Séminaire Lotharingien de Combinatoire **81** (2020), Article B81i

HANKEL DETERMINANTS, HAHN POLYNOMIALS, AND A FORMULA OF KRATTENTHALER

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Dedicated to Professor Christian Krattenthaler on the occasion of his 60th birthday

Abstract. In this note, we find a connection between an identity of C. Krattenthaler and some Hankel determinants related to the Hahn polynomials. We also consider some limiting cases related to the Meixner and Charlier polynomials.

1. Introduction

Let $\mathbb{N}_0$ denote the set

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

If $\{\mu_n\}$ is a sequence of complex numbers and $L : \mathbb{C}[x] \to \mathbb{C}$ is the linear functional defined by

$$L[x^n] = \mu_n, \quad n \in \mathbb{N}_0,$$

then L is called the *moment functional* [9] determined by the formal moment sequence $\{\mu_n\}$. The number μ_n is called the *moment* of order n.

Suppose that $\{P_n\}$ is a family of monic polynomials, with $\deg(P_n) = n$. If the polynomials $P_n(x)$ satisfy

$$L[P_n P_m] = h_n \delta_{n,m}, \quad \text{for } n, m \in \mathbb{N}_0, \tag{1.1}$$

where $h_0 = \mu_0$, $h_n \neq 0$ and $\delta_{n,m}$ is Kronecker's delta, then $\{P_n\}$ is called a *sequence of orthogonal polynomials* with respect to L. Since

$$L[x P_n P_k] = 0, \quad \text{for } k \notin \{n-1, n, n+1\},$$

the monic orthogonal polynomials $P_n(x)$ satisfy the *three-term recurrence relation*

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

with initial conditions $P_{-1} = 0$ and $P_0 = 1$, where $\beta_0 = \frac{\mu_1}{\mu_0}$ and

$$\beta_n = \frac{1}{h_n} L\left[x P_n^2\right], \quad \gamma_n = \frac{1}{h_{n-1}} L[x P_n P_{n-1}], \quad \text{for } n \in \mathbb{N}.$$

Since $L[x P_n P_{n-1}] = L\left[P_n^2\right]$, we have

$$\gamma_n = \frac{h_n}{h_{n-1}}, \quad \text{for } n \in \mathbb{N}, \tag{1.2}$$

and it follows that

$$h_n = \mu_0 \prod_{i=1}^{n} \gamma_i. \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 15A15; Secondary 33C45 33C20.

Key words and phrases. Hankel determinants, orthogonal polynomials, moments.

Let the lower triangular matrix A_n be defined by

$$(A_n)_{i,j} = \begin{cases} a_{i,j}, & i \geq j, \\ 0, & i < j, \end{cases} \qquad \text{for } 0 \leq i, j \leq n-1,$$

where

$$x^i = \sum_{k=0}^{i} a_{i,k} P_k(x), \quad a_{i,i} = 1.$$

If we define the diagonal matrix D_n by

$$(D_n)_{i,j} = h_i \delta_{i,j}, \quad \text{for } 0 \leq i, j \leq n-1,$$

and the *Hankel matrix* H_n by

$$(H_n)_{i,j} = \mu_{i+j}, \quad \text{for } 0 \leq i, j \leq n-1, \tag{1.4}$$

then we have the $\mathrm{LDL}^{\mathrm{T}}$ factorization (see [29, Section 4.1])

$$H_n = A_n D_n A_n^T. \tag{1.5}$$

We define the corresponding *Hankel determinants* by $\Delta_0 = 1$ and

$$\Delta_n = \det(H_n), \quad \text{for } n \in \mathbb{N}.$$

Using (1.5), we see that

$$\Delta_n = \prod_{j=0}^{n-1} h_j, \tag{1.6}$$

and using (1.3) in (1.6), we get

$$\Delta_n = \prod_{j=0}^{n-1} \mu_0 \prod_{i=1}^{j} \gamma_i = \mu_0^n \prod_{k=1}^{n-1} \gamma_k^{n-k}. \tag{1.7}$$

The identity (1.7) is sometimes called "Heilermann formula" [39], since J. B. H. Heilermann considered the J-fraction expansion [31] [1]

$$\sum_{n=0}^{\infty} \frac{\mu_n}{w^{n+1}} = \cfrac{\mu_0}{w - \beta_1 - \cfrac{\gamma_1}{w - \beta_2 - \cfrac{\gamma_2}{w - \beta_3 - \cfrac{\gamma_3}{\ddots}}}}, \tag{1.8}$$

in his 1845 Ph.D. thesis "De transformatione serierum in fractiones continuas" [6, Eq. (5.2)].

Determinants have a long history and an extensive literature, see [3, 7, 47, 53, 59, 60, 61, 64], and the impressive monographs [39] and [41].

The theories of Hankel determinants and orthogonal polynomials are deeply connected, see [9, 14, 19, 27, 33, 36, 38, 42, 57].

For some applications of Hankel determinants to combinatorial problems, see [1, 10, 11, 21, 28, 30, 35, 55, 56, 62].

[1] We are indebted to one of the anonymous referees for pointing out this reference.

Some authors[2] have computed Hankel determinants related to continuous Hahn polynomials [25], q-Hahn polynomials [8], and little q-Jacobi polynomials [32]. For extensions to continuous and discrete elliptic Selberg integrals, see [52].

In [40, Eq. (3.5)], C. Krattenthaler showed (among many other results), the identity

$$\sum_{0\le k_1<k_2<\cdots<k_n\le N}\prod_{i=1}^{n}\left[\frac{(x)_{k_i}(y)_{N-k_i}}{k_i!\,(N-k_i)!}\right]\prod_{1\le i<j\le n}(k_j-k_i)^2 = \prod_{k=0}^{n-1}\left[\frac{k!}{(N-k)!}(x)_k(y)_k(x+y+k+n-1)_{N-n+1}\right], \quad (1.9)$$

as a limiting case of the q-analog

$$\sum_{0\le k_1<k_2<\cdots<k_n\le N}\prod_{i=1}^{n}\left[\frac{(x;q)_{k_i}\,(y;q)_{N-k_i}}{(q;q)_{k_i}\,(q;q)_{N-k_i}}y^{k_i}\right]\prod_{1\le i<j\le n}\left(q^{k_j}-q^{k_i}\right)^2 = \prod_{k=0}^{n-1}\left[y^k q^{k(k-1)}\frac{(q;q)_k}{(q;q)_{N-k}}\,(x;q)_k\,(y;q)_k\,\left(xyq^{k+n-1};q\right)_{N-n+1}\right]. \quad (1.10)$$

He gave two different proofs of (1.10) using: 1) a Schur function identity from [44], and 2) a q-integral evaluation from [22, 37].

The purpose of this note is to give a different proof of (1.9) related to the theory of orthogonal polynomials. We also study some limiting cases, and consider some possible generalizations.

2. Main result

Suppose that the linear functional L has the form

$$L[p]=\sum_{k=0}^{N}c_k p(k), \quad \text{for } p(x)\in\mathbb{C}[x], \quad (2.1)$$

for some sequence $\{c_k\}$. Then, the moments μ_l are given by

$$\mu_l=\sum_{k=0}^{N}k^l c_k, \quad \text{for } l\in\mathbb{N}_0,$$

and the entries of the Hankel matrix (1.4) are

$$(H_n)_{i,j}=\mu_{i+j}=\sum_{k=0}^{N}k^{i+j}c_k, \quad \text{for } 0\le i,j\le n-1. \quad (2.2)$$

We can obtain a representation for the determinants of H_n.

Proposition 1. *The Hankel determinants Δ_n are given by*

$$\Delta_n=\sum_{0\le k_1<k_2<\cdots<k_n\le N}\left(\prod_{i=1}^{n}c_{k_i}\right)V_n(k_1,\ldots,k_n), \quad (2.3)$$

[2]We are indebted to one of the anonymous referee for suggesting these references.

where $V_n(k_1, \ldots, k_n)$ *denotes the polynomial*

$$V_n(k_1, \ldots, k_n) = \prod_{1 \le i < j \le n} (k_j - k_i)^2.$$

Proof. If we rewrite (2.2) as

$$(H_n)_{i,j} = \sum_{k=0}^{N} \sum_{l=0}^{N} k^i c_k \delta_{k,l} l^j,$$

we see that H_n has the form

$$H_n = V^T C V, \tag{2.4}$$

where V is the $(N+1) \times n$ Vandermonde matrix

$$(V)_{i,j} = i^j, \quad \text{for } 0 \le i \le N,\ 0 \le j \le n-1,$$

and C is the $(N+1) \times (N+1)$ diagonal matrix

$$(C)_{i,j} = c_i \delta_{i,j}, \quad \text{for } 0 \le i, j \le N.$$

Using the Cauchy–Binet formula [26] in (2.4), we have

$$\det(H_n) = \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \det(c_i \delta_{i,k_i}) \left[\det \left(k_i^j \right)_{0 \le j \le n-1} \right]^2,$$

and since

$$\left[\det_{1 \le i,j \le n} \left(k_i^{j-1} \right) \right]^2 = \prod_{1 \le i < j \le n} (k_j - k_i)^2,$$

the result follows. □

Remark 2. The expression (2.3) is a particular case of Heine's formula [34]

$$\det_{1 \le i,j \le n} (\mu_{i+j}) = \frac{1}{n!} \int_a^b \int_a^b \cdots \int_a^b V_n(x_1, \ldots, x_n) d\alpha(x_1) d\alpha(x_2) \cdots d\alpha(x_n),$$

where

$$\mu_i = \int_a^b x^i d\alpha(x), \quad \text{for } i \in \mathbb{N}_0.$$

Heine's formula is used extensively in random matrix theory [4, 45, 12].

In the following, we shall use standard hypergeometric notation: the Pochhammer symbol (or rising factorial) $(u)_k$ [49, Eq. (5.2.4)] is defined by $(u)_0 = 1$ and

$$(u)_k = u(u+1) \cdots (u+k-1), \quad \text{for } k \in \mathbb{N}. \tag{2.5}$$

We can also write [49, Eq. (5.2.5)]

$$(u)_z = \frac{\Gamma(u+z)}{\Gamma(u)}, \quad \text{if } -(u+z) \notin \mathbb{N}_0, \tag{2.6}$$

where $\Gamma(z)$ is the Gamma function. Then the generalized hypergeometric function [49, Ch. 16] is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{2.7} $$

The Stieltjes transform of the moments μ_n is given by

$$ S(w) = \sum_{n=0}^{\infty} \frac{\mu_n}{w^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{w^{n+1}} \sum_{k=0}^{N} k^n c_k = \sum_{k=0}^{N} \frac{c_k}{w-k}. $$

If

$$ c_k = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \quad \text{for } k \in \mathbb{N}_0, $$

we have (see [17])

$$ S(w) = \frac{1}{w} \, {}_{p+1}F_{q+1}\left(\begin{matrix} -w, \ a_1, \dots, a_p \\ 1-w, \ b_1, \dots, b_q \end{matrix} ; z \right). \tag{2.8} $$

If $a_1 = -N$, with $N \in \mathbb{N}$, then the series (2.7) terminates, and the Stieltjes transform (2.8) is a rational function (of w). It was shown by Wall in [63, Theorem 43.1][3] that a rational function has the form

$$ R(w) = \sum_{k=0}^{N} \frac{c_k}{w - x_k}, \quad c_k > 0, \quad x_k \in \mathbb{R}, \quad x_i \neq x_j, $$

if and only if

$$ R(w) = \cfrac{\mu_0}{\beta_1 + w - \cfrac{\gamma_1}{\beta_2 + w - \cfrac{\gamma_2}{\ddots \cfrac{}{\beta_{N-1} + w - \cfrac{\gamma_{N-1}}{\beta_N + w}}}}}, $$

where $\beta_k \in \mathbb{R}$ and $\mu_0, \gamma_k > 0$. Thus, we recover the J-fraction expansion (1.8) and could compute the Hankel determinant Δ_n using Heilermann's formula (1.7). This was a technique used (among others) by Flajolet in [23].

2.1. **Hahn polynomials.** Here we apply the previous considerations to the monic Hahn polynomials, which are defined by [49, Eq. (18.20.5)]

$$ Q_n(x) = \frac{(\alpha+1)_n \, (-N)_n}{(n+\alpha+\beta+1)_n} \, {}_3F_2\left(\begin{matrix} -n, -x, n+\alpha+\beta+1 \\ \alpha+1, -N \end{matrix} ; 1 \right). $$

For $\alpha, \beta \in \mathbb{R} \setminus [-N, -1]$, the monic Hahn polynomials satisfy the orthogonality relation [16]

$$ \sum_{k=0}^{N} Q_n(k) Q_m(k) \binom{N}{k} (\alpha+1)_k \, (\beta+1)_{N-k} = h_n \delta_{n,m}, \tag{2.9} $$

[3]We are indebted to one of the reviewers for pointing out this reference.

where

$$h_n = (n!)^2 \binom{N}{n} (\alpha+1)_n \, (\beta+1)_n \frac{(\alpha+\beta+2+2n)_{N-n}}{(\alpha+\beta+1+n)_n}. \tag{2.10}$$

In order to prove our main theorem, we will need the identity

$$\prod_{k=0}^{n} \frac{(x+2k+1)_{N-k}}{(x+k)_k} = \prod_{k=0}^{n} (x+k+n+1)_{N-n}, \quad \text{for } 0 \le n \le N, \tag{2.11}$$

which is easy to verify by making use of (2.6).

We now have all the elements necessary to show our main result.

Theorem 3. *For all* $1 \le n \le N$ *and* $\alpha, \beta \in \mathbb{R} \setminus [-N, -1]$, *we have*

$$\sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\binom{N}{k_i} (\alpha+1)_{k_i} \, (\beta+1)_{N-k_i} \right] V_n(k_1, \ldots, k_n)$$
$$= \prod_{k=0}^{n-1} \left[(k!)^2 \binom{N}{k} (\alpha+1)_k \, (\beta+1)_k \, (\alpha+\beta+1+k+n)_{N-n+1} \right]. \tag{2.12}$$

Proof. Let

$$c_k = \binom{N}{k} (\alpha+1)_k \, (\beta+1)_{N-k}.$$

From (2.9), we see that the monic orthogonal polynomials associated with the linear functional L defined by (2.1) are the Hahn polynomials.

On the other hand, we have from (2.3)

$$\det_{0 \le i,j \le n-1} \left(L\left[x^{i+j}\right] \right) = \sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \left(\prod_{i=1}^{n} c_{k_i} \right) V_n(k_1, \ldots, k_n).$$

Using (1.6) and (2.10), we get

$$\sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\binom{N}{k_i} (\alpha+1)_{k_i} \, (\beta+1)_{N-k_i} \right] V_n(k_1, \ldots, k_n)$$
$$= \prod_{k=0}^{n-1} \left[(k!)^2 \binom{N}{k} (\alpha+1)_k \, (\beta+1)_k \frac{(\alpha+\beta+2+2k)_{N-k}}{(\alpha+\beta+1+k)_k} \right].$$

If we use (2.11), with $x = \alpha+\beta+1$, we have

$$\prod_{k=0}^{n-1} \frac{(\alpha+\beta+2+2k)_{N-k}}{(\alpha+\beta+1+k)_k} = \prod_{k=0}^{n-1} (\alpha+\beta+1+k+n)_{N-n+1},$$

and the result follows. □

Corollary 4. *For all* $1 \le n \le N$, *we have*

$$\sum_{0 \le k_1 < k_2 < \cdots < k_n \le N} \prod_{i=1}^{n} \left[\frac{(x)_{k_i} \, (y)_{N-k_i}}{k_i! \, (N-k_i)!} \right] V_n(k_1, \ldots, k_n)$$
$$= \prod_{k=0}^{n-1} \left[\frac{k!}{(N-k)!} (x)_k \, (y)_k \, (x+y+k+n-1)_{N-n+1} \right]. \tag{2.13}$$

Proof. If we set $\alpha = x-1$, $\beta = y-1$ in (2.12) and divide both sides of by $(N!)^n$, we obtain (2.13). To remove any restrictions on x, y, we observe that (2.13) is an identity between polynomials in x and y of degree

$$\sum_{k=0}^{n-1}(k+k+N-n+1) = Nn.$$

According to Theorem 3, Equation (2.13) is true for $x, y \notin [-N+1, 0]$, and therefore it is true for all x, y. □

2.2. **Meixner polynomials.**

Lemma 5. *Let $0 \le k \le n \le N$ and $w > 0$. Then, as $N \to \infty$, we have*

$$\frac{N!}{(N-k)!}\frac{(1+wN)_{N-k}}{(1+wN)_N} \sim (w+1)^{-k}, \tag{2.14}$$

and, for all $a > 0$,

$$\frac{N!}{(N-k)!}\frac{(1+wN)_k}{(1+wN)_N}(a+k+n+wN)_{N-n+1} \sim w^{1-a-n}(w+1)^{a+k}N^{2k-n+1}. \tag{2.15}$$

Proof. From (2.6), we have

$$\frac{N!}{(N-k)!}\frac{(1+wN)_{N-k}}{(1+wN)_N} = \frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+N+1-k)}{\Gamma(Nw+N+1)}.$$

Using Stirling's formula [49, Eq. (5.11.1)]

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + O\left(z^{-1}\right), \quad \text{for } z \to \infty, \tag{2.16}$$

we obtain

$$\ln\left[\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+N+1-k)}{\Gamma(Nw+N+1)}\right] = -k\ln(w+1) + O\left(N^{-1}\right), \quad \text{for } N \to \infty.$$

Similarly, from (2.6) we have

$$\frac{N!}{(N-k)!}\frac{(1+wN)_k}{(1+wN)_N}(a+k+n+wN)_{N-n+1}$$
$$= \frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+k+1)}{\Gamma(Nw+N+1)}\frac{\Gamma(Nw+N+k+a+1)}{\Gamma(Nw+n+k+a)}.$$

Then, using (2.16), we get

$$\ln\left[\frac{\Gamma(N+1)}{\Gamma(N+1-k)}\frac{\Gamma(Nw+k+1)}{\Gamma(Nw+N+1)}\frac{\Gamma(Nw+N+k+a+1)}{\Gamma(Nw+n+k+a)}\right]$$
$$= (2k-n+1)\ln(N) + (a+k)\ln(w+1) - (a+n-1)\ln(w) + O\left(N^{-1}\right)$$

as $N \to \infty$. □

Corollary 6. *Let $0 < z < 1$, $a > 0$ and $n = 1, 2, \ldots$. Then*

$$\sum_{0 \le k_1 < k_2 < \cdots < k_n} \prod_{i=1}^{n}\left[\frac{(a)_{k_i}\, z^{k_i}}{k_i!}\right] V_n(k_1, \ldots, k_n) = \prod_{k=0}^{n-1}\left[k!(a)_k\, z^k\,(1-z)^{-a-2k}\right]. \tag{2.17}$$

Proof. Multiplying both sides of (2.13) by

$$\left[\frac{N!}{(y)_N}\right]^n,$$

setting

$$x=a, \quad y=1+\frac{1-z}{z}N,$$

and using (2.14)–(2.15), we obtain

$$\sum_{0\leq k_1<k_2<\cdots<k_n\leq N}\prod_{i=1}^{n}\left[\frac{N!}{(y)_N}\frac{(a)_{k_i}(y)_{N-k_i}}{k_i!\,(N-k_i)!}\right]V_n(k_1,\ldots,k_n)$$
$$\sim\sum_{0\leq k_1<k_2<\cdots<k_n}\prod_{i=1}^{n}\left[\frac{(a)_{k_i}\,z^{k_i}}{k_i!}\right]V_n(k_1,\ldots,k_n), \quad \text{for } N\to\infty,$$

and

$$\prod_{k=0}^{n-1}\left[\frac{N!}{(y)_N}\frac{k!}{(N-k)!}(a)_k\,(y)_k\,(a+y+k+n-1)_{N-n+1}\right]$$
$$\sim\prod_{k=0}^{n-1}\left[k!\,(a)_k\left(\frac{1-z}{z}\right)^{1-a-n}z^{-a-k}N^{2k-n+1}\right], \quad \text{for } N\to\infty.$$

However,

$$\prod_{k=0}^{n-1}(1-z)^{1-a-n}=(1-z)^{-n(a+n-1)}=\prod_{k=0}^{n-1}(1-z)^{-a-2k},$$
$$\prod_{k=0}^{n-1}z^{n-1-k}=z^{\frac{1}{2}n(n-1)}=\prod_{k=0}^{n-1}z^k,$$

and

$$\prod_{k=0}^{n-1}N^{2k-n+1}=1.$$

Therefore,

$$\lim_{N\to\infty}\sum_{0\leq k_1<k_2<\cdots<k_n\leq N}\prod_{i=1}^{n}\left[\frac{(a)_{k_i}\,z^{k_i}}{k_i!}\right]V_n(k_1,\ldots,k_n)=\prod_{k=0}^{n-1}\left[k!\,(a)_k\,z^k\,(1-z)^{-a-2k}\right]. \quad \square$$

The monic Meixner polynomials are defined by [49, Eq. (18.20.7)]

$$M_n(x)=(a)_n\left(1-z^{-1}\right)^{-n}\,{}_2F_1\left[\begin{matrix}-n,\ -x\\ a\end{matrix};1-z^{-1}\right].$$

For $a>0$ and $0<z<1$, the monic Meixner polynomials satisfy the orthogonality relation [16]

$$\sum_{k=0}^{\infty}M_n(k)M_m(k)\frac{(a)_k}{k!}z^k=n!\,(a)_n\,z^n(1-z)^{-a-2n}\delta_{n,m}. \tag{2.18}$$

If we choose

$$c_k = \frac{(a)_k}{k!} z^k, \quad \text{for } k \in \mathbb{N}_0, \tag{2.19}$$

we see from (2.18) that the monic orthogonal polynomials associated with the linear functional L defined by

$$L[p] = \lim_{N\to\infty} \sum_{k=0}^{N} c_k p(k) \tag{2.20}$$

are the Meixner polynomials. Now, from (2.3), we have

$$\det_{0\le i,j\le n-1} \left(L\left[x^{i+j}\right] \right) = \lim_{N\to\infty} \sum_{0\le k_1<k_2<\cdots<k_n\le N} (\prod_{i=1}^{n} c_{k_i}) V_n(k_1,\ldots,k_n), \tag{2.21}$$

and using (1.6) and (2.18), we get

$$\lim_{N\to\infty} \sum_{0\le k_1<k_2<\cdots<k_n\le N} \prod_{i=1}^{n} \left[\frac{(a)_{k_i}\, z^{k_i}}{k_i!} \right] V_n(k_1,\ldots,k_n) = \prod_{k=0}^{n-1} \left[k!\,(a)_k\, z^k (1-z)^{-a-2k} \right],$$

in agreement with (2.17).

Remark 7. The moments associated with (2.19) are given by [17]

$$\mu_n(z) = \sum_{k=0}^{\infty} k^n\, (a)_k \frac{z^k}{k!} = (1-z)^{-a-n} P_n(z),$$

where $P_n(z)$ is the polynomial

$$P_n(z) = \sum_{k=0}^{n} S(n,k)\,(a)_k\, z^k\,(1-z)^{n-k}, \tag{2.22}$$

and the coefficients $S(n,k)$ are the Stirling numbers of the second kind defined by [49, Eq. (26.8)]

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n.$$

Since

$$\Delta_n = (1-z)^{-an-\frac{1}{2}n(n-1)} \det_{0\le i,j\le n-1} (P_{i+j}(z)),$$

we see from (2.17) that

$$\det_{0\le i,j\le n-1} (P_{i+j}(z)) = z^{\frac{1}{2}n(n-1)} \prod_{k=0}^{n-1} k!\,(a)_k. \tag{2.23}$$

The polynomials (2.22) and their Hankel determinants (2.23) seem not to have been studied before.

2.3. Charlier polynomials.

Corollary 8. *Let $z>0$ and $n=1,2,\ldots$. Then*

$$\sum_{0\le k_1<k_2<\cdots<k_n}\prod_{i=1}^{n}\left[\frac{z^{k_i}}{k_i!}\right]V_n(k_1,\ldots,k_n)=\prod_{k=0}^{n-1}\left[(k!)^2\frac{z^k}{k!}e^z\right]. \tag{2.24}$$

Proof. If we do the replacement

$$z\to\frac{z}{z+a}$$

in (2.17), we obtain

$$\sum_{0\le k_1<k_2<\cdots<k_n}\prod_{i=1}^{n}\left[\frac{(a)_{k_i}\left(\frac{z}{z+a}\right)^{k_i}}{k_i!}\right]V_n(k_1,\ldots,k_n)$$
$$=\prod_{k=0}^{n-1}\left[(k!)^2\frac{(a)_k\left(\frac{z}{z+a}\right)^{k}}{k!}\left(1-\frac{z}{z+a}\right)^{-a-2k}\right].$$

The result follows from the limits

$$\lim_{a\to\infty}\frac{(a)_k}{(z+a)^k}=\lim_{a\to\infty}\prod_{j=0}^{k-1}\frac{a+j}{a+z}=1,$$

and

$$\lim_{a\to\infty}\left(1-\frac{z}{z+a}\right)^{-a-2k}\lim_{a\to\infty}\left(1+\frac{z}{a}\right)^{a+2k}=e^z.$$ □

The monic Charlier polynomials are defined by [49, Eq. (18.20.7)]

$$C_n(x)=(-z)^n\ {}_2F_0\left[\begin{matrix}-n,\ -x\\ -\end{matrix};-z^{-1}\right].$$

For $z>0$, the monic Charlier polynomials satisfy the orthogonality relation [49, Eq. (18.19.1)]

$$\sum_{k=0}^{\infty}C_n(k)C_m(k)\frac{z^k}{k!}=n!z^ne^z\delta_{n,m}. \tag{2.25}$$

If we choose

$$c_k=\frac{z^k}{k!},\quad \text{for } k\in\mathbb{N}_0, \tag{2.26}$$

we see from (2.25) that the monic orthogonal polynomials associated with the linear functional L defined by (2.20) are the Charlier polynomials. Using (1.6), (2.3) and (2.25), we get

$$\lim_{N\to\infty}\sum_{0\le k_1<k_2<\cdots<k_n\le N}\prod_{i=1}^{n}\left[\frac{z^{k_i}}{k_i!}\right]V_n(k_1,\ldots,k_n)^2=\prod_{k=0}^{n-1}\left(k!z^ke^z\right),$$

in agreement with (2.24).

Remark 9. The moments associated with (2.26) are given by [17]

$$\mu_n(z) = \sum_{k=0}^{\infty} k^n \frac{z^k}{k!} = e^z T_n(z),$$

where $T_n(z)$ is the Touchard (or exponential, or Bell) polynomial

$$T_n(z) = \sum_{k=0}^{n} S(n,k) z^k.$$

We clearly have

$$\Delta_n = e^{nz} \det_{0 \le i,j \le n-1} (T_{i+j}).$$

The determinant

$$\det_{0 \le i,j \le n-1} (T_{i+j}) = z^{\frac{1}{2}n(n-1)} \prod_{k=0}^{n-1} k!$$

has been computed by several authors in many different ways, see [5, 20, 24, 36, 46, 50, 51, 54]. The special case $z = 1$ (Bell numbers), was considered in [2, 13, 43, 58, 65].

3. Conclusions

We have established a connection between C. Krattenthaler's identity (1.9) and the Hankel determinants of moments of Hahn polynomials. As we mentioned at the end of the last section, the corresponding identity for Hankel determinants of Charlier polynomials has appeared in the literature multiple times.

We have not been able to find any other instance of the determinants

$$\det_{0 \le i,j \le n-1} \left(\sum_{k=0}^{N} k^{i+j} c_k \right), \quad \det_{0 \le i,j \le n-1} \left(\sum_{k=0}^{\infty} k^{i+j} c_k \right),$$

for general c_k, or at least for c_k being a hypergeometric term (we do not claim that they do not exist, but we have not uncovered a single reference). That is why we were so amazed to learn about (1.9).

The next case of interest will be

$$c_k = \frac{(\alpha_1)_k (\alpha_2)_k (\alpha_3)_k}{(\beta_1)_k (\beta_2)_k} \frac{1}{k!}, \quad \text{for } k \in \mathbb{N}_0,$$

which is the weight function for the Generalized Hahn polynomials of type II introduced in [18].

In his work [40], Krattenthaler employed Schur functions to establish (1.9). In [15], we used Schur functions to compute the Hankel determinants of the Meixner and Charlier polynomials, and consider some extensions.

Acknowledgement

This work was done while visiting the Johannes Kepler Universität Linz and supported by the strategic program "Innovatives OÖ – 2010 plus" from the Upper Austrian Government.

I thank George Andrews, Christian Krattenthaler and Veronika Pillwein for their helpful suggestions and encouragement.

Finally, I also wish to express our gratitude to the anonymous referees, who provided us with invaluable suggestions and comments that greatly improved our first draft of the paper.

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