Chromatic symmetric functions on graphs and polytopes

80th Séminaire Lotharingien de Combinatoire, Lyon

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March 27th, 2018

The chromatic symmetric function on graphs

A *colouring* on a graph G is a map $f : V(G) \to \mathbb{N}$.
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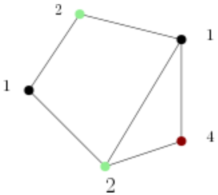

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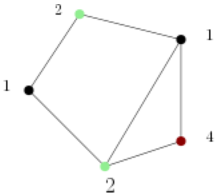

Figure: Example of a proper colouring f of a graph

Set $x_f = \prod_v x_{f(v)}$. We have $x_f = x_1^2 x_2^2 x_4$ in the figure.

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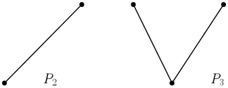

Figure: The line graph P_2 and the path P_3

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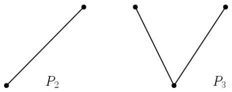

Figure: The line graph P_2 and the path P_3

Their CSF are

$$\Psi_{\mathbf{G}}(P_2) = 2 \sum_{1 \leq i < j} x_i x_j \,, \quad \Psi_{\mathbf{G}}(P_3) = 6 \left(\sum_{1 \leq i < j < k} x_i x_j x_k \right) + \left(\sum_{i \neq j} x_i^2 x_j \right) .$$

Evaluating $x_1 = \cdots = x_t = 1$ and $x_i = 0$ for $i > t$ we obtain the chromatic polynomial $\chi_G(t)$.

Tree conjecture on graphs

Given the CSF of a graph we can compute the amount of **edges**, **connected components**, decide if it is a **tree** and compute the **degree sequence** for trees, but

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Figure: Non-isomorphic graphs with the same CSF[1]

Conjecture (Tree conjecture - Stanley and Stembridge)

Any two non-isomorphic trees T_1, T_2 have distinct CSF.

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Think about the chromatic polynomial

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CF on graphs - The kernel problem

Question (The kernel problem on graphs)

Describe all linear relations of the form

$$\sum_i a_i \Psi_{\mathbf{G}}(G_i) = 0\,.$$

Theorem (RP-2017)

The space $\ker \Psi_{\mathbf{G}}$ *is spanned by the modular relations and isomorphism relations.*

Outline

Graphs terminology

The edge deletion of a graph: $H \setminus \{e\}$.

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The edge deletion of a graph: $H \setminus \{e\}$.

The edge addition of a graph: $G + \{e\}$.

Modular relations

$$\Psi_{\mathbf{G}}(G) = \sum_{f \text{ proper on } G} x_f \,.$$

Proposition (Modular relations - Guay-Paquet, Orellana, Scott, 2013)

Let G be a graph that contains an edge e_3 and does not contain e_1, e_2 such that the edges $\{e_1, e_2, e_3\}$ form a triangle. Then,

$$\Psi_{\mathbf{G}}(G) - \Psi_{\mathbf{G}}(G + \{e_1\}) - \Psi_{\mathbf{G}}(G + \{e_2\}) + \Psi_{\mathbf{G}}(G + \{e_1, e_2\}) = 0 \,.$$

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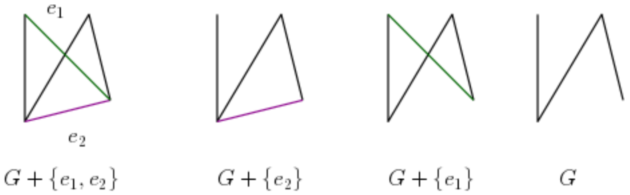

The kernel problem

For G_1, G_2 isomorphic graphs, we have $G_1 - G_2 \in \ker \Psi_{\mathbf{G}}$. These are called *isomorphism relation*.

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The kernel of $\Psi_{\mathbf{G}}$ is generated by modular relations and isomorphism relations.

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Theorem (RP-2017)

The kernel of $\Psi_{\mathbf{G}}$ is generated by modular relations and isomorphism relations.

Let $\mathcal{M} = \langle$ modular relations, isomorphism relations $\rangle$.
Goal: $\ker \Psi_{\mathbf{G}} = \mathcal{M}$.

Idea of proof - Rewriting graph combinations

$$e_3 \in G \Rightarrow G - (G + \{e_1\}) - (G + \{e_2\}) + (G + \{e_1, e_2\}) \in \mathcal{M}\,.$$

- Take $z = \sum_i G_i a_i$ in the kernel of $\Psi_{\mathbf{G}}$.

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- Linear algebra magic. Cash in the theorem.

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Proposition (Non-extendible graphs)

A graph is non-extendible if and only if any connected component of G^c, the complement graph of G, is a complete graph.

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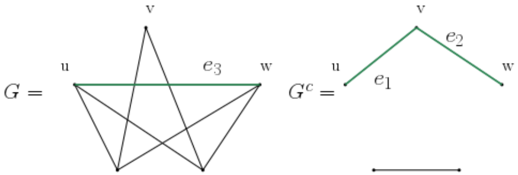

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Note: Up to isomorphism, we can identify a partition λ with a non-extendible graph K_λ^c in such a way $\lambda = \lambda(G^c)$.

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Note: Up to isomorphism, we can identify a partition λ with a non-extendible graph K_λ^c in such a way $\lambda = \lambda(G^c)$.
Consequence: Our original z can be rewritten, using modular relations and isomorphic relations, as

$$z = \sum_\lambda K_\lambda^c a_\lambda \in \ker \Psi_{\mathbf{G}} \, .$$

Idea of proof - Rewriting graph combinations

So

$$z = \sum_{\lambda} K_{\lambda}^{c} a_{\lambda} \in \ker \Psi_{\mathbf{G}} \, ,$$

Apply $\Psi_{\mathbf{G}}$ to get

$$0 = \sum_{\lambda} \Psi_{\mathbf{G}}(K_{\lambda}^{c}) a_{\lambda} \Rightarrow a_{\lambda} = 0 \, .$$

Possible to show: the set $\{\Psi_{\mathbf{G}}(K_{\lambda}^{c})\}_{\lambda}$ is linearly independent. So $z = 0$, as desired.

Polytopes

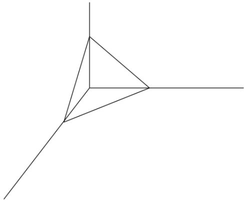
Fix a dimension n. A polytope is a bounded set of the form $\mathfrak{q} = \{x \in \mathbb{R}^n | Ax \leq b\}$.
Given a colouring $f : [n] \to \mathbb{N}$ of the **coordinates**, the face $\mathfrak{q}_f$ is

$$\mathfrak{q}_f = \arg\min_{x \in \mathfrak{q}} \sum_{i=1}^{n} x_i f(i) \,.$$

Polytopes: Examples

Simplexes and its dilations: Consider $J \subseteq [n]$ non empty.

$$\lambda \mathfrak{s}_J = \mathrm{conv}\{\lambda e_i | i \in J\}\,.$$



The permutahedron and its generalisations

The n order permutahedron: $\mathfrak{per} = \text{conv}\{(\sigma(1), \ldots, \sigma(n)) | \sigma \in S_n\}$.
Is $(n-1)$-dimensional.

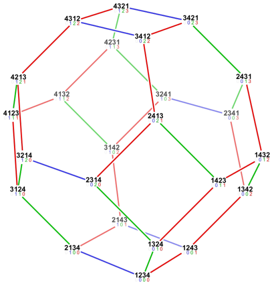
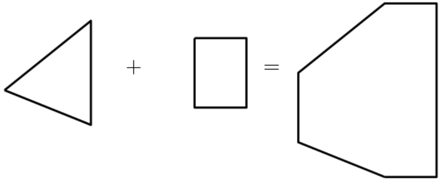


Figure: The 4-permutahedron[2]

[2] https://en.wikipedia.org/wiki/Permutohedron

Minkowsky sum

$$A +_M B = \{a + b | a \in A, b \in B\} .$$



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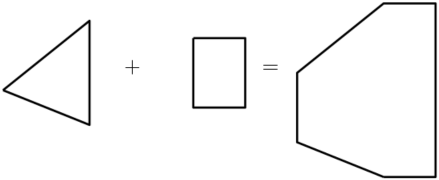

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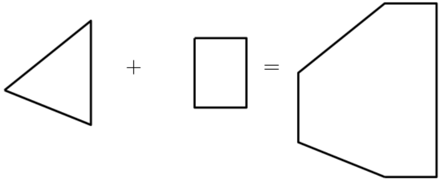

$C := A -_M B$ if $A = C +_M B$.
C may not exist but if exists it is **unique** (only for polytopes).

The permutahedron and its generalisations

A *generalised permutahedron* is a polytope $\mathfrak{q}$ of the form

$$\mathfrak{q} = \left({\sum_{\substack{J \neq \emptyset \\ a_J > 0}}}^{M} a_J \mathfrak{s}_J \right) -_M \left({\sum_{\substack{J \neq \emptyset \\ a_J < 0}}}^{M} |a_J| \mathfrak{s}_J \right),$$

A *nestohedron* is only the positive part:

$$\mathfrak{q} = {\sum_{\substack{J \neq \emptyset \\ a_J > 0}}}^{M} a_J \mathfrak{s}_J \, .$$

Generalised permutahedra - Examples

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We define the *chromatic quasisymmetric function* (CF) as

$$\Psi_{\mathbf{GP}}(\mathfrak{q}) = \sum_{\mathfrak{q}_f = \mathrm{pt}} x_f .$$

Zonotopes and other embedings

Given a graph G, its zonotope is defined as

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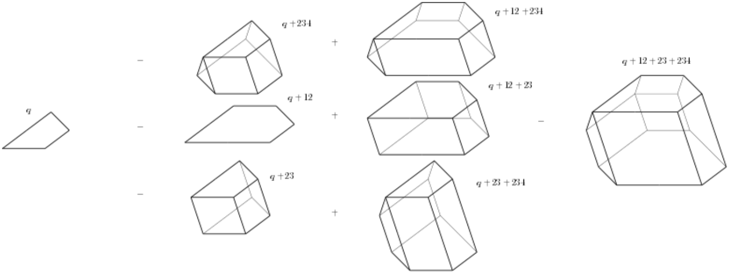
Faces of nestohedra

Proposition (Modular relations on nestohedra)

Consider a nestohedron $\mathfrak{q}$*,* $\{B_j | j \in T\}$ *a family of subsets on* $\{1, \cdots n\}$ *and* $\{a_j | j \in T\}$ *some positive scalars. Suppose "some magic" happens. Then,* $\sum_{T \subseteq J} (-1)^{\# T} \Psi_{\mathbf{GP}} \left[\mathfrak{q} +_M \sum_{j \in T}^{M} a_j \mathfrak{s}_{B_j} \right] = 0.$



K_π^c parallel and conclusion of proof

Theorem (RP 2017)

The modular relations, the isomorphism relations and the simple relations span the kernel of the restriction of $\Psi_{\mathbf{GP}}$ to the nestohedra.

Tree conjecture on graphs

This is a graph invariant:

$$\chi'(G) = \sum_f x_f \prod_i q_i^{\# \text{ monochromatic edges in } f \text{ of colour } i}$$

where the sum runs over all colourings.

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Conjecture (Tree conjecture)

Any two non-isomorphic trees T_1, T_2 have distinct χ'.

Further questions

- From nestohedra to generalised permutahedra?
- The image of the CF on graphs $\Psi_{\mathbf{G}}$ is spanned by $\{\Psi_{\mathbf{G}}(K_\lambda^c)\}_\lambda$, which forms a basis of $\operatorname{im}\Psi_{\mathbf{G}}$. Combinatorial meaning of the coefficients?

Thank you