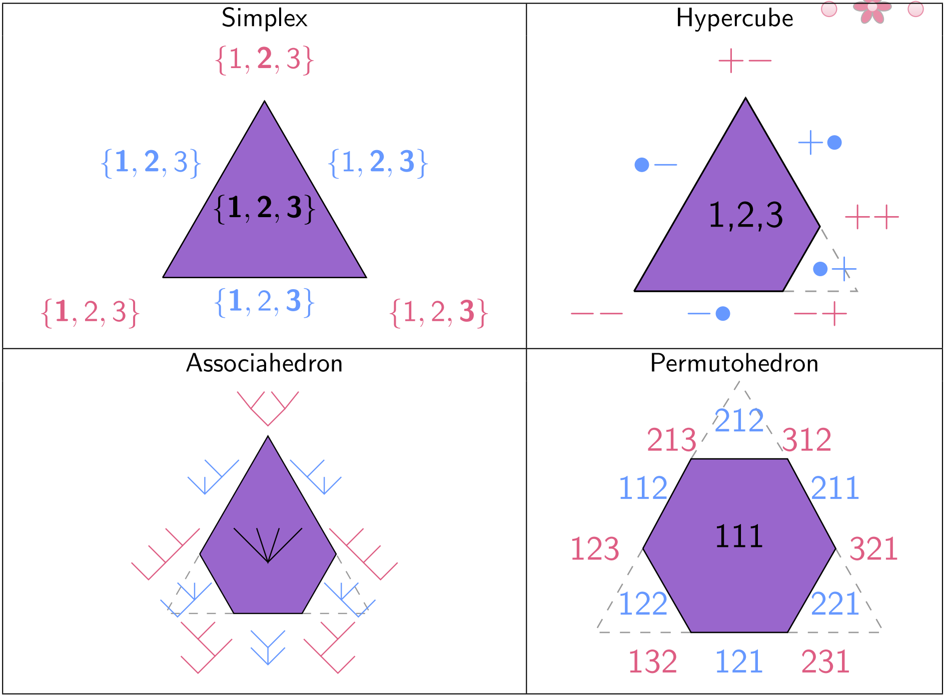




**Cube** To a face of dimension $k$ is associated a sequence of length $n-1$ $+$, $-$ and $k$ $\bullet$ (or left-combshaped trees)

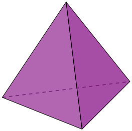
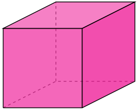
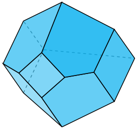
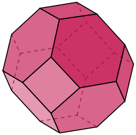
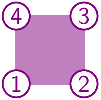
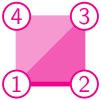
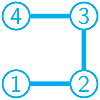
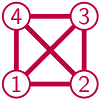
**Associahedron** To any face is associated a planar tree




**Permutahedron** To any face of dimension $k$ is associated a surjection of highness $k$



Simplex
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
{1, 2, 3}
Hypercube
+−
•−
+•
1,2,3
++
•+
−−
−•
−+
Associahedron
Permutohedron
212
213
312
112
211
111
123
321
122
221
132
121
231

| Polytopes | Simplex | Hypercube | Associahedron | Permutohedron |
|---|---|---|---|---|
| Picture |  |  |  |  |
| Associated Hypergraphs |   |   |   |   |
| Combinatorial objects | Multipointed sets | Paths with steps E, NE+ et NE- | Planar trees | Surjections |
| Cardinality | $2^{n+1} - 1$ (A074909) | $3^n$ (A013609) | Super-Catalan (A001003) | Fubini nbrs (A000670) |

# Tristructures on constructs

# Families of hypergraphs (non symmetric case)

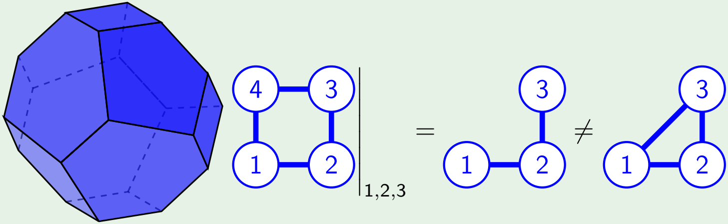
## Definition

*Let* $G = \{G_n, n \geq 1\}$ *be a family of hypergraphs. This family is compatible if*

- $G_n$ *is a hypergraph on* $n$ *vertices* $\{1, \ldots, n\}$
- $\forall k, l \geq 1,\ k + l = n,\ G_n\big|_{\{1,\ldots,k\}} = G_k$ *and* $G_n\big|_{\{k+1,\ldots,n\}} = \tilde{G}_l$*, where* $\tilde{G}_l$ *is obtained from* $G_l$ *by relabelling* $(1, \ldots, l)$ *to* $(k+1, \ldots, n)$*.*

It implies if some vertices belong to the same edge in a hypergraph, they also belong to the same edge in higher hypergraph.

## Cyclohedron



# Tristructures

Given two constructs $T = \begin{array}{c} T_1 \ \cdots \ T_n \\ \diagdown \ | \ \diagup \\ T_0 \end{array}$ and $S = \begin{array}{c} S_1 \ \cdots \ S_m \\ \diagdown \ | \ \diagup \\ S_0 \end{array}$ of hypergraphs $G_k$ and $G_l$, we define:

- $T \prec S$ is the sum of constructs of $G_{k+l}$ with root $T_0$
- $T \succ S$ is the sum of constructs of $G_{k+l}$ with root $S_0$
- $T \cdot S$ is the sum of constructs of $G_{k+l}$ with root $T_0 \cup S_0$,

which preserve partial orders given by $T$ and $S$.

# Tristructures

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## Conjecture

*It endows the graded vector space of constructs of a compatible family of hypergraphs with a structure of free trialgebra over one generator (associated to an operad).*

# Example : Trialgebra of the simplex (=Trias)

Constructs of the simplicies are multipointed sets.

The previous operations are then given by:

$$T \prec S = T \cup \bar{S},\ \ T \succ S = \bar{T} \cup S,\ \ T \cdot S = T \cup S,$$

where $\bar{T}$ (resp. $\bar{S}$) is the underlying set of the multipointed set $T$ (resp. $S$).

This algebra is algebra Trias introduced by Loday and Ronco.

# Example : Trialgebra of the cube (=new !)

Constructs of the cube are sequences of $\{+, -, \bullet\}$ of length $n-1$.
$=$ sequences of $\{+, -, \bullet\}$ of length $n$ starting by $+$

The previous operations are then given by:

$$u \prec v = u(-^{|v|}),$$
$$u \succ (v_1 + v_2) = (u * v_1) + v_2,$$
$$u \cdot (v_1 + v_2) = u(-^{|v_1|}) \bullet v_2,$$

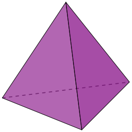
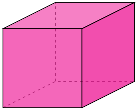
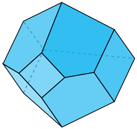
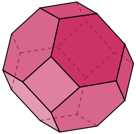
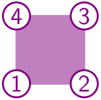
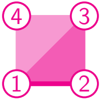
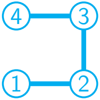
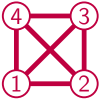
where $v_2$ is a sequence of $\{-, \bullet\}$, $* = \prec + \succ + \cdot$ and $u * \epsilon = u$.

This algebra is called Tricube.

# Tricube algebra

$$
\begin{aligned}
(u \prec v) \prec w &= u \succ (v \prec w) \\
u \prec (v \# w) &= (u \prec v) \prec w \\
(u * v) \succ w &= u \succ (v \succ w) \\
(u \succ v) \cdot w &= u \succ (v \cdot w) \\
(u \prec v) \cdot w &= u \cdot (v \succ w) \\
(u \cdot v) \cdot w &= u \cdot (v \cdot w) \\
(u \cdot v) \prec w &= u \cdot (v \prec w),
\end{aligned}
$$

where $\# \in \{\prec, \succ, \cdot\}$ and $* = \succ + \cdot + \prec$.

| Polytopes | Simplex | Hypercube | Associahedron | Permutohedron |
|---|---|---|---|---|
| Picture |  |  |  |  |
| Associated Hypergraphs |  |  |  |  |
| Algèbres | Trias [Loday] | Tricube | Tridendriform [Loday-Ronco, Chapoton] | ST (graded version of [Chapoton]) |

# Check list

## Done

- Unified frame for tristructure on hypergraph polytopes
- New examples of operads
- Blue print method

## To do

- Prove that the necessary condition on the hypergraph family is sufficient
- Endow the algebras with a Hopf algebra structure,
- Study quantified variants of these algebras with:

$$a * b = a \prec b + q\ a \cdot b + a \succ b,$$

- Look at link between algebras coming from truncations of polytopes (for instance tridendriform structure on surjections),
- Examine other examples, ...

Thank you for your attention !

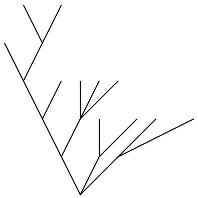


Figure: Left-combshaped trees with every non-leftmost child being the root of only corollas