Coefficientwise Hankel-total positivity

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Lyon, 26 March 2018

A big project in collaboration with
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Key references:

1. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32**, 125–161 (1980).

2. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (UQAM, 1983).

Total positivity

A (finite or infinite) matrix of real numbers is called *totally positive* if all its minors are nonnegative.

Applications:

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Statistics
- **Stieltjes moment problem**
- **Enumerative combinatorics**

$\vdots$

Hankel-total positivity

Given a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$, we define its *Hankel matrix*

$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j \ge 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

Fundamental Characterization (Stieltjes 1894, Gantmakher–Krein 1937):

For a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure μ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$.
[That is, $(a_n)_{n \ge 0}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \ge 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

From numbers to polynomials

Consider **polynomials** that enumerate some combinatorial objects with **weights** for some statistics: e.g.

- Counting permutations of $[n]$ by number of cycles:

  $S_n(x) = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$ (Stirling cycle polynomial)

- Counting permutations of $[n]$ by number of descents:

  $A_n(x) = \sum_{k=0}^{n} \left\langle{n \atop k}\right\rangle x^k$ (Eulerian polynomial)

- Counting partitions of $[n]$ by number of blocks:

  $B_n(x) = \sum_{k=0}^{n} \left\{{n \atop k}\right\} x^k$ (Bell polynomial)

- Counting non-crossing partitions of $[n]$ by number of blocks:

  $N_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$ (Narayana polynomial)

These polynomials can also be **multivariate**!

An industry in combinatorics: q-Narayana polynomials, (p,q)-Bell polynomials, ...

Coefficientwise total positivity

- Consider sequences and matrices whose entries are *polynomials* with real coefficients in one or more indeterminates $\mathbf{x}$.
- A matrix is *coefficientwise totally positive* if every minor is a polynomial with nonnegative coefficients.
- A sequence is *coefficientwise Hankel-totally positive* if its Hankel matrix is coefficientwise totally positive.
- More generally, can consider sequences and matrices with entries in a *partially ordered commutative ring*.

But now there is no analogue of the Fundamental Characterization!

Coefficientwise Hankel-TP is **combinatorial**, not analytic.

Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n\geq 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \geq 0$, but it is *stronger*.

Coefficientwise Hankel-TP in combinatorics

Many interesting sequences of combinatorial polynomials $(P_n(x))_{n\geq 0}$ have been proven in recent years to be coefficientwise *log-convex*:

- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} {n \brace k} x^k$
  (Liu–Wang 2007, Chen–Wang–Yang 2011)
- Narayana polynomials $N_n(x) = \sum\limits_{k=0}^{n} N(n,k)\, x^k$
  (Chen–Wang–Yang 2010)
- Narayana polynomials of type B: $W_n(x) = \sum\limits_{k=0}^{n} \binom{n}{k}^2 x^k$
  (Chen–Tang–Wang–Yang 2010)
- Eulerian polynomials $A_n(x) = \sum\limits_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$
  (Liu–Wang 2007, Zhu 2013)

Might these sequences actually be coefficientwise *Hankel-totally positive*?

- In many cases I can prove that the answer is **yes**, by using the Flajolet–Viennot method of *continued fractions*.
- In several other cases I have strong **empirical evidence** that the answer is **yes**, but no proof.
- The continued-fraction approach gives a *sufficient but not necessary* condition for coefficientwise Hankel-total positivity.

Combinatorics of continued fractions (Flajolet 1980)

We consider two types of continued fractions:

- Stieltjes type (S-fractions):

$$f(t) \;=\; \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}$$

- Jacobi type (J-fractions):

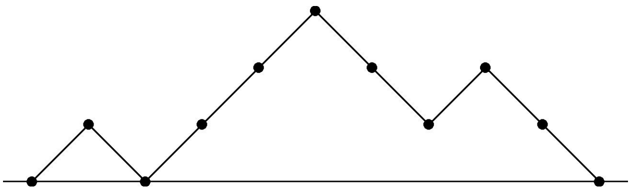
$$f(t) \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \gamma_3 t - \cdots}}}}$$

For lack of time I will show here only Stieltjes.

Jacobi can also be handled, but Hankel-TP is more subtle.

Combinatorics of Stieltjes-type continued fractions

A *Dyck path* of length $2n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ ["rise"] and $(1,-1)$ ["fall"]:



Theorem (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$, we have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \;=\; \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n)\, t^n$$

where $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height i gets weight α_i.

$S_n(\boldsymbol{\alpha})$ is called the *Stieltjes–Rogers polynomial* of order n.

Theorem (A.S. 2014, based on Viennot 1983): The sequence $(S_n(\boldsymbol{\alpha}))_{n \ge 0}$ of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

Proof uses the Karlin–McGregor–Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

Can now specialize $\boldsymbol{\alpha}$ to *nonnegative* elements in any partially ordered commutative ring, and get Hankel-TP.

Example 1: Narayana polynomials

- Narayana numbers $N(n,k) = \dfrac{1}{n}\binom{n}{k}\binom{n}{k-1}$
- Count numerous objects of combinatorial interest:
  - Dyck paths of length $2n$ with k peaks
  - Non-crossing partitions of $[n]$ with k blocks
  - Non-nesting partitions of $[n]$ with k blocks
- Narayana polynomials $N_n(x) = \sum\limits_{k=0}^{n} N(n,k)\, x^k$
- Ordinary generating function $\mathcal{N}(t,x) = \sum\limits_{n=0}^{\infty} N_n(x)\, t^n$
- Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} \;=\; \frac{1}{1 - tx - t(\mathcal{N} - 1)}$$

  which can be rewritten as

$$\mathcal{N} \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - t\mathcal{N}}}$$

- Leads immediately to S-type continued fraction

$$\sum_{n=0}^{\infty} N_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cdots}}}}}$$

Conclusion: The sequence $(N_n(x))_{n \ge 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive.

Example 2: Bell polynomials

- Stirling number $\left\{ {n \atop k} \right\}$ = # of partitions of $[n]$ with k blocks
- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
- Ordinary generating function $\mathcal{B}(t,x) = \sum\limits_{n=0}^{\infty} B_n(x)\, t^n$
- Flajolet (1980) expressed $\mathcal{B}(t,x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} B_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

Conclusion: The sequence $(B_n(x))_{n \ge 0}$ of Bell polynomials is coefficientwise Hankel-totally positive.

- Can extend to polynomial $B_n(x,p,q)$ that enumerates set partitions w.r.t. blocks (x), crossings (p) and nestings (q):

$$\sum_{n=0}^{\infty} B_n(x,p,q)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{[1]_{p,q} t}{1 - \cfrac{xt}{1 - \cfrac{[2]_{p,q} t}{1 - \cdots}}}}}$$

- Implies coefficientwise Hankel-TP jointly in x, p, q

Example 3: Narayana polynomials of type B

The polynomials

$$W_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

- Grand Dyck paths with weight x for each peak
- Coordinator polynomial of the classical root lattice A_n
- Rank generating function of the lattice of noncrossing partitions of type B on $[n]$

- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1+x,\; \frac{2x}{1+x},\; \frac{1+x^2}{1+x},\; \frac{x+x^2}{1+x^2},\; \frac{1+x^3}{1+x^2},\; \frac{x+x^3}{1+x^3},\; \ldots$$

- However, there *is* a nice *J-type* continued fraction:

$$\sum_{n=0}^{\infty} W_n(x)\, t^n \;=\; \cfrac{1}{1-(1+x)t - \cfrac{2xt^2}{1-(1+x)t - \cfrac{xt^2}{1-(1+x)t - \cfrac{xt^2}{1-\cdots}}}}$$

- By J-fraction theory (not explained here) one can show:

  Theorem (A.S. unpublished 2014, Wang–Zhu 2016):
  The sequence $(W_n(x))_{n\ge 0}$ is coefficientwise Hankel-TP.

Some cases I am *unable* (as yet) to prove ...

Finally, there are *many* cases where I find **empirically** that a sequence $(P_n(x))_{n\geq 0}$ is coefficientwise Hankel-TP, but I am unable to prove it because there is neither an S-type nor a J-type continued fraction in the ring of polynomials:

- Rook polynomials
- Domb polynomials
- Apéry polynomials
- Boros–Moll polynomials
- Ramanujan polynomials
- Inversion enumerators for trees
- Reduced binomial discriminant polynomials

⋮

Example 1: Rook polynomials

- Non-attacking rooks on $n \times n$ chessboard with weight x per rook:

$$R_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 k!\, x^k$$

- Can *prove*: Stieltjes moment sequence for each $x \geq 0$.
  (Can find explicit moment representation)
- *Empirical*: Hankel matrix is coefficientwise TP up to 11×11.
- **Conjecture**: Hankel matrix is coefficientwise TP.

Example 2: Apéry polynomials

- Apéry numbers $A_n \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$
- **Theorem** (conjectured by me, 2014; proven G. Edgar, unpub. 2016): $(A_n)_{n \geq 0}$ is a Stieltjes moment sequence.
- Define Apéry polynomials $A_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$
- **Conjecture 1**: $(A_n(x))_{n \geq 0}$ is a Stieltjes moment sequence for all $x \geq 1$ (but not for $0 < x < 1$).
- **Conjecture 2**: $(A_n(1+y))_{n \geq 0}$ is coefficientwise Hankel-TP in y.
  (Tested up to 12×12)

(Tentative) Conclusion

- Many interesting sequences $(P_n(\mathbf{x}))_{n \ge 0}$ of combinatorial polynomials are (or appear to be) coefficientwise Hankel-totally positive.

- In some cases this can be proven by the Flajolet–Viennot method of continued fractions.
  - When S-fractions exist, they give the simplest proofs.
  - Sometimes S-fractions don't exist, but J-fractions can work.

- Alas, in many cases neither S-fractions nor J-fractions exist (in the ring of polynomials).

- **New methods of proof will be needed:**
  - Branched continued fractions ($\Longrightarrow$ Pétréolle talk)
  - Differential operators?
  - Direct study of Hankel minors?
  - . . . ???

- Coefficientwise Hankel-TP is a big phenomenon that we understand, at present, only very incompletely.

Dédié à la mémoire de Philippe Flajolet (1948–2011)