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STACK-SORTING PREIMAGES OF PERMUTATION CLASSES

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Abstract. We extend and generalize many of the enumerative results concerning West's stack-sorting map s. First, we prove a useful theorem that allows one to efficiently compute $|s^{-1}(\pi)|$ for any permutation π, answering a question of Bousquet-Mélou. This method relies on combinatorial objects called "valid hook configurations." We then enumerate permutations in various sets of the form $s^{-1}(\mathrm{Av}(\tau^{(1)},\ldots,\tau^{(r)}))$, where $\mathrm{Av}(\tau^{(1)},\ldots,\tau^{(r)})$ is the set of permutations avoiding the patterns $\tau^{(1)},\ldots,\tau^{(r)}$. In many cases studied in this paper, these preimage sets are permutation classes themselves. In one case, we solve a problem previously posed by Bruner. We are often able to refine our counts by enumerating these permutations according to their number of descents or peaks. Our investigation not only provides several new combinatorial interpretations and identities involving known sequences, but also paves the way for several new enumerative problems.

1. Introduction

Throughout this paper, we write permutations as words in one-line notation. Let S_n denote the set of permutations of $\{1,\ldots,n\}$. A *descent* of a permutation $\pi=\pi_1\cdots\pi_n\in S_n$ is an index $i\in[n-1]$ such that $\pi_i>\pi_{i+1}$. A *peak* of π is an index $i\in\{2,\ldots,n-1\}$ such that $\pi_{i-1}<\pi_i>\pi_{i+1}$.

Definition 1. We say the permutation $\sigma=\sigma_1\cdots\sigma_n$ *contains the pattern* $\tau=\tau_1\cdots\tau_m$ if there are indices $i_1<\cdots<i_m$ such that $\sigma_{i_1}\cdots\sigma_{i_m}$ has the same relative order as τ. Otherwise, we say σ *avoids* τ. Denote by $\mathrm{Av}(\tau^{(1)},\ldots,\tau^{(r)})$ the set of permutations that avoid the patterns $\tau^{(1)},\ldots,\tau^{(r)}$. Let $\mathrm{Av}_n(\tau^{(1)},\ldots,\tau^{(r)})=\mathrm{Av}(\tau^{(1)},\ldots,\tau^{(r)})\cap S_n$. Let $\mathrm{Av}_{n,k}(\tau^{(1)},\ldots,\tau^{(r)})$ be the set of permutations in $\mathrm{Av}_n(\tau^{(1)},\ldots,\tau^{(r)})$ with exactly k descents.

A set of permutations is called a *permutation class* if it is the set of permutations avoiding some (possibly infinite) collection of patterns. Equivalently, a permutation class is a downset in the poset of all permutations ordered by containment. The *basis* of a class is the set of minimal permutations (in the containment ordering) not in the class.

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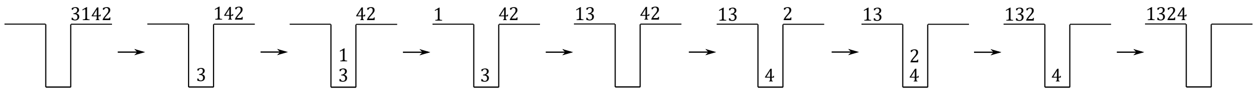


FIGURE 1. The stack-sorting map s sends 3142 to 1324.

The notion of pattern avoidance in permutations, which has blossomed into an enormous area of research (see [4, 31, 36]) and which plays a lead role in the present article, began its development with Knuth's book *The Art of Computer Programming* [32]. In this book, Knuth described a so-called *stack-sorting algorithm*; it was the study of the combinatorial properties of this algorithm that led him to introduce the idea of pattern avoidance. In his 1990 Ph.D. thesis [40], West defined a deterministic version of Knuth's stack-sorting algorithm, which we call the *stack-sorting map* and denote by s. The stack-sorting map is a function defined by the following procedure.

Suppose we are given an input permutation $\pi \in S_n$. Place this permutation on the right side of a vertical "stack." Throughout this process, if the next entry in the input permutation is smaller than the entry at the top of the stack or if the stack is empty, the next entry in the input permutation is placed at the top of the stack. Otherwise, the entry at the top of the stack is appended to the end of the growing output permutation. This procedure stops when the output permutation has size n. We then define $s(\pi)$ to be this output permutation. Figure 1 illustrates this procedure and shows that $s(3142) = 1324$.

If $\pi \in S_n$, we can write $\pi = LnR$, where L (respectively, R) is the (possibly empty) substring of π to the left (respectively, right) of the entry n. West observed that the stack-sorting map can be defined recursively by $s(\pi) = s(L)s(R)n$ (here, we also have to allow s to take permutations of arbitrary finite sets of positive integers as arguments). There is also a natural definition of the stack-sorting map in terms of tree traversals of decreasing binary plane trees (see [4, 21, 22]).

The "purpose" of the stack-sorting map is to sort the input permutation into increasing order. Hence, we say a permutation $\pi \in S_n$ is *sortable* if $s(\pi) = 123 \cdots n$. The above example illustrates that the stack-sorting map does not always do its job. In other words, not all permutations are sortable. In fact, the following characterization of sortable permutations follows from Knuth's work.

Theorem 2 ([32])**.** *A permutation π is sortable if and only if it avoids the pattern* 231*.*

Even if a permutation is not sortable, we can still try to sort it via iterated use of the stack-sorting map. In what follows, s^t denotes the composition of s with itself t times.

Definition 3. A permutation $\pi \in S_n$ is called *t-stack-sortable* if $s^t(\pi) = 123 \cdots n$. Let $\mathcal{W}_t(n)$ denote the set of t-stack-sortable permutations in S_n. Let $W_t(n) = |\mathcal{W}_t(n)|$.

Theorem 2 states that $\mathcal{W}_1(n) = \mathrm{Av}_n(231)$, so it follows from the well-known enumeration of 231-avoiding permutations that $W_1(n) = C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number. In his thesis, West proved [40] that a permutation is 2-stack-sortable if and only if it avoids the pattern 2341 and also avoids any 3241 pattern that is not part of a 35241 pattern. He also conjectured the following theorem, which Zeilberger proved in 1992.

Theorem 4 ([42])**.** *We have*

$$W_2(n) = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \tag{1}$$

Combinatorial proofs of this theorem arose later in [18, 26, 27, 30]. Some authors have studied the enumerations of 2-stack-sortable permutations according to certain statistics [5, 9, 12, 26]. Recently, the authors of [25] introduced new combinatorial objects known as *fighting fish* and showed that they are counted by the numbers $W_2(n)$. Fang has now given a bijection between fighting fish and 2-stack-sortable permutations [28]. The authors of [3] study what they call *n-point dominoes*, and they have made the fascinating discovery that the number of these objects is $W_2(n+1)$.

The primary purpose of this article is to enumerate preimages of permutation classes under the stack-sorting map. This is a natural generalization of the study of sortable and 2-stack-sortable permutations since $\mathcal{W}_1(n) = s^{-1}(\text{Av}_n(21))$ and $\mathcal{W}_2(n) = s^{-1}(\text{Av}_n(231))$. In fact, Bouvel and Guibert [12] have already considered stack-sorting preimages of certain classes in their study of permutations that are sortable via multiple stacks and D_8 symmetries (we state some of their results in Section 3). Claesson and Úlfarsson [17] have also studied this problem in relation to a generalization of classical permutation patterns known as *mesh patterns*, which were introduced in [15]. They showed that each set of the form $s^{-1}(\text{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ can be described as the set of permutations avoiding a specific collection of mesh patterns, and they provided an algorithm for computing this collection. In specific cases, $s^{-1}(\text{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ is a genuine permutation class. For example, $s^{-1}(\text{Av}(m(m-1)\cdots 321))$ is a permutation class.

The idea to count the preimages of a permutation under the stack-sorting map dates back to West, who called $|s^{-1}(\pi)|$ the *fertility* of the permutation π and went to great lengths to compute the fertilities of the permutations of the forms

$$23\cdots k1(k+1)\cdots n, \quad 12\cdots(k-2)k(k-1)(k+1)\cdots n, \quad \text{and} \quad k12\cdots(k-1)(k+1)\cdots n.$$

The very specific forms of these permutations indicates the initial difficulty of computing fertilities. We define the fertility of a set of permutations to be the sum of the fertilities of the permutations in that set. With this terminology, our main goal in this paper is to compute the fertilities of sets of the form $\text{Av}_n(\tau^{(1)}, \ldots, \tau^{(r)})$. Let us stress that although these sets are often easy to understand (for example, $\text{Av}_n(231, 312, 321)$ consists of the "layered permutations" in which each layer has size 1 or 2), computing the fertilities of these sets is a much more difficult problem.

Bousquet-Mélou [10] studied permutations with positive fertilities, which she termed *sorted* permutations. She mentioned that it would be interesting to find a method for computing the fertility of any given permutation. This was achieved in [21] using new combinatorial objects called "valid hook configurations." The theory of valid hook configurations was the key ingredient used in [22] in order to improve the best-known upper bounds for $W_3(n)$ and $W_4(n)$, and it will be one of our primary tools in subsequent sections. Recently, the authors of [23] used valid hook configurations to establish connections between permutations with fertility 1 and free probability theory. The current author has also investigated which numbers arise as the fertilities of permutations [19].

In Section 2, we review the definitions and necessary theorems concerning valid hook configurations. We also prove a theorem that ameliorates the computation of fertilities in many cases. This theorem was stated in [22], but the proof was omitted because the result was not needed in that paper. We have decided to prove the result here because we will make use of it in our computations. This result is also used in [19].

Section 3 reviews some facts about generalized patterns and stack-sorting preimages of permutation classes. In Section 4, we enumerate the set $s^{-1}(\text{Av}(132, 231, 312, 321))$, which is a permutation class. In Section 5, we study $s^{-1}(\text{Av}(132, 231, 321))$ and $s^{-1}(\text{Av}(132, 312, 321))$, the latter of which is a permutation class. We show that these sets are both enumerated by central binomial coefficients. A corollary of the results in this section actually settles a problem of Bruner [16]. In Section 6, we consider $s^{-1}(\text{Av}(231, 312, 321))$, which turns out to be a permutation class. We enumerate this class both directly and by using valid hook configurations, leading to a new identity involving well-studied orderings on integer compositions and integer partitions. Section 7 considers the set $s^{-1}(\text{Av}(132, 231, 312))$. Finding the fertilities of permutations in $\text{Av}(132, 231, 312)$ allows us to prove that some of the estimates used in [22] are sharp (see Section 7 for details). In addition, we will find that the permutations in $s^{-1}(\text{Av}(132, 231, 312))$ are enumerated by the Fine numbers, giving a new interpretation for this well-studied sequence. Section 8 is brief and is merely intended to state that $s^{-1}(\text{Av}(312, 321))$ is the permutation class $\text{Av}(3412, 3421)$, which Kremer [33] has proven to be enumerated by the large Schröder numbers. Section 9 enumerates the permutations in $s^{-1}(\text{Av}(132, 321))$. In Section 10, we prove that $|s^{-1}(\text{Av}_n(132, 312))| = |s^{-1}(\text{Av}_n(231, 312))|$. Finally, we prove that

$$8.4199 \leq \lim_{n\to\infty} |s^{-1}(\text{Av}_n(321))|^{1/n} \leq 11.6569$$

in Section 11. This is notable because $s^{-1}(\text{Av}(321))$ is a permutation class. In most of these sections, we actually refine our counts by enumerating stack-sorting preimages of permutation classes according to the number of descents and according to the number of peaks. These results are summarized in Table 1.

The basic idea in most of our proofs is to express the fertility of a set of permutations as a sum of products of Catalan numbers. The sum ranges over compositions indexed by valid hook configurations, as explained in the next section. Our results lead to several open problems and conjectures, which we accumulate in Section 12.

2. Valid Hook Configurations and Valid Compositions

To construct a valid hook configuration, begin by choosing a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$. Recall that a descent of π is an index i such that $\pi_i > \pi_{i+1}$. Let $d_1 < \cdots < d_k$ be the descents of π. We use the example permutation 3142567 to illustrate the construction. The *plot* of π is the graph displaying the points (i, π_i) for $1 \leq i \leq n$. The left image in Figure 2

$\tau^{(1)},\ldots,\tau^{(r)}$	$\lvert s^{-1}(\mathrm{Av}_n(\tau^{(1)},\ldots,\tau^{(r)}))\rvert$	Section	How proved	OEIS
123	0 for $n \geq 4$	3	Easy	A130713
213	Catalan numbers	3	Knuth [32]	A000108
231	Closed formula	3	Zeilberger [42]	A000139
132	Closed formula	3	Bouvel–Guibert [12]	A000139
312	Baxter numbers	3	Bouvel–Guibert [12]	A001181
321	Estimates for growth rate	11	VHCs	A319027
$132, 231$	Unknown	10		A071356†
$132, 312$	Unknown	10		A071356†
$231, 312$	Equal to $\lvert s^{-1}(\mathrm{Av}_n(132, 312))\rvert$	10	VHCs	A071356†
$231, 321$	Unknown	12		A165543†
$132, 321$	Explicit gen. function*	9	VHCs	A319028
$312, 321$	Large Schröder numbers	8	Kremer's result [33]	A006318
$132, 231, 312$	Fine numbers*	7	VHCs	A000957
$231, 312, 321$	Complicated sum formula Simple sum formula	6 6	VHCs Classical argument	A049124
$132, 231, 321$	Closed formula*	5	VHCs	A000984
$132, 312, 321$	Closed formula*	5	VHCs	A000984
$132, 231, 312, 321$	Closed formula*	4	VHCs	A071721

Table 1. Summary of the enumeration of stack-sorting preimages of permutation classes with bases consisting of size-3 patterns. The appearance of a * indicates that we can also refine the enumeration according to certain permutation statistics. The symbol † indicates that the given OEIS sequence is only conjectured to count the corresponding preimage set. The abbreviation "VHCs" stands for "valid hook configurations."

shows the plot of our example permutation. A point (i, π_i) is a *descent top* if i is a descent. The descent tops in our example are $(1, 3)$ and $(3, 4)$.

A *hook* of π is drawn by starting at a point (i, π_i) in the plot of π, moving vertically upward, and then moving to the right until reaching another point (j, π_j). In order for this to make sense, we must have $i < j$ and $\pi_i < \pi_j$. The point (i, π_i) is called the *southwest endpoint* of the hook, while (j, π_j) is called the *northeast endpoint*. The right image in Figure 2 shows our example permutation with a hook that has southwest endpoint $(3, 4)$ and northeast endpoint $(6, 6)$.

Definition 5. Let π be a permutations with descents $d_1 < \cdots < d_k$. A *valid hook configuration* of π is a tuple $\mathcal{H} = (H_1, \ldots, H_k)$ of hooks of π subject to the following constraints:

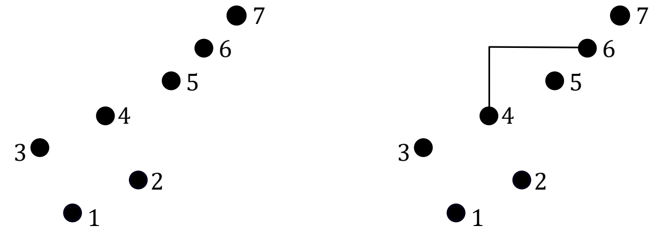


FIGURE 2. The left image is the plot of 3142567. The right image shows this plot along with a single hook.

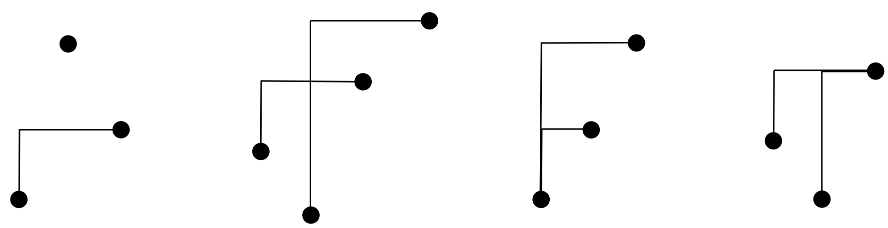


FIGURE 3. Four configurations of hooks that are forbidden in a valid hook configuration.

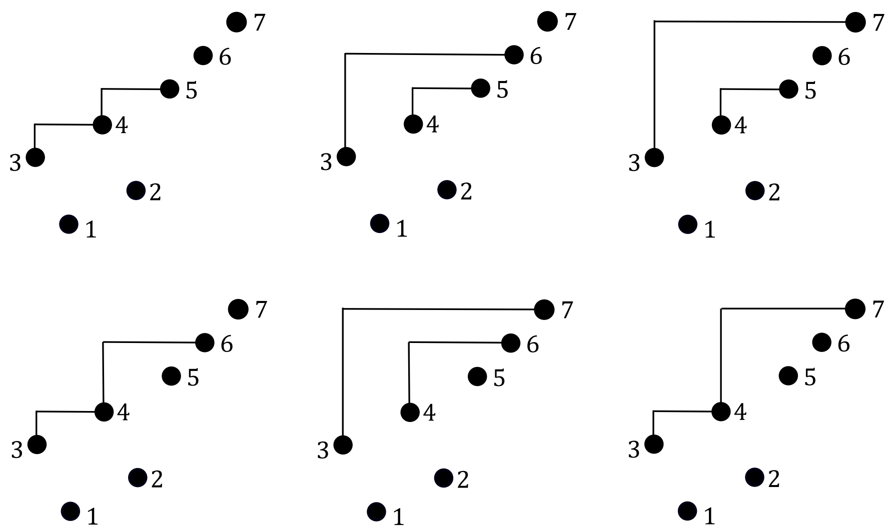


FIGURE 4. All of the valid hook configurations of 3142567.

1. The southwest endpoint of H_i is (d_i, π_{d_i}).
2. A point in the plot cannot lie above a hook.
3. Hooks cannot intersect each other except in the case that the northeast endpoint of one hook is the southwest endpoint of the other.

Figure 3 shows four placements of hooks that are forbidden by Conditions 2 and 3. Figure 4 shows all of the valid hook configurations of 3142567.

A valid hook configuration of π induces a coloring of the plot of π. To color the plot, draw a "sky" over the entire diagram and color the sky blue. Assign arbitrary distinct colors other than blue to the k hooks in the valid hook configuration. There are k northeast endpoints of hooks, and these points remain uncolored. However, all of the other $n-k$ points will be colored. In order to decide how to color a point (i, π_i) that is not a northeast endpoint, imagine that this point looks upward. If this point sees a hook when looking upward, it receives the same color as the hook that it sees. If the point does not see a hook, it must see the sky, so it receives the color blue. However, if (i, π_i) is the southwest endpoint of a hook, then it must look around (on the left side of) the vertical part of that hook. See Figure 5 for the colorings induced by the valid hook configurations in Figure 4. Note that the leftmost point $(1, 3)$ is blue in each of these colorings because this point looks around the first (red) hook and sees the sky.

To summarize, we started with a permutation π with exactly k descents. We chose a valid hook configuration of π by drawing k hooks according to Conditions 1, 2, and 3 in Definition 5. This valid hook configuration then induced a coloring of the plot of π. Specifically, $n-k$ points were colored, and $k+1$ colors were used (one for each hook and one for the sky). Let q_i be the number of points colored the same color as the i^{th} hook, and let q_0 be the number of points colored blue (the color of the sky). Then $(q_0, \dots, q_k)$ is a composition[1] of $n-k$ into $k+1$ parts; we say the valid hook configuration *induces* this composition. Let $\mathcal{V}(\pi)$ be the set of compositions induced by valid hook configurations of π. We call the elements of $\mathcal{V}(\pi)$ the *valid compositions* of π.

We will often make implicit use of the following result, which is Lemma 3.1 in [22].

Theorem 6 ([22]). *Each valid composition of a permutation $\pi \in S_n$ is induced by a unique valid hook configuration of π.*

Let $C_j = \frac{1}{j+1}\binom{2j}{j}$ denote the j^{th} Catalan number. Given a composition $(q_0, \dots, q_k)$, let

$$C_{(q_0,\dots,q_k)} = \prod_{t=0}^{k} C_{q_t}.$$

The following theorem explains why valid hook configurations are so useful when studying the stack-sorting map.

Theorem 7 ([21]). *If $\pi \in S_n$ has exactly k descents, then the fertility of π is given by the formula*

$$|s^{-1}(\pi)| = \sum_{(q_0,\dots,q_k)\in\mathcal{V}(\pi)} C_{(q_0,\dots,q_k)}.$$

Example 8. The permutation 3142567 has six valid hook configurations, which are shown in Figure 4. The colorings induced by these valid hook configurations are displayed in Figure 5. The valid compositions induced by these valid hook configurations are (reading the first row before the second row, each from left to right)

$$(3,1,1), \quad (2,2,1), \quad (1,3,1), \quad (2,1,2), \quad (1,2,2), \quad (1,1,3).$$

It follows from Theorem 7 that

$$|s^{-1}(3142567)| = C_{(3,1,1)} + C_{(2,2,1)} + C_{(1,3,1)} + C_{(2,1,2)} + C_{(1,2,2)} + C_{(1,1,3)} = 27.$$

[1]Throughout this paper, a *composition of b into a parts* is an a-tuple of *positive* integers whose sum is b.

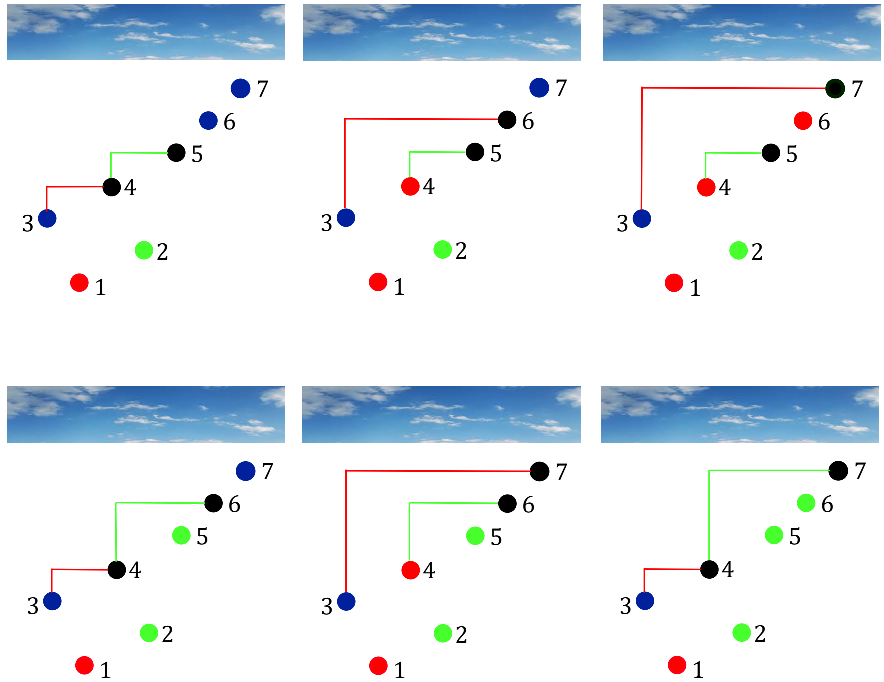


FIGURE 5. The different colorings induced by the valid hook configurations of 3142567.

Remark 9. One immediate consequence of Theorem 7 is that a permutation is sorted if and only if it has a valid hook configuration.

It is also possible to refine Theorem 7 according to certain permutation statistics such as the number of descents and the number of peaks[2]. Recall that a peak of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ is an index i such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. In what follows, we consider the *Narayana numbers* $N(i,j) = \frac{1}{i}\binom{i}{j}\binom{i}{j-1}$. Let us also define

$$V(i,j) = 2^{i-2j+1}\binom{i-1}{2j-2}C_{j-1}.$$

It is known[3] that $V(i,j)$ is the number of decreasing binary plane trees with i vertices and j leaves, and this is actually why these numbers arise in this context. In what follows, let $\text{Comp}_a(b)$ be the set of compositions of b into a parts (that is, a-tuples of positive integers that sum to b).

Theorem 10 ([21]). *If $\pi \in S_n$ has exactly k descents, then the number of permutations in $s^{-1}(\pi)$ with exactly m descents is*

$$\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)} \sum_{(j_0,\ldots,j_k)\in\text{Comp}_{k+1}(m+1)} \prod_{t=0}^{k} N(q_t, j_t).$$

[2]Theorem 10 was originally stated in [21] in terms of "valleys" instead of peaks, but the formulation we give here is equivalent.

[3]See sequence A091894 in [37].

The number of permutations in $s^{-1}(\pi)$ with exactly m peaks is

$$\sum_{(q_0,\ldots,q_k)\in\mathcal{V}(\pi)}\ \sum_{(j_0,\ldots,j_k)\in\mathrm{Comp}_{k+1}(m+1)}\ \prod_{t=0}^{k} V(q_t,j_t).$$

In her study of sorted permutations (permutations with positive fertilities), Bousquet-Mélou introduced the notion of the *tree of the canonical preimage* of a sorted permutation π and showed that this tree determines the fertility of π [10]. She asked for an explicit method for computing the fertility of a permutation from the tree of its canonical preimage. The current author reformulated the notion of a canonical tree in the language of valid hook configurations, defining the *canonical hook configuration* of a permutation [22].[4] Here, we describe a method for computing a permutation's fertility from its canonical hook configuration. Specifically, we show how to describe all valid compositions of π from the canonical hook configuration. This method was stated in [21], but the proof was omitted.

As before, let $d_1 < \cdots < d_k$ be the descents of π. We will construct the canonical hook configuration of π, which we denote by $\mathcal{H}^* = (H_1^*,\ldots,H_k^*)$. That is, H_i^* is the hook in $\mathcal{H}^*$ whose southwest endpoint is (d_i,π_{d_i}). In order to define $\mathcal{H}^*$, we need to choose the northeast endpoints of the hooks $H_1^*,\ldots,H_k^*$. To start, consider all possible points that could be northeast endpoints of H_k^*; because d_k is the *largest* descent of π, these are precisely the points above and to the right of (d_k,π_{d_k}). Among these points, choose the leftmost one (equivalently, the lowest one) to be the northeast endpoint of H_k^*. Next, consider all possible points that could be northeast endpoints of H_{k-1}^* (given that we already know H_k^* and that we need to satisfy the conditions in Definition 5). Among these points, choose the leftmost one to be the northeast endpoint of H_{k-1}^*. Continue in this fashion, always choosing the leftmost possible point as the northeast endpoint of H_ℓ^* given that $H_{\ell+1}^*,\ldots,H_k^*$ have already been chosen. If it is ever impossible to find a northeast endpoint for H_ℓ^*, then π has no valid hook configurations (meaning π is not sorted by Remark 9). Otherwise, we obtain the canonical hook configuration of π from this process. Figure 6 shows the canonical hook configuration of a permutation.

Let us assume π is sorted so that it has a canonical hook configuration $\mathcal{H}^* = (H_1^*,\ldots,H_k^*)$. We extend the sequence $d_1 < \cdots < d_k$ of descents of π by making the conventions $d_0 = 0$ and $d_{k+1} = n$. For $1 \le i \le k+1$, the i^{th} ascending run of π is the string $\pi_{d_{i-1}+1}\cdots\pi_{d_i}$. We use $\mathcal{H}^*$ to define certain parameters as follows.

- Let $(b_i^*,\pi_{b_i^*})$ be the northeast endpoint of H_i^*.
- Let $(q_0^*,\ldots,q_k^*)$ be the valid composition of π induced by $\mathcal{H}^*$.
- For $1 \le i \le k$, define e_i by requiring that $\pi_{b_i^*}$ is in the e_i^{th} ascending run of π. In other words, $d_{e_i-1} < b_i^* \le d_{e_i}$. Furthermore, put $e_0 = k+1$.

[4]Given a valid hook configuration of a permutation π, we obtain a binary plane tree on the vertex set $\{1,\ldots,n\}$ as follows. If i is a descent of π, then (i,π_i) is the southwest endpoint of a hook. Let (j,π_j) be the northeast endpoint of this hook. In this case, we make π_i the left child of π_j in the tree. If $i \in \{1,\ldots,n-1\}$ is not a descent of π, we make π_i the left child of π_{i+1}. The labeled tree obtained from the canonical hook configuration of π is the tree of the canonical preimage of π defined by Bousquet-Mélou.

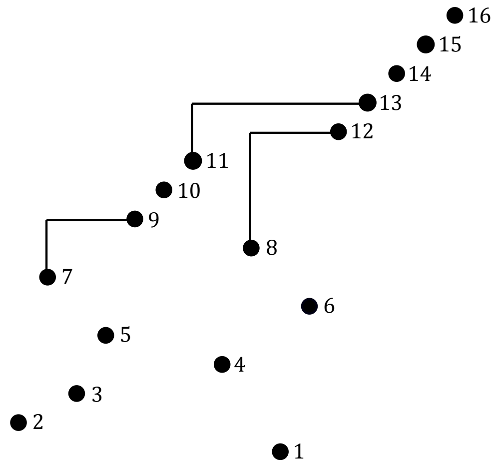


FIGURE 6. The canonical hook configuration of
2 7 3 5 9 10 11 4 8 1 6 12 13 14 15 16.

- For $1 \leq j \leq k+1$, let $\alpha_j = |\{i \in \{1, \ldots, k\} : e_i = j\}|$ denote the number of northeast endpoints $(b_i^*, \pi_{b_i^*})$ such that $\pi_{b_i^*}$ is in the j^{th} ascending run of π.

Example 11. Let $\pi = 2\ 7\ 3\ 5\ 9\ 10\ 11\ 4\ 8\ 1\ 6\ 12\ 13\ 14\ 15\ 16$ be the permutation whose canonical hook configuration appears in Figure 6. We have $d_0 = 0$, $d_1 = 2$, $d_2 = 7$, $d_3 = 9$, and $d_4 = 16$. Furthermore,

- $(b_1^*, b_2^*, b_3^*) = (5, 13, 12)$;
- $(q_0^*, q_1^*, q_2^*, q_3^*) = (7, 2, 2, 2)$;
- $(e_0, e_1, e_2, e_3) = (4, 2, 4, 4)$;
- $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 1, 0, 2)$.

Notice that the fact that $e_0 = k+1$ does not influence the value of α_{k+1}.

We are now in a position to state and prove the theorem that allows one to combine the above pieces of data in order to describe the valid compositions of π. The reader who is interested only in the enumerative results of the forthcoming sections can safely bypass the proof of this theorem.

Theorem 12. *Let $\pi \in S_n$ be a sorted permutation with k descents, and preserve the notation from above. A composition $(q_0, \ldots, q_k)$ of $n-k$ into $k+1$ parts is a valid composition of π if and only if the following two conditions hold:*

(a) For every $m \in \{0, \ldots, k\}$,
$$\sum_{j=m}^{e_m-1} q_j \geq \sum_{j=m}^{e_m-1} q_j^*.$$
(b) For all $m, p \in \{0, 1, \ldots, k\}$ with $m \leq p \leq e_m - 2$, we have
$$\sum_{j=m}^{p} q_j \geq d_{p+1} - d_m - \sum_{j=m+1}^{p+1} \alpha_j.$$

Proof. To ease notation, let $P(i) = (i, \pi_i)$. Suppose $(q_0, \ldots, q_k) \in \mathcal{V}(\pi)$, and let $\mathcal{H} = (H_1, \ldots, H_k)$ be the valid hook configuration inducing $(q_0, \ldots, q_k)$. Let $P(b_i)$ be the northeast endpoint of H_i. Put $\pi_0 = n+1$, $\pi_{n+1} = n+2$, and $b_0 = b_0^* = n+1$. It will be convenient to view the "sky" as a hook $H_0 = H_0^*$ with southwest endpoint $(d_0, \pi_{d_0}) = (0, n+1)$ and northeast endpoint $(b_0, \pi_{b_0}) = (n+1, n+2)$ (this conflicts with our definition of a hook since these points are not in the plot of π, but we will ignore this technicality). Put $\mathcal{B}^* = \{b_0^*, \ldots, b_k^*\}$ and $\mathcal{B} = \{b_0, \ldots, b_k\}$. If we build $\mathcal{H}$ by choosing the northeast endpoints $P(b_k), \ldots, P(b_1)$ in this order, then every point lying below H_i also lies below H_i^* (since we chose b_i^* as small as possible). This means that every possible choice for $P(b_m)$ was also a choice for $P(b_m^*)$ when we built $\mathcal{H}^*$. It follows from our choice of $P(b_m^*)$ that $b_m^* \leq b_m$ and $\pi_{b_m^*} \leq \pi_{b_m}$. This implies that H_m lies above H_m^* or is equal to H_m^* for every $m \in \{0, \ldots, k\}$.

Suppose $m, p \in \{0, \ldots, k\}$ and $m \leq p \leq e_m - 1$ (recall that $\pi_{b_m^*}$ is in the e_m^{th} ascending run of π). Let $X = \{d_m + 1, d_m + 2, \ldots, \min\{b_m^*, d_{p+1}\}\}$. Suppose $b_\gamma \in X \cap \mathcal{B}$, where $\gamma \neq m$. Because $b_m^* \leq b_m$, we must have $d_m < b_\gamma < b_m$. This means that b_γ lies below the hook H_m, so H_γ lies below H_m. Deducing that $m + 1 \leq \gamma$, we find that $d_m < b_\gamma^*$. We also know that $b_\gamma^* \leq b_\gamma$, so $d_m < b_\gamma^* \leq \min\{b_m^*, d_{p+1}\}$ (because $b_\gamma \in X$). This proves the implication $b_\gamma \in X \cap \mathcal{B} \implies b_\gamma^* \in X \cap \mathcal{B}^*$. The map $X \cap \mathcal{B} \to X \cap \mathcal{B}^*$ given by $b_j \mapsto b_j^*$ is an injection, so

$$|X \cap \mathcal{B}| \leq |X \cap \mathcal{B}^*|. \tag{2}$$

Choose $x \in X \setminus \mathcal{B}$. Recall that $\mathcal{H}$ induces a coloring of the plot of π. The point $P(x)$ lies below the hook H_m. None of the hooks $H_0, H_1, \ldots, H_{m-1}$ lie below H_m, and all of the hooks $H_{p+1}, H_{p+2}, \ldots, H_k$ appear to the right of $P(x)$ (although H_{p+1} could have $P(x)$ as its southwest endpoint). Therefore, if $P(x)$ looks upward, it sees one of the hooks $H_m, H_{m+1}, \ldots, H_p$. Letting $\mathcal{A}_{m,p}$ be the set of points that are given the same color as one of the hooks $H_m, H_{m+1}, \ldots, H_p$, we see that $x \in \mathcal{A}_{m,p}$. This shows that $(X \setminus \mathcal{B}) \subseteq \mathcal{A}_{m,p}$. Hence,

$$\sum_{j=m}^{p} q_j = |\mathcal{A}_{m,p}| \geq |X \setminus \mathcal{B}| \geq |X \setminus \mathcal{B}^*|, \tag{3}$$

where the last inequality follows from (2).

Suppose $p \leq e_m - 2$ (meaning $\min\{b_m^*, d_{p+1}\} = d_{p+1}$). We have $|X \cap \mathcal{B}^*| = \displaystyle\sum_{j=m+1}^{p+1} \alpha_j$ and $|X| = d_{p+1} - d_m$, so we can use (3) to find that

$$\sum_{j=m}^{p} q_j \geq |X| - |X \cap \mathcal{B}^*| = d_{p+1} - d_m - \sum_{j=m+1}^{p+1} \alpha_j.$$

This yields (b).

Next, suppose $p = e_m - 1$ (meaning $\min\{b_m^*, d_{p+1}\} = b_m^*$). In this case, the elements of $X \setminus \mathcal{B}^*$ are the indices x such that $P(x)$ lies below H_m^* and is not a northeast endpoint of a hook in $\mathcal{H}^*$. These points are precisely those that are given the same color as one of the hooks $H_m^*, \ldots, H_{e_m-1}^*$ in the coloring induced by $\mathcal{H}^*$ (by the definition of e_m, these are the hooks equal to or lying below H_m^*). The number of such points is $\displaystyle\sum_{j=m}^{e_m-1} q_j^*$, so we deduce from (3) that

$$\sum_{j=m}^{e_m-1} q_j \geq |X \setminus \mathcal{B}^*| = \sum_{j=m}^{e_m-1} q_j^*. \tag{4}$$

This proves (a).

To prove the converse, suppose we are given a composition $(q_0, \ldots, q_k)$ of $n-k$ into $k+1$ parts that satisfies (a) and (b). We wish to construct a valid hook configuration $\mathcal{H} = (H_1, \ldots, H_k)$ of π that induces the composition $(q_0, \ldots, q_k)$. To do so, it suffices to specify the northeast endpoints of the hooks. Letting $P(b_i)$ denote the northeast endpoint of H_i as before, we see that we need only choose the indices b_i. We will choose them in the order $b_k, \ldots, b_1$.

Let $\ell \in \{1, \ldots, k\}$. Suppose that we have already chosen $b_k, \ldots, b_{\ell+1}$ and that we are now ready to choose b_ℓ. Let $Z = Z_\ell$ be the set of indices $z \in \{d_\ell + 1, \ldots, n\} \setminus \{b_{\ell+1}, \ldots, b_k\}$ such that $P(z)$ does not lie below any of the hooks $H_{\ell+1}, \ldots, H_k$. Let us write $Z = \{z_\ell(1), \ldots, z_\ell(\theta_\ell)\}$, where $\theta_\ell = |Z|$ and $z_\ell(1) < \cdots < z_\ell(\theta_\ell)$. Put $b_\ell = z_\ell(q_\ell + 1)$ (we will see in the next paragraph that $\theta_\ell \geq q_\ell + 1$ so that this definition makes sense). This choice of b_ℓ is actually forced upon us. Indeed, we *must* put $b_\ell = z_\ell(w)$ for some w. The points $P(z_\ell(1)), \ldots, P(z_\ell(w-1))$ are precisely the points that see the hook H_ℓ when they look upward. Therefore, if we can show that this construction actually produces a valid hook configuration $\mathcal{H}$, we will know that $(q_0, \ldots, q_k)$ is the valid composition of π induced by $\mathcal{H}$.

Let us verify that $\theta_\ell \geq q_\ell + 1$ so that $z_\ell(q_\ell + 1)$ actually exists. There are $n - d_\ell - (k - \ell)$ indices $z \in \{d_\ell + 1, \ldots, n\} \setminus \{b_{\ell+1}, \ldots, b_k\}$, and $\displaystyle\sum_{j=\ell+1}^{k} q_j$ of them are such that $P(z)$ lies below one of the hooks $H_{\ell+1}, \ldots, H_k$. Consequently,

$$\theta_\ell = |Z| = n - d_\ell - (k - \ell) - \sum_{j=\ell+1}^{k} q_j. \tag{5}$$

We know that $b_i^* > d_i$ for all i. Therefore, among the numbers $b_1^*, \ldots, b_k^*$, only $b_1^*, \ldots, b_{\ell-1}^*$ could possibly lie in the first ℓ ascending runs of π. This shows that $\displaystyle\sum_{j=1}^{\ell} \alpha_j \leq \ell - 1$. Combining this with (5) and the fact that $\displaystyle\sum_{j=0}^{\ell} q_j = n - k - \sum_{j=\ell+1}^{k} q_j$, we get

$$\theta_\ell = \ell - d_\ell + \sum_{j=0}^{\ell} q_j = \ell - 1 - d_\ell + \sum_{j=0}^{\ell-1} q_j + q_\ell + 1 \geq \sum_{j=0}^{\ell-1} q_j - \left(d_\ell - \sum_{j=1}^{\ell} \alpha_j\right) + q_\ell + 1.$$

Setting $m = 0$ and $p = \ell - 1$ in condition (b), we obtain

$$\sum_{j=0}^{\ell-1} q_j \geq d_\ell - \sum_{j=1}^{\ell} \alpha_j,$$

so $\theta_\ell \geq q_\ell + 1$.

Now that we have defined the indices b_ℓ, we can construct the candidate hook H_ℓ (we use the term "candidate hook" because we still need to prove that it is a hook). Specifically, H_ℓ is the candidate hook with southwest endpoint $P(d_\ell)$ and northeast endpoint $P(b_\ell)$. To check that this is in fact a hook, we must verify that $d_\ell < b_\ell$ and $\pi_{d_\ell} < \pi_{b_\ell}$ for all ℓ. We constructed b_ℓ so that $d_\ell < b_\ell$. We also know that $\pi_{d_\ell} < \pi_{b_\ell^*}$ because H_ℓ^* is a hook with southwest endpoint $P(d_\ell)$ and northeast endpoint $P(b_\ell^*)$. Hence, it suffices to show that $\pi_{b_\ell^*} \leq \pi_{b_\ell}$.

Observe that $b_k^* = d_k + q_k^* + 1$ because the points lying below H_k^* are precisely $P(d_k + 1), \ldots, P(d_k + q_k^*)$. Likewise, $b_k = d_k + q_k + 1$. Setting $m = k$ in condition (a), we get $q_k \geq q_k^*$ because $e_k = k + 1$. This shows that $b_k^* \leq b_k$. This also forces the inequality $\pi_{b_k^*} \leq \pi_{b_k}$ since $\pi_{b_k^*}$ and π_{b_k} both lie in the $(k+1)^{\text{th}}$ ascending run of π. It follows that H_k lies above H_k^* or is equal to H_k^*.

Now, choose some $\ell \in \{1, \ldots, k-1\}$, and suppose that $b_r^* \leq b_r$ and $\pi_{b_r^*} \leq \pi_{b_r}$ for all $r \in \{\ell + 1, \ldots, k\}$. For each such r, this means that H_r lies above or is equal to H_r^*. Recall the definition of Z from above. The entries $\pi_{z_\ell(1)}, \ldots, \pi_{z_\ell(\theta_\ell)}$ are left-to-right maxima of the string $\pi_{d_\ell+1} \cdots \pi_n$ (a left-to-right maximum of a string of positive integers $w_1 \cdots w_r$ is an entry w_j such that $w_j > w_i$ for all $i \in \{1, \ldots, j-1\}$). Indeed, if there were some $\pi_a > \pi_{z_\ell(t)}$ with $a \geq d_\ell + 1$, then by choosing a maximally, we would find that $P(a)$ is the southwest endpoint of a hook that lies above $P(z_\ell(t))$, contradicting the definition of $z_\ell(t)$. We know from our definition of e_ℓ that

$$b_\ell^* \geq d_{e_\ell - 1} + 1. \tag{6}$$

We wish to show that $b_\ell \geq b_\ell^*$, which will of course imply that $b_\ell \geq d_{e_\ell-1} + 1$. By way of contradiction, let us first assume $b_\ell \leq d_{e_\ell-1}$. We have $d_p + 1 \leq b_\ell \leq d_{p+1}$ for some $p \in \{\ell, \ell+1, \ldots, e_\ell - 2\}$. As mentioned above, H_ℓ is the candidate hook with southwest endpoint $P(d_\ell)$ and northeast endpoint $P(b_\ell)$. Because $\pi_{b_\ell} = \pi_{z_\ell(q_\ell+1)}$ is a left-to-right maximum of $\pi_{d_\ell+1} \cdots \pi_n$, every point $P(x)$ with $d_\ell + 1 \leq x \leq d_p$ lies below H_ℓ. Furthermore, each of the hooks $H_{\ell+1}, \ldots, H_p$ must lie (entirely) below H_ℓ because $P(b_\ell)$ cannot lie above any of these hooks (our construction guarantees that no point in the plot of π lies above any of these hooks). It follows from our construction that there are precisely $\displaystyle\sum_{j=\ell}^{p} q_j$ points that lie below the candidate hook H_ℓ and are not in $\{P(b_1), \ldots, P(b_k)\}$. Each of the points $P(b_{\ell+1}), \ldots, P(b_p)$ lies below H_ℓ because the hooks $H_{\ell+1}, \ldots, H_p$ lie below H_ℓ. This means that the total number of points lying below H_ℓ is at least $p - \ell + \displaystyle\sum_{j=\ell}^{p} q_j$. For each such point

$P(z)$, we have $d_\ell + 1 \le z \le d_{p+1}$, so

$$p - \ell + \sum_{j=\ell}^{p} q_j < d_{p+1} - d_\ell. \tag{7}$$

Note that the inequality here is strict because b_ℓ is an element of $\{d_\ell + 1, \ldots, d_{p+1}\}$ and $P(b_\ell)$ does not lie below H_ℓ.

If δ is an index such that $d_\ell + 1 \le b^*_\delta \le d_{p+1}$, then $\delta \le p$. The point $P(b^*_\delta)$ must lie below H^*_ℓ because $d_\ell < b^*_\delta \le d_{p+1} \le d_{e_\ell - 1} < b^*_\ell$ (using (6)). Hence, $\ell + 1 \le \delta \le p$. This shows that there are at most $p - \ell$ possible choices for δ, so

$$\sum_{j=\ell+1}^{p+1} \alpha_j \le p - \ell.$$

It now follows from (7) that

$$\sum_{j=\ell+1}^{p+1} \alpha_j + \sum_{j=\ell}^{p} q_j < d_{p+1} - d_\ell,$$

which we can see is a contradiction by setting $m = \ell$ in condition (b). We conclude that $b_\ell \ge d_{e_\ell - 1} + 1$.

A consequence of the previous paragraph is that H_ℓ lies above the hooks $H_{\ell+1}, \ldots, H_{e_\ell - 1}$, so the number of points not in the set $\{P(b_1), \ldots, P(b_k)\}$ that lie below H_ℓ is at least $\sum_{j=\ell}^{e_\ell - 1} q_j$. By condition (a), this is at least $\sum_{j=\ell}^{e_\ell - 1} q^*_j$. Moreover, each of the points $P(b_{\ell+1}), \ldots, P(b_{e_\ell - 1})$ lies below H_ℓ. This shows that there are at least

$$e_\ell - 1 - \ell + \sum_{j=\ell}^{e_\ell - 1} q^*_j$$

points below H_ℓ.

If η is an index such that $P(b^*_\eta)$ lies below H^*_ℓ, then $d_\ell < d_\eta < b^*_\eta < b^*_\ell \le d_{e_\ell}$. This guarantees that $\ell < \eta \le e_\ell - 1$, so there are at most $e_\ell - 1 - \ell$ such indices η. The number of points below H^*_ℓ that are not of the form $P(b^*_\eta)$ is $\sum_{j=\ell}^{e_\ell - 1} q^*_j$, so the total number of points below H^*_ℓ is at most

$$e_\ell - 1 - \ell + \sum_{j=\ell}^{e_\ell - 1} q^*_j.$$

According to the previous paragraph, the number of points below H_ℓ is at least the number of points below H^*_ℓ. Therefore, H_ℓ lies above H^*_ℓ or is equal to H^*_ℓ. In other words, $b^*_\ell \le b_\ell$ and $\pi_{b^*_\ell} \le \pi_{b_\ell}$. It follows by induction that $b^*_i \le b_i$ and $\pi_{b^*_i} \le \pi_{b_i}$ for all $i \in \{1, \ldots, k\}$.

We have shown that the candidate hooks $H_1, \ldots, H_k$ are in fact genuine hooks. Our construction guarantees that $\mathcal{H} = (H_1, \ldots, H_k)$ is a valid hook configuration of π, so the proof is complete. □

3. Background on Preimages of Permutation Classes

In this section, we review some known results concerning preimages of permutation classes. We also establish some notation for subsequent sections.

First, suppose $\pi \in S_n$ for some $n \geq 4$, and write $\pi = LnR$ so that $s(\pi) = s(L)s(R)n$. Either L or R has size at least 2, and the permutations $s(L)$ and $s(R)$ each must end in their last entries. It follows that $s(\pi)$ contains the pattern 123, so

$$|s^{-1}(\mathrm{Av}_n(123))| = 0 \quad \text{whenever} \quad n \geq 4.$$

It is easy to see that

$$|s^{-1}(\mathrm{Av}_n(213))| = C_n.$$

Indeed, suppose $\pi \in s^{-1}(\mathrm{Av}_n(213))$. Since $s(\pi)$ avoids 213 and has last entry n, $s(\pi)$ must be the identity permutation of size n. In other words, $s^{-1}(\mathrm{Av}(213)) = s^{-1}(\mathrm{Av}(21)) = \mathrm{Av}(231)$ by Theorem 2. Theorem 4 tells us that

$$|s^{-1}(\mathrm{Av}_n(231))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

Theorem 3.2 in [12] states that we also have

$$|s^{-1}(\mathrm{Av}_n(132))| = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}.$$

Part of Theorem 3.4 in [12] states that

$$|s^{-1}(\mathrm{Av}_n(312))| = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}.$$

This last expression is also the number of so-called *Baxter permutations* of size n, and it produces the sequence A001181 in [37]. Both Theorems 3.2 and 3.4 in [12] are proved via explicit bijections. The only size-3 pattern τ for which $|s^{-1}(\mathrm{Av}_n(\tau))|$ is not known is 321. The sequence $(|s^{-1}(\mathrm{Av}_n(321))|)_{n\geq 1}$ appears to be new.[5] In Section 11, we use valid hook configurations to derive estimates for the exponential growth rate of this sequence.

The above remarks show that the sets $s^{-1}(\mathrm{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ are relatively uninteresting when one of the patterns $\tau^{(i)}$ is an element of $\{123, 213\}$. Indeed, if $\tau^{(i)} = 213$, then none of the permutations in $\mathrm{Av}(\tau^{(1)}, \ldots, \tau^{(r)})$ except possibly identity permutations have preimages under s. Hence, we will focus our attention on permutation classes whose bases are subsets of $\{132, 231, 312, 321\}$.

As mentioned in the introduction, it is always possible to describe $s^{-1}(\mathrm{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ as the set of permutations avoiding some finite collection of *mesh patterns* [17]. We will see

[5] I have added it as sequence A319027 in [37].

shortly that if the patterns $\tau^{(1)}, \ldots, \tau^{(r)}$ are all of size 3, then we can actually describe our stack-sorting preimage sets in terms of barred patterns and vincular patterns.

A *barred pattern* is a permutation pattern in which some entries are overlined. Saying a permutation contains a barred pattern means that it contains a copy of the pattern formed by the unbarred entries that is not part of a pattern that has the same relative order as the full barred pattern. For example, saying a permutation contains the barred pattern $3\overline{5}241$ means that it contains a 3241 pattern that is not part of a 35241 pattern. In fact, West [40] introduced barred patterns in order to describe 2-stack-sortable permutations, showing that a permutation is 2-stack-sortable if and only if it avoids the classical pattern 2341 and the barred pattern $3\overline{5}241$.

A *vincular pattern* is a permutation pattern in which some consecutive entries can be underlined. We say a permutation *contains* a vincular pattern if it contains an occurrence of the permutation pattern in which underlined entries are consecutive. For example, saying that a permutation $\sigma = \sigma_1 \cdots \sigma_n$ contains the vincular pattern $\underline{32}41$ means that there are indices $i_1 < i_2 < i_3 < i_4$ such that $\sigma_{i_4} < \sigma_{i_2} < \sigma_{i_1} < \sigma_{i_3}$ and $i_2 = i_1 + 1$. Vincular patterns appeared first in [1] and have received a large amount of attention ever since [39].

If τ is a classical, barred, or vincular pattern, then we say a permutation *avoids* τ if it does not contain τ. Let $\text{Av}(\tau^{(1)}, \ldots, \tau^{(r)})$ be the set of permutations avoiding $\tau^{(1)}, \ldots, \tau^{(r)}$, and let $\text{Av}_n(\tau^{(1)}, \ldots, \tau^{(r)}) = \text{Av}(\tau^{(1)}, \ldots, \tau^{(r)}) \cap S_n$.

In order to characterize stack-sorting preimages of many permutation classes, we make use of the following theorem due to Claesson and Úlfarsson (the part concerning $s^{-1}(\text{Av}(231))$ is due to West). These results are stated in [17] in terms of mesh patterns, but it is straightforward to rephrase them in terms of barred and vincular patterns.

Theorem 13 ([17, 40]). *We have*

- $s^{-1}(\text{Av}(132)) = \text{Av}(1342, \underline{31}42)$;
- $s^{-1}(\text{Av}(231)) = \text{Av}(2341, 3\overline{5}241)$;
- $s^{-1}(\text{Av}(312)) = \text{Av}(3412, 34\underline{21})$;
- $s^{-1}(\text{Av}(321)) = \text{Av}(34251, 35241, 45231)$.

We can intersect the sets appearing in Theorem 13 to obtain the following characterizations of preimage sets.

Corollary 14. *We have*

(i) $s^{-1}(\text{Av}(132, 231)) = \text{Av}(1342, 2341, \underline{31}42, 3\overline{5}241)$;
(ii) $s^{-1}(\text{Av}(132, 312)) = \text{Av}(1342, 3142, 3412, 34\underline{21})$;
(iii) $s^{-1}(\text{Av}(231, 312)) = \text{Av}(2341, 3412, 34\underline{21}, 3\overline{5}241)$;
(iv) $s^{-1}(\text{Av}(231, 321)) = \text{Av}(2341, 3241, 45231)$;
(v) $s^{-1}(\text{Av}(132, 321)) = \text{Av}(1342, 34251, 35241, 45231, \underline{31}42)$;
(vi) $s^{-1}(\text{Av}(312, 321)) = \text{Av}(3412, 3421)$;
(vii) $s^{-1}(\text{Av}(132, 231, 312)) = \text{Av}(1342, 2341, 3412, 3142, 34\underline{21}, \underline{32}41)$;
(viii) $s^{-1}(\text{Av}(231, 312, 321)) = \text{Av}(2341, 3241, 3412, 3421)$;

(ix) $s^{-1}(\text{Av}(132, 231, 321)) = \text{Av}(1342, 2341, 3241, 45231, \underline{31}42)$;
(x) $s^{-1}(\text{Av}(132, 312, 321)) = \text{Av}(1342, 3142, 3412, 3421)$;
(xi) $s^{-1}(\text{Av}(132, 231, 312, 321)) = \text{Av}(1342, 2341, 3142, 3241, 3412, 3421)$.

Proof. Note that $(i), (iii), (iv), (v), (ix)$ are immediate from Theorem 13. To prove (vi), we must show that[6]

$$\text{Av}(3412, 34\underline{21}, 34251, 35241, 45231) = \text{Av}(3412, 3421).$$

Certainly the left-hand side contains the right-hand side, so we must prove the reverse containment. Suppose $\pi \in \text{Av}(3412, 34\underline{21}, 34251, 35241, 45231)$. We need to show that π avoids 3421. Suppose instead that π contains 3421. This means that there are indices $i_1 < i_2 < i_3 < i_4$ with $\pi_{i_4} < \pi_{i_3} < \pi_{i_1} < \pi_{i_2}$. We may assume these indices were chosen in such a way as to minimize $i_4 - i_3$. Because π avoids $34\underline{21}$, we must have $i_4 - i_3 \geq 2$. If $\pi_{i_3+1} > \pi_{i_2}$, then $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_3+1}\pi_{i_4}$ is an occurrence of the pattern 34251, which is a contradiction. If $\pi_{i_1} < \pi_{i_3+1} < \pi_{i_2}$, then $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_3+1}\pi_{i_4}$ is an occurrence of the pattern 35241 in π, which is also impossible. If $\pi_{i_4} < \pi_{i_3+1} < \pi_{i_1}$, then $\pi_{i_1}\pi_{i_2}\pi_{i_3+1}\pi_{i_4}$ is another occurrence of the pattern 3421, contradicting the minimality of $i_4 - i_3$. The last remaining possibility is that $\pi_{i_3+1} < \pi_{i_4}$. In this case, $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_3+1}$ is an occurrence of the pattern $34\underline{21}$, which is a contradiction. This proves (vi).

Note that $(viii)$ follows from (iv) and (vi). The proofs of $(ii), (vii), (x), (xi)$ are similar to the proof of (vi) that we just described; we omit the details because we will not need these results in the rest of the paper. $\square$

We end this section by fixing some notation concerning specific power series. Recall the notation from Theorem 7 and Theorem 10. When counting preimages of permutations according to numbers of descents and peaks, we will make use of the generating functions

$$F(x, y) = \sum_{n \geq 1} \sum_{m \geq 1} N(n, m) x^n y^{m-1} \quad \text{and} \quad G(x, y) = \sum_{n \geq 1} \sum_{m \geq 1} V(n, m) x^n y^{m-1}.$$

It is known (see sequences A001263 and A091894 in [37]) that

$$F(x, y) = \frac{1 - x(y+1) - \sqrt{1 - 2x(y+1) + x^2(y-1)^2}}{2xy} \tag{8}$$

and

$$G(x, y) = \frac{1 - 2x - \sqrt{(1-2x)^2 - 4x^2 y}}{2xy}. \tag{9}$$

We let $[z_1^{n_1} \cdots z_r^{n_r}]A(z_1, \ldots, z_r)$ denote the coefficient of $z_1^{n_1} \cdots z_r^{n_r}$ in the generating function $A(z_1, \ldots, z_r)$.

[6]The author thanks Chetak Hossain for pointing him to this fact.

4. Enumeration of $s^{-1}(\mathrm{Av}(132, 231, 312, 321))$

We saw in Corollary 14 that $s^{-1}(\mathrm{Av}(132, 231, 312, 321))$ is an actual permutation class. One could enumerate this class directly, but we will use valid hook configurations in order to illustrate this uniform method for finding fertilities.

Theorem 15. *For $n \geq 2$, we have*

$$|s^{-1}(\mathrm{Av}_n(132, 231, 312, 321))| = 2C_n - 2C_{n-1}.$$

The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 312, 321))$ with exactly m descents is

$$N(n, m+1) + \sum_{i=1}^{n-2}\sum_{j=1}^{m} N(i, j)N(n-i-1, m-j+1).$$

The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 312, 321))$ with exactly m peaks is

$$V(n, m+1) + \sum_{i=1}^{n-2}\sum_{j=1}^{m} V(i, j)V(n-i-1, m-j+1).$$

Proof. The only elements of $\mathrm{Av}_n(132, 231, 312, 321)$ are $123\cdots n$ and $2134\cdots n$. The only valid composition of $123\cdots n$ is (n). Each valid hook configuration of $2134\cdots n$ has exactly one hook. This hook has southwest endpoint $(1, 2)$ and has northeast endpoint $(n+1-i, n+1-i)$ for some $i \in \{1, \ldots, n-2\}$. This valid hook configuration induces the valid composition $(i, n-i-1)$. It follows from Theorem 7 and the standard Catalan number recurrence relation that

$$|s^{-1}(\mathrm{Av}_n(132, 231, 312, 321))| = C_n + \sum_{i=1}^{n-2} C_iC_{n-i-1} = C_n + \sum_{i=0}^{n-1} C_iC_{n-i-1} - 2C_{n-1}$$
$$= 2C_n - 2C_{n-1}.$$

The second and third statements of the theorem follow immediately from Theorem 10. □

5. Enumeration of $s^{-1}(\mathrm{Av}(132, 231, 321))$ and $s^{-1}(\mathrm{Av}(132, 312, 321))$

Let

$$\sigma_{n,\ell} = \ell 12\cdots(\ell-1)(\ell+1)\cdots n \quad \text{and} \quad \gamma_{n,\ell} = 23\cdots\ell 1(\ell+1)\cdots n.$$

It is straightforward to check that

$$\mathrm{Av}_n(132, 231, 321) = \{\sigma_{n,1}, \ldots, \sigma_{n,n}\} \quad \text{and} \quad \mathrm{Av}_n(132, 312, 321) = \{\gamma_{n,1}, \ldots, \gamma_{n,n}\}.$$

West [40] found formulas for the fertilities of $\sigma_{n,\ell}$ and $\gamma_{n,\ell}$ and found that they are equal. It follows that

$$|s^{-1}(\mathrm{Av}_n(132, 231, 321))| = |s^{-1}(\mathrm{Av}_n(132, 312, 321))|.$$

This equality is easy to verify with the theory of valid hook configurations. Indeed, for $2 \leq \ell \leq n$, we have

$$\mathcal{V}(\sigma_{n,\ell}) = \{(n-\ell-i+1, \ell+i-2) : 1 \leq i \leq n-\ell\}$$

and

$$\mathcal{V}(\gamma_{n,\ell}) = \{(\ell+i-2, n-\ell-i+1) : 1 \le i \le n-\ell\}.$$

That is, the valid compositions of $\sigma_{n,\ell}$ are obtained by interchanging the two parts in the valid compositions of $\gamma_{n,\ell}$. Along with Theorem 10, this also implies the following.

Theorem 16. *The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ with exactly m descents is equal to the number of elements of $s^{-1}(\mathrm{Av}_n(132, 312, 321))$ with exactly m descents. The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ with exactly m peaks is equal to the number of elements of $s^{-1}(\mathrm{Av}_n(132, 312, 321))$ with exactly m peaks.*

Note that West did not prove these refined equalities.

Theorem 17. *We have*

$$|s^{-1}(\mathrm{Av}_n(132, 231, 321))| = |s^{-1}(\mathrm{Av}_n(132, 312, 321))| = \binom{2n-2}{n-1}.$$

The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ (equivalently, of $s^{-1}(\mathrm{Av}_n(132, 312, 321))$) with exactly m descents is

$$\binom{n-1}{m}^2.$$

The number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ (equivalently, of $s^{-1}(\mathrm{Av}_n(132, 312, 321))$) with exactly m peaks is

$$2^{n-2m-2}\binom{n}{2m+2}\binom{2m+2}{m+1}.$$

Proof. By Theorem 16, we need only consider the preimage sets $s^{-1}(\mathrm{Av}_n(132, 231, 321))$. We will prove the second and third statements; the first statement will then follow from the second and the well-known identity $\sum_{m=0}^{n-1}\binom{n-1}{m}^2 = \binom{2n-2}{n-1}$. The only valid composition of $\sigma_{n,1} = 123\cdots n$ is (n). For $2 \le \ell \le n$, the valid compositions of $\sigma_{n,\ell}$ are $(n-\ell-i+1, \ell+i-2)$ for $1 \le i \le n-\ell$. In particular, we can ignore $\sigma_{n,n}$ because it has no valid compositions (that is, $\sigma_{n,n}$ is not sorted).

Using the first part of Theorem 10, we find that the number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ with exactly m descents is

$$N(n, m+1) + \sum_{\ell=2}^{n-1} \sum_{(q_0,q_1)\in\mathcal{V}(\sigma_{n,\ell})} \sum_{j_0+j_1=m+1} N(q_0, j_0)N(q_1, j_1)$$

$$= N(n, m+1) + \sum_{\ell=2}^{n-1}\sum_{i=1}^{n-\ell}\sum_{j=1}^{m} N(n-\ell-i+1, j)N(\ell+i-2, m-j+1).$$

Letting $r = \ell + i$, this becomes

$$\begin{aligned} N(n,m+1) &+ \sum_{\ell=2}^{n-1}\sum_{r=\ell+1}^{n}\sum_{j=1}^{m} N(n-r+1,j)N(r-2,m-j+1) \\ &= N(n,m+1) + \sum_{\ell=2}^{n-1}\sum_{r=\ell-1}^{n-2}\sum_{j=1}^{m} N(n-r-1,j)N(r,m-j+1) \\ &= N(n,m+1) + \sum_{r=1}^{n-2} r \sum_{j=1}^{m} N(n-r-1,j)N(r,m-j+1) \\ &= [x^n y^m]\left(F(x,y) + x^2 y\, F(x,y)\cdot \frac{\partial}{\partial x}F(x,y)\right), \end{aligned}$$

where $F(x,y) = \sum_{n\geq 1}\sum_{m\geq 1} N(n,m)x^n y^{m-1}$. Applying (8) and some algebraic manipulations, we find that this is equal to

$$[x^n y^m]\left(\frac{x}{\sqrt{1+x^2(y-1)^2-2x(y+1)}}\right).$$

This is known to equal $\binom{n-1}{m}^2$ (see sequence A008459 in [37]).

The proof of the third statement in the theorem proceeds exactly as in the proof of the second statement. In this case, we find that the number of elements of $s^{-1}(\mathrm{Av}_n(132, 231, 321))$ with exactly m peaks is

$$[x^n y^m]\left(G(x,y) + x^2 y\, G(x,y)\cdot \frac{\partial}{\partial x}G(x,y)\right).$$

Applying (9) and some algebraic manipulations, we find that this is equal to

$$[x^n y^m]\left(\frac{x}{\sqrt{1-4x-4x^2(y-1)}}\right).$$

It is known that this expression is equal to $2^{n-2m-2}\binom{n}{2m+2}\binom{2m+2}{m+1}$ (see sequence A051288 in [37]). $\square$

Recently, Bruner proved that

$$|\operatorname{Av}_n(2431, 4231, 1432, 4132)| = \binom{2n-2}{n-1}.$$

She also listed several other permutation classes that appear to be enumerated by central binomial coefficients but did not prove that this is the case. One of these classes is $\mathrm{Av}(1243, 2143, 2413, 2431)$. Of course, a permutation is in this class if and only if its reverse is in the class

$$\mathrm{Av}(1342, 3142, 3412, 3421) = s^{-1}(\mathrm{Av}(132, 312, 321)).$$

Therefore, the next corollary, which follows immediately from Theorem 17 and Corollary 14, settles one of the enumerative problems that Bruner listed.

Corollary 18. *We have*

$$|\operatorname{Av}_n(1342, 3142, 3412, 3421)| = \binom{2n-2}{n-1}.$$

The number of elements of $\operatorname{Av}_n(1342, 3142, 3412, 3421)$ *with exactly* m *descents is*

$$\binom{n-1}{m}^2.$$

The number of elements of $\operatorname{Av}_n(1342, 3142, 3412, 3421)$ *with exactly* m *peaks is*

$$2^{n-2m-2}\binom{n}{2m+2}\binom{2m+2}{m+1}.$$

6. Enumeration of $s^{-1}(\operatorname{Av}(231, 312, 321))$

It is standard to identify a permutation with a configuration of points in the plane via its plot. Doing so, we can build permutations by placing the plots of smaller permutations in various configurations. For example, if $\lambda = \lambda_1 \cdots \lambda_\ell \in S_\ell$ and $\mu = \mu_1 \cdots \mu_m \in S_m$, then the *sum* of λ and μ, denoted $\lambda \oplus \mu$, is obtained by placing the plot of μ above and to the right of the plot of λ. More formally, the i^{th} entry of $\lambda \oplus \mu$ is

$$(\lambda \oplus \mu)_i = \begin{cases} \lambda_i, & \text{if } 1 \le i \le \ell; \\ \mu_{i-\ell} + \ell, & \text{if } \ell + 1 \le i \le \ell + m. \end{cases}$$

Let $\operatorname{Dec}_a = a(a-1)\cdots 1 \in S_a$ denote the decreasing permutation of size a. The permutations in $\operatorname{Av}(231, 312)$ are often called *layered* [36]; each is of the form $\operatorname{Dec}_{a_1} \oplus \cdots \oplus \operatorname{Dec}_{a_t}$ for some composition $(a_1, \ldots, a_t)$. For example, $32154687 = 321 \oplus 21 \oplus 1 \oplus 21$ is the layered permutation corresponding to the composition $(3, 2, 1, 2)$. Under this correspondence between layered permutations and compositions, the permutations in $\operatorname{Av}(231, 312, 321)$ correspond to compositions whose parts are all at most 2. It follows that these permutations are counted by the Fibonacci numbers (this is well known).

As we saw in Corollary 14, the set $s^{-1}(\operatorname{Av}(231, 312, 321))$ is the same as the permutation class $\operatorname{Av}(2341, 3241, 3412, 3421)$. A permutation σ is in this class if and only if the first and third entries in every 231 pattern in σ are consecutive integers. In this section, we use valid hook configurations to derive a formula for $|s^{-1}(\operatorname{Av}_n(231, 312, 321))|$. We then enumerate this class directly; showing that these permutations are counted by the terms in sequence A049124 in [37]. Together, these results give a new formula and a new combinatorial interpretation for the terms in this sequence.

Recall that $\operatorname{Comp}_a(b)$ is the set of all compositions of b into a parts. Define a partial order $\preceq$ on $\operatorname{Comp}_a(b)$ by declaring that $(x_1, \ldots, x_a) \preceq (y_1, \ldots, y_a)$ if $\sum_{i=1}^{\ell} x_i \le \sum_{i=1}^{\ell} y_i$ for all $\ell \in \{1, \ldots, a\}$. A *partition* is a composition whose parts are nonincreasing. Following [38], we let $L(u, v)$ denote the set of all partitions (including the empty partition) with at most u parts and with largest part at most v. Endow $L(u, v)$ with a partial order $\le$ by declaring that $(\lambda_1, \ldots, \lambda_\ell) \le (\mu_1, \ldots, \mu_m)$ if $\ell \le m$ and $\lambda_i \le \mu_i$ for all $i \in \{1, \ldots, \ell\}$. Geometrically,

$L(u,v)$ is the set of all partitions whose Young diagrams fit inside a $u\times v$ rectangle, and $\lambda\leq\mu$ if and only if the Young diagram of λ fits inside of the Young diagram of μ.

Given $x=(x_1,\ldots,x_a)\in\text{Comp}_a(b)$, let $\psi(x)\in L(b-a,a-1)$ be the partition that has exactly x_i-1 parts of size $a-i$ for all $i\in\{1,\ldots,a-1\}$. For example, if $a=4$, $b=12$, and $x=(3,1,5,3)$, then $\psi(x)=(3,3,1,1,1,1)$. The map $\psi:\text{Comp}_a(b)\to L(b-a,a-1)$ is an isomorphism of posets. For $x\in\text{Comp}_a(b)$, let

$$D_x=|\{y\in\text{Comp}_a(b):y\preceq x\}|.$$

Equivalently, $D_{\psi^{-1}(\lambda)}$ is the number of partitions (including the empty partition) whose Young diagrams fit inside of the Young diagram of the partition λ. Recall the notation $C_{(x_0,\ldots,x_k)}=\prod_{t=0}^k C_{x_t}$, where C_j is the j^{th} Catalan number.

Theorem 19. *Preserving the notation of the preceding paragraph with $a=k+1$ and $b=n-k$, we have*

$$|s^{-1}(\text{Av}_n(231,312,321))|=\sum_{k=0}^{n-1}\sum_{q\,\in\,\text{Comp}_{k+1}(n-k)}C_qD_q=\sum_{k=0}^{n-1}\sum_{\lambda\in L(n-2k-1,k)}C_{\psi^{-1}(\lambda)}D_{\psi^{-1}(\lambda)}.$$

Proof. Recall that each permutation in $\text{Av}(231,312,321)$ is of the form $\text{Dec}_{a_1}\oplus\cdots\oplus\text{Dec}_{a_t}$, where $a_i\in\{1,2\}$ for all i. Suppose $\pi\in\text{Av}_n(231,312,321)$ has descents $d_1<\cdots<d_k$. For every $p\in\{0,\ldots,k-1\}$, let $u_p=d_{p+1}-p$. Defining $y_0=u_0$, $y_i=u_i-u_{i-1}$ for $1\leq i\leq k-1$, and $y_k=n-k-u_{k-1}$, we obtain a composition $y=(y_0,\ldots,y_k)\in\text{Comp}_{k+1}(n-k)$. Given this composition y, we can easily reconstruct the permutation π. Thus, there is a bijective correspondence between compositions in $\text{Comp}_{k+1}(n-k)$ and permutations in $\text{Av}_n(231,312,321)$ with k descents.

We are going to use Theorem 12 to describe all of the valid compositions of π. Preserve the notation from that theorem and the discussion immediately preceding it. We must compute the canonical hook configuration $\mathcal{H}^*=(H_1^*,\ldots,H_k^*)$ of π. This is fairly simple to do: the hook H_i^* has southwest endpoint (d_i,π_{d_i}) and northeast endpoint (d_i+2,π_{d_i+2}). Thus, $b_i^*=d_i+2$. We also have $(q_0^*,\ldots,q_k^*)=(n-2k,1,\ldots,1)$, $e_0=k+1$, and $e_i=i+1$ for all $i\in\{1,\ldots,k\}$. Finally, $\alpha_1=0$, and $\alpha_i=1$ for all $i\in\{2,\ldots,k+1\}$.

Every composition $(q_0,\ldots,q_k)\in\text{Comp}_{k+1}(n-k)$ satisfies condition (a) in Theorem 12. In condition (b), the inequality $m\leq p\leq e_m-2$ is only satisfied when $m=0$. In this case, we have $e_m-2=e_0-2=k-1$, $d_m=d_0=0$, and $\sum_{j=m+1}^{p+1}\alpha_j=p$, so

$$d_{p+1}-d_m-\sum_{j=m+1}^{p+1}\alpha_j=d_{p+1}-p=u_p.$$

Hence, Theorem 12 tells us that a composition $q=(q_0,\ldots,q_k)\in\text{Comp}_{k+1}(n-k)$ is a valid composition of π if and only if

$$\sum_{j=0}^{p}q_j\geq u_p$$

for all $p\in\{0,\ldots,k-1\}$. Because $u_p=\sum_{i=0}^p y_i$, this occurs if and only if $y\preceq q$.

Combining these observations with Theorem 7, we find that (recall the definition of $\text{Av}_{n,k}(\tau^{(1)},\ldots,\tau^{(r)})$ from Definition 1)

$$|s^{-1}(\text{Av}_n(231,312,321))|=\sum_{k=0}^{n-1}\sum_{\pi\in\text{Av}_{n,k}(231,312,321)}\sum_{q\in\mathcal{V}(\pi)}C_q$$
$$=\sum_{k=0}^{n-1}\sum_{y\in\text{Comp}_{k+1}(n-k)}\sum_{\substack{q\in\text{Comp}_{k+1}(n-k)\\ y\preceq q}}C_q=\sum_{k=0}^{n-1}\sum_{q\in\text{Comp}_{k+1}(n-k)}C_qD_q.$$

The identity

$$\sum_{k=0}^{n-1}\sum_{q\in\text{Comp}_{k+1}(n-k)}C_qD_q=\sum_{k=0}^{n-1}\sum_{\lambda\in L(n-2k-1,k)}C_{\psi^{-1}(\lambda)}D_{\psi^{-1}(\lambda)}$$

follows from the discussion immediately preceding this theorem (with $a=k+1$ and $b=n-k$). □

The following theorem gives a summation formula that is much more explicit than the one in Theorem 19. One of the main motivations behind Theorem 19 will come from Conjecture 21, which states that the equality between the summations in Theorems 19 and 20 holds term-by-term.

Theorem 20. *For $n\geq 1$, we have*

$$|s^{-1}(\text{Av}_n(231,312,321))|=\sum_{k=0}^{n-1}\frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}.$$

Proof. Let $h(n)=|s^{-1}(\text{Av}_n(231,312,321))|=|\text{Av}_n(2341,3241,3412,3421)|$ (with $h(0)=1$), and put

$$H(x)=\sum_{n\geq 0}h(n)x^n.$$

We will prove that

$$H(x)=1+\frac{xH(x)^2}{1-x^2H(x)^2}. \tag{10}$$

This will imply that the power series $A(x)$ defined by $A(x)=x+\dfrac{A(x)^2}{1-A(x)^2}$ satisfies $A(x)=xH(x)$. It is known that

$$\sum_{k=0}^{n-1}\frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}$$

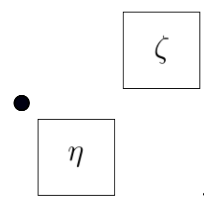
is the $(n+1)^{\text{th}}$ term of the sequence A049124 in [37] and that the generating function of that sequence is $A(x)$, so this will complete the proof.

We can rewrite (10) as

$$H(x)=1+xH(x)^2+x^2H(x)^2(H(x)-1); \tag{11}$$

it is this form of the equation that we will prove. For convenience, say a permutation is *nice* if it is an element of $\mathrm{Av}(2341, 3241, 3412, 3421)$ (including the empty permutation). As mentioned earlier, a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is nice if and only if the first and third entries in every 231 pattern in σ are consecutive integers. Note that the empty permutation is nice and accounts for the term 1 on the right-hand side of (11). We now show how to construct nonempty nice permutations. It is convenient to split these permutations into two types.

The first type of nonempty nice permutation σ is that in which the first entry σ_1 is not part of a 231 pattern. As mentioned above, we identify permutations with their plots so that we can build large permutations by arranging plots of smaller permutations. Each nonempty nice permutation of the first type is formed by choosing two nice permutations η and ζ and arranging them as follows:



Accordingly, the nonempty nice permutations of the first type contribute the term $xH(x)^2$ to the right-hand side of (11).

The second type of nonempty nice permutation σ is that in which σ_1 is part of a 231 pattern. To build such a permutation, begin by choosing nice permutations $\lambda = \lambda_1 \cdots \lambda_\ell$, $\tau = \tau_1 \cdots \tau_t$, and $\mu = \mu_1 \cdots \mu_m$ such that μ is nonempty. Write $\mu' = \mu_2 \cdots \mu_m$ (so μ' is empty if $m = 1$). Write $\lambda = \lambda'\lambda''$, where λ' is the first descending run of λ. In other words, $\lambda' = \lambda_1 \cdots \lambda_j$, where j is the smallest index that is not a descent of λ. If λ is empty, then λ' and λ'' are also empty. The plot of σ is formed by arranging the plots of λ, τ, and μ as in Figure 7.

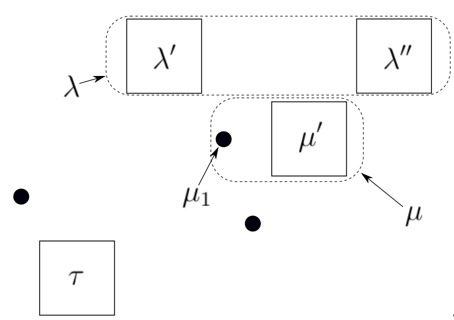


FIGURE 7. The plot of a nonempty nice permutation σ of the second type.

The points in the section labeled μ are arranged vertically so that they form a permutation that is order isomorphic to μ. For example, the point labeled μ_1 should be placed higher than exactly $\mu_1 - 1$ of the points in the box labeled μ'. Similarly, the points in the section labeled λ should form a permutation that is order isomorphic to λ.

We claim that every nonempty nice permutation of the second type is built uniquely via this procedure. If we can justify this claim, then it will follow that the nonempty nice permutations of the second type contribute the term $x^2H(x)^2(H(x)-1)$ to the right-hand side of (11).

Suppose σ is a nice permutation of the second type. We will show that the plot of σ has the shape shown in Figure 7. First, locate points that form a 231 pattern that contains σ_1. The last such point must have height σ_1-1, as indicated in Figure 7. Among all choices for the highest point in this pattern, choose the one that is farthest to the right. Label this point μ_1. Because σ is nice, the points with heights $1,2,\ldots,\sigma_1-2$ must appear to the left of all points in the plot of σ except $(1,\sigma_1)$. These points form the box τ in Figure 7. We recover the points in the box labeled λ' by taking the points that lie horizontally between τ and the point labeled μ_1. The entries in λ' must be decreasing because σ is nice. The assumption that σ is nice also tells us that the point labeled μ_1 and the dot appearing below and to its right must represent consecutive entries in σ (this also uses the fact that the point labeled μ_1 was chosen farthest to the right). Now consider P, the leftmost point that appears to the right of the lowest point in λ'. Define λ'' to consist of the points equal to and to the right of P. The remaining points of the plot will form μ'.

We want to ensure that none of the points in λ'' lie below any of the points in μ. If this were the case, then we could consider the lowest point in λ', the point P (the leftmost point in λ''), and this point of λ'' lying below a point in μ. These three points would form a 231 pattern in which the first and third entries have heights that are not consecutive integers, contradicting the fact that σ is nice. Thus, the plot of σ must be as portrayed in Figure 7.

Let us emphasize that this decomposition is unique. The only difficult part is verifying that there is a unique way to determine which points are in μ and which are in λ''. This is guaranteed by the assumption that λ' is the first descending run of λ. Indeed, this means that the highest point in μ is precisely the point whose height is 1 less than the lowest point in λ'. □

Combining Theorems 19 and 20, we obtain the identity

$$\sum_{k=0}^{n-1}\sum_{q\,\in\,\mathrm{Comp}_{k+1}(n-k)} C_qD_q=\sum_{k=0}^{n-1}\frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}.$$

The numerical data for $n\leq 12$ suggests the following conjecture.

Conjecture 21. *In the notation of Theorem 19, we have*

$$\sum_{q\,\in\,\mathrm{Comp}_{k+1}(n-k)} C_qD_q=\frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}$$

for all nonnegative integers n and k.

When combined with Theorem 19, a proof of Conjecture 21 would yield an alternative proof of Theorem 20. We know from the proof of Theorem 19 that

$$|s^{-1}(\mathrm{Av}_{n,k}(231,312,321))| = \sum_{q\,\in\,\mathrm{Comp}_{k+1}(n-k)} C_q D_q,$$

so Conjecture 21 is equivalent to the identity

$$|s^{-1}(\mathrm{Av}_{n,k}(231,312,321))| = \frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}.$$

7. Enumeration of $s^{-1}(\mathrm{Av}(132,231,312))$

In this section, we make use of the *generalized Narayana numbers*

$$N_k(n,r) = \frac{k+1}{n}\binom{n}{r+k}\binom{n}{r-1}.$$

Note that the standard Narayana numbers are simply $N(n,r) = N_0(n,r)$.

Finding the fertilities of permutations in $\mathrm{Av}(132,231,312)$ is interesting because it proves the tightness of certain estimates that were used in [22] in order to obtain upper bounds for $W_3(n)$ and $W_4(n)$. More specifically, the following theorem is an immediate consequence of Corollary 3.1 and Lemma 4.1 in [22].

Theorem 22 ([22]). *Suppose $\pi \in S_n$ has k descents. We have*

$$|s^{-1}(\pi)| \leq \frac{2k+2}{n+1}\binom{2n-2k-1}{n}.$$

Furthermore, the number of elements of $s^{-1}(\pi)$ with exactly m descents is at most

$$N_k(n-k, m-k+1) = \frac{k+1}{n-k}\binom{n-k}{m+1}\binom{n-k}{m-k}.$$

We prove below that these estimates are sharp when $\pi \in \mathrm{Av}_n(132,231,312)$. In fact, it is straightforward to check that the only permutation in $\mathrm{Av}_n(132,231,312)$ with exactly k descents is

$$\theta_{n,k} = (k+1)k(k-1)\cdots 321(k+2)(k+3)\cdots n$$

(this permutation is the sum of a decreasing permutation of size $k+1$ and an increasing permutation of size $n-k-1$). For example, $\theta_{7,2} = 3214567$ is the only permutation of size 7 that has 2 descents and avoids the patterns $132, 231, 312$.

Theorem 23. *With notation as above,*

$$|s^{-1}(\theta_{n,k})| = \frac{2k+2}{n+1}\binom{2n-2k-1}{n}.$$

Furthermore, the number of permutations in $s^{-1}(\theta_{n,k})$ with exactly m descents is

$$N_k(n-k, m-k+1).$$

Proof. As in Section 6, let $\mathrm{Comp}_a(b)$ denote the set of compositions of b into a parts. Corollary 3.1 in [22] states that

$$\sum_{(q_0,\ldots,q_k)\in\mathrm{Comp}_{k+1}(n-k)} C_{(q_0,\ldots,q_k)} = \frac{2k+2}{n+1}\binom{2n-2k-1}{n}.$$

Lemma 4.1 in that same paper tells us that

$$\sum_{(q_0,\ldots,q_k)\in\mathrm{Comp}_{k+1}(n-k)} \quad \sum_{(j_0,\ldots,j_k)\in\mathrm{Comp}_{k+1}(m+1)} \quad \prod_{t=0}^{k} N(q_t,j_t) = N_k(n-k,m-k+1).$$

Invoking Theorems 7 and 10, we see that it suffices to show that $\mathcal{V}(\theta_{n,k}) = \mathrm{Comp}_{k+1}(n-k)$. We know that $\mathcal{V}(\theta_{n,k}) \subseteq \mathrm{Comp}_{k+1}(n-k)$, so it remains to prove the reverse containment. Choose $(q_0,\ldots,q_k) \in \mathrm{Comp}_{k+1}(n-k)$. The fact that this composition exists implies that $n-k \geq k+1$. For every $i \in \{1,\ldots,k\}$, draw a hook on the plot of $\theta_{n,k}$ with southwest endpoint $(i, k+2-i)$ and northeast endpoint (u_i, u_i), where $u_i = n+1-\sum_{j=0}^{i-1} q_j$. This forms a valid hook configuration of $\theta_{n,k}$ that induces the valid composition $(q_0,\ldots,q_k)$ (see Figure 8 for an example). $\square$

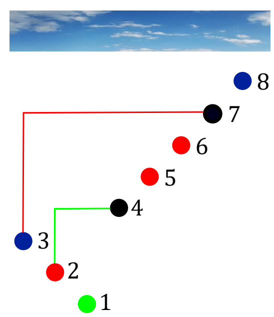


FIGURE 8. A valid hook configuration of $\theta_{2,8}$ that induces the valid composition $(2,3,1)$.

The following theorem gives a new combinatorial interpretation for the Fine numbers F_n, which are defined by

$$\sum_{n\geq 0} F_n x^n = \frac{1}{x}\frac{1-\sqrt{1-4x}}{3-\sqrt{1-4x}}.$$

The sequence of Fine numbers, which is sequence A000957 in [37], has many combinatorial connections with Catalan numbers. See the survey [24] for more on this ubiquitous sequence. The following theorem also involves two refinements of the Fine numbers. These are the numbers

$$g_{n,m} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} N_k(n-k, m-k+1)$$

and

$$h_{n,m} = \frac{2^{n-2m-1}}{n+2} n + 2m + 1\binom{n-m-1}{m}.$$

These numbers, which have combinatorial interpretations in terms of Dyck paths, appear as sequences A100754 and A114593 in [37]. It is known that[7]

$$\sum_{m=0}^{n-1} g_{n,m} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} h_{n,m} = F_{n+1}. \tag{12}$$

Theorem 24. *In the notation of the preceding paragraph, we have*

$$|s^{-1}(\mathrm{Av}_n(132, 231, 312))| = F_{n+1}.$$

Moreover, the number of permutations in $s^{-1}(\mathrm{Av}_n(132, 231, 312))$ *with exactly* m *descents is* $g_{n,m}$. *The number of permutations in* $s^{-1}(\mathrm{Av}_n(132, 231, 312))$ *with exactly* m *peaks is* $h_{n,m}$.

Proof. Recall that $\mathrm{Av}_n(132, 231, 312) = \{\theta_{n,0}, \theta_{n,1}, \ldots, \theta_{n,n-1}\}$. It follows from Theorem 23 that $s^{-1}(\theta_{n,k})$ is empty if $k > \lfloor \frac{n-1}{2} \rfloor$. It now follows immediately from Theorem 7 that the number of permutations in $s^{-1}(\mathrm{Av}_n(132, 231, 312))$ with exactly m descents is $g_{n,m}$. Along with (12), this implies that $|s^{-1}(\mathrm{Av}_n(132, 231, 312))| = F_{n+1}$.

Recall the generating function $G(x, y)$ from (9) (on page 17). According to the preceding paragraph and Theorem 10, the number of permutations in $s^{-1}(\theta_{n,k})$ with exactly m peaks is

$$\sum_{(q_0,\ldots,q_k)\in\mathrm{Comp}_{k+1}(n-k)} \ \sum_{(j_0,\ldots,j_k)\in\mathrm{Comp}_{k+1}(m+1)} \ \prod_{t=0}^{k} V(q_t, j_t).$$

This is nothing else than the coefficient of $x^n y^m$ in $x^k y^k\, G(x, y)^{k+1}$. Consequently, the number of permutations in $s^{-1}(\mathrm{Av}_n(132, 231, 312))$ with exactly m peaks is

$$[x^n y^m] \left(\sum_{k=0}^{n-1} x^k y^k\, G(x, y)^{k+1} \right) = [x^n y^m] \left(\sum_{k=0}^{\infty} x^k y^k\, G(x, y)^{k+1} \right) = [x^n y^m] \left(\frac{G(x, y)}{1 - xy\, G(x, y)} \right).$$

Straightforward algebraic manipulations show that

$$\frac{G(x, y)}{1 - xy\, G(x, y)} = \frac{1 - 2x - \sqrt{1 - 4x + 4x^2 - 4x^2 y}}{xy(1 + 2x + \sqrt{1 - 4x + 4x^2 - 4x^2 y})}.$$

This expression satisfies the functional equation given for the generating function of the numbers $h(n, m)$ in [37] (sequence A114593). Therefore, it is equal to the generating function

$$\sum_{n\geq 1} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} h(n, m) x^n y^m. \qquad \square$$

8. Enumeration of $s^{-1}(\mathrm{Av}(312, 321))$

We showed in the proof of Corollary 14 that $s^{-1}(\mathrm{Av}(312, 321)) = \mathrm{Av}(3412, 3421)$. Chetak Hossain, who pointed out this equality of sets to the current author, also mentioned the

[7] These identities are stated without proof in [37], but they can be proven by standard (yet somewhat tedious) arguments involving generating functions.

paper [33]. In this paper, Kremer proved that $|\operatorname{Av}_n(3412, 3421)|$ is the $(n-1)^{\text{th}}$ large Schröder number. In other words, we have the following theorem.[8]

Theorem 25 ([33])**.** *We have*

$$\sum_{n\geq 1} |s^{-1}(\operatorname{Av}_n(312, 321))|x^n = \sum_{n\geq 1} |\operatorname{Av}_n(3412, 3421)|x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2}.$$

9. Enumeration of $s^{-1}(\operatorname{Av}(132, 321))$

It appears as though the sequence enumerating the permutations in $s^{-1}(\operatorname{Av}(132, 321))$ has not been studied before, but we will see that its generating function is fairly simple. Let $C(x) = \sum_{n\geq 0} C_n x^n = \dfrac{1-\sqrt{1-4x}}{2x}$ be the generating function of the sequence of Catalan numbers. Recall the generating functions $F(x, y)$ and $G(x, y)$ from (8) and (9) (page 17). Let $\mathfrak{a}(n, m)$ denote the number of elements of $s^{-1}(\operatorname{Av}_n(132, 321))$ with exactly m descents, and let $\mathfrak{b}(n, m)$ denote the number of elements of $s^{-1}(\operatorname{Av}_n(132, 321))$ with exactly m peaks.

Theorem 26. *In the notation of the preceding paragraph, we have*

$$\sum_{n\geq 1} |s^{-1}(\operatorname{Av}_n(132, 321))|\, x^n = C(x) - 1 + x^3(C'(x))^2.$$

Furthermore,

$$\sum_{n\geq 1}\sum_{m\geq 0} \mathfrak{a}(n, m)x^n y^m = F(x, y) + x^3 y\left(\frac{\partial}{\partial x}F(x, y)\right)^2, \tag{13}$$

and

$$\sum_{n\geq 1}\sum_{m\geq 0} \mathfrak{b}(n, m)x^n y^m = G(x, y) + x^3 y\left(\frac{\partial}{\partial x}G(x, y)\right)^2. \tag{14}$$

Proof. We will prove (13); the proof of (14) is similar. In addition, the first statement of the theorem follows from (13) and the fact that $F(x, 1) = C(x) - 1$.

For $h, i \geq 1$ and $t \geq 0$, let

$$\delta_{h,i,t} = (h+1)(h+2)\cdots(h+i)12\cdots h(h+i+1)(h+i+2)\cdots(h+i+t) \in S_{h+i+t}.$$

For example, $\delta_{1,3,2} = 234156$. It is straightforward to check that

$$\operatorname{Av}_n(132, 321) = \{123\cdots n\} \cup \{\delta_{h,i,t} : h, i \geq 1,\ t \geq 0,\ h+i+t = n\}.$$

Moreover, the set of valid compositions of $\delta_{h,i,t}$ is

$$\mathcal{V}(\delta_{h,i,t}) = \{(i+t-\ell, h+\ell-1) : 1 \leq \ell \leq t\}.$$

[8]Kremer did not mention the stack-sorting map in her theorem.

In what follows, we only consider positive values of t since $\mathcal{V}(\delta_{h,i,0}) = \emptyset$. Invoking Theorem 10, we find that

$$\begin{aligned}
\mathfrak{a}(n,m) &= N(n,m+1) + \sum_{\substack{h,i,t\geq 1\\ h+i+t=n}} \sum_{\ell=1}^{t}\sum_{j=1}^{m} N(i+t-\ell,j)N(h+\ell-1,m+1-j) \\
&= N(n,m+1) + \sum_{h=1}^{n-2}\sum_{i=1}^{n-h-1}\sum_{\ell=1}^{n-h-i}\sum_{j=1}^{m} N(n-h-\ell,j)N(h+\ell-1,m+1-j) \\
&= N(n,m+1) + \sum_{h=1}^{n-2}\sum_{\ell=1}^{n-h-1}\sum_{i=1}^{n-h-\ell}\sum_{j=1}^{m} N(n-h-\ell,j)N(h+\ell-1,m+1-j) \\
&= N(n,m+1) + \sum_{h=1}^{n-2}\sum_{\ell=1}^{n-h-1}\sum_{j=1}^{m} (n-h-\ell)N(n-h-\ell,j)N(h+\ell-1,m+1-j).
\end{aligned}$$

The substitution $r = n-h-\ell$ gives

$$\begin{aligned}
\mathfrak{a}(n,m) &= N(n,m+1) + \sum_{h=1}^{n-2}\sum_{r=1}^{n-h-1}\sum_{j=1}^{m} r\,N(r,j)N(n-r-1,m+1-j) \\
&= N(n,m+1) + \sum_{r=1}^{n-2}\sum_{j=1}^{m} r(n-r-1)\,N(r,j)N(n-r-1,m+1-j).
\end{aligned}$$

It is now routine to verify that this last expression is the coefficient of $x^n y^m$ in

$$F(x,y) + x^3 y\left(\frac{\partial}{\partial x}F(x,y)\right)^2. \qquad \square$$

10. Enumerative Equivalence of $s^{-1}(\mathrm{Av}(132,312))$ and $s^{-1}(\mathrm{Av}(231,312))$

We saw in Corollary 14 that

$$s^{-1}(\mathrm{Av}(132,312)) = \mathrm{Av}(1342, 3142, 3412, 34\underline{21})$$

and

$$s^{-1}(\mathrm{Av}(231,312)) = \mathrm{Av}(2341, 3412, 34\underline{21}, 3\overline{5}241).$$

From these descriptions of these sets, there is no obvious reason to expect that

$$|s^{-1}(\mathrm{Av}_n(132,312))| = |s^{-1}(\mathrm{Av}_n(231,312))|.$$

However, this is indeed the case; valid hook configurations make the proof quite painless.

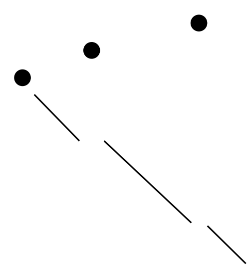
Theorem 27. *For all positive integers n, we have*

$$|s^{-1}(\mathrm{Av}_n(132,312))| = |s^{-1}(\mathrm{Av}_n(231,312))|.$$

In fact, the number of permutations in $s^{-1}(\mathrm{Av}_n(132,312))$ with exactly m descents is the same as the number of permutations in $s^{-1}(\mathrm{Av}_n(231,312))$ with exactly m descents. Moreover, the number of permutations in $s^{-1}(\mathrm{Av}_n(132,312))$ with exactly m peaks is the same as the number of permutations in $s^{-1}(\mathrm{Av}_n(231,312))$ with exactly m peaks.

Proof. Invoking Theorem 7 and Theorem 10, we see that it suffices to exhibit a bijection $\varphi : \mathrm{Av}_n(132, 312) \to \mathrm{Av}_n(231, 312)$ with the property that $\mathcal{V}(\pi) = \mathcal{V}(\varphi(\pi))$ for all $\pi \in \mathrm{Av}_n(132, 312)$.

Suppose we are given a permutation $\pi \in \mathrm{Av}_n(132, 312)$. Every entry of π is either a left-to-right maximum (i.e., larger than everything preceding it) or a left-to-right minimum (i.e., smaller than everything preceding it). Thus, the plot of π has the following basic shape:



The line segments in this diagram represent decreasing consecutive subsequences, which could be empty. This means that π is uniquely determined by the lengths of its descending runs. Each permutation in $\mathrm{Av}_n(231, 312)$ is also uniquely determined by the lengths of its descending runs. This is because the entries in each descending run of the permutation must be consecutive integers. Indeed, the permutations in $\mathrm{Av}(231, 312)$ are the layered permutations; as described at the beginning of Section 6, a permutation is in $\mathrm{Av}(231, 312)$ if and only if it can be written as a sum of decreasing permutations.

For example, the unique permutation in $\mathrm{Av}_{10}(132, 312)$ whose descending runs have lengths $2, 3, 1, 2, 1, 1$ is 5 4 6 3 2 7 8 1 9 10 (whose plot is shown on the left in Figure 9). The unique permutation in $\mathrm{Av}_{10}(231, 312)$ whose descending runs have lengths $2, 3, 1, 2, 1, 1$ is 2 1 5 4 3 6 8 7 9 10 (whose plot is shown on the right in Figure 9). We now define $\varphi : \mathrm{Av}_n(132, 312) \to \mathrm{Av}_n(231, 312)$ by declaring $\varphi(\pi)$ to be the unique permutation in $\mathrm{Av}_n(231, 312)$ whose i^{th} descending run has the same length as the i^{th} descending run of π for all i.

Figure 9 illustrates the map φ. In this figure, we have drawn a valid hook configuration on each of the plots. Notice that we can obtain the plot of $\varphi(\pi)$ by vertically sliding some of the points in the plot of π; we keep the hooks attached to their endpoints throughout this sliding motion. Let us check that this preserves the validity of the configuration of hooks. First, the northeast endpoints of hooks in a valid hook configuration of a 312-avoiding permutation must be left-to-right maxima. Since π and $\varphi(\pi)$ both avoid 312 and have their left-to-right maxima in the same (horizontal) positions, every hook of π remains a hook in $\varphi(\pi)$ (and vice versa). This also shows that a point never moves "through" a hook during this sliding process. Therefore, Conditions 2 and 3 in Definition 1 are preserved by the sliding process. It is immediate from the definition of φ that π and $\varphi(\pi)$ have the same set of descents. This shows that Condition 1 in Definition 1 is also preserved. Consequently, the valid hook configurations of $\pi \in \mathrm{Av}_n(132, 312)$ correspond bijectively to the valid hook configurations of $\varphi(\pi)$. Corresponding valid hook configurations induce the same valid compositions, so we have $\mathcal{V}(\pi) = \mathcal{V}(\varphi(\pi))$ for all $\pi \in \mathrm{Av}_n(132, 312)$. $\square$

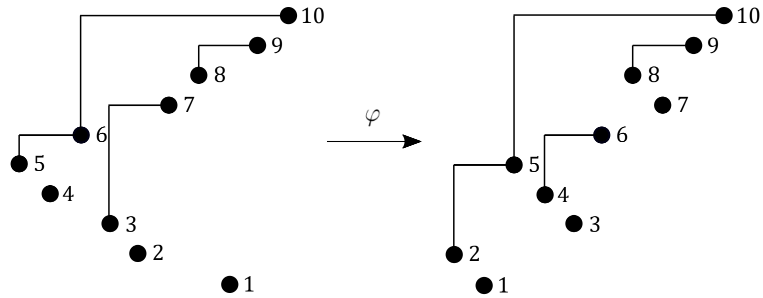


FIGURE 9. The map φ from the proof of Theorem 27 sends the permutation 5 4 6 3 2 7 8 1 9 10 to the permutation 2 1 5 4 3 6 8 7 9 10. The valid hook configuration drawn on the plot of 5 4 6 3 2 7 8 1 9 10 corresponds to the one drawn on the plot of 2 1 5 4 3 6 8 7 9 10. Both valid hook configurations induce the valid composition $(1, 1, 2, 1, 1)$.

We have seen that the sequences $(|s^{-1}(\mathrm{Av}_n(132, 312))|)_{n\geq 1}$ and $(|s^{-1}(\mathrm{Av}_n(231, 312))|)_{n\geq 1}$ are identical. Numerical evidence suggests that this sequence is, up to reindexing, the same as the sequence A071356 in [37]. The latter sequence is defined as the expansion of a relatively simple generating function, but it also has some combinatorial interpretations. In addition, it appears as though $(|s^{-1}(\mathrm{Av}_n(132, 231))|)_{n\geq 1}$ is the same sequence. We state these observations formally in the following conjecture.

Conjecture 28. *We have*

$$\sum_{n\geq 1}|s^{-1}(\mathrm{Av}_n(132,312))|x^n=\sum_{n\geq 1}|s^{-1}(\mathrm{Av}_n(132,231))|x^n=\frac{1-2x-\sqrt{1-4x-4x^2}}{4x}.$$

11. Estimates for $s^{-1}(\mathrm{Av}(321))$

As discussed in the introduction, there are known formulas for $|s^{-1}(\mathrm{Av}_n(\tau))|$ whenever τ is a permutation pattern of size 3 other than 321. By contrast, the sequence

$$(|s^{-1}(\mathrm{Av}_n(321))|)_{n\geq 1}$$

appears to be new (it is now sequence A319027 in [37]). This sequence is of interest because $s^{-1}(\mathrm{Av}(321))$ is equal to the permutation class $\mathrm{Av}(34251, 35241, 45231)$ (see Theorem 13). We will use valid hook configurations to establish nontrivial estimates for the growth rate of this sequence. Note that the trivial estimates for this growth rate are given by

$$4\leq \lim_{n\to\infty}|s^{-1}(\mathrm{Av}_n(321))|^{1/n}\leq 16. \tag{15}$$

The acute reader might beg for a proof of the existence of the limit defining this growth rate. The multiplicative version of Fekete's lemma [29] states that if $(a_m)_{m=1}^{\infty}$ is a supermultiplicative[9] sequence, then $\lim_{m\to\infty}\sqrt[m]{a_m}$ exists. It is straightforward to show (in the notation of Section 6) that $s(\sigma\oplus\mu)=s(\sigma)\oplus s(\mu)$ for any $\sigma\in S_m$ and $\mu\in S_n$. It follows that there is an injective map $s^{-1}(\mathrm{Av}_m(321))\times s^{-1}(\mathrm{Av}_n(321))\to s^{-1}(\mathrm{Av}_{m+n}(321))$ given by $(\sigma,\mu)\mapsto\sigma\oplus\mu$.

[9]We say a sequence of real numbers $(a_m)_{m=1}^{\infty}$ is *supermultiplicative* if $a_m a_n\leq a_{m+n}$ for all positive integers m, n.

Hence, $(|s^{-1}(\mathrm{Av}_n(321))|)_{n\geq 1}$ is supermultiplicative. The lower bound in (15) follows from the fact that $|s^{-1}(123\cdots n)| = C_n$ has growth rate 4. We know the upper bound because $|\mathrm{Av}_n(321)| = C_n$ has growth rate 4 and because each permutation of size n has fertility at most C_n (this is proven in the solution to Exercise 23 in Chapter 8 of [4]).

The proof of the lower bound in the following theorem requires the following purely technical lemma.

Lemma 29. *Let* $C_x = \dfrac{\Gamma(2x+1)}{\Gamma(x+2)\Gamma(x+1)}$, *where* Γ *denotes the gamma function. We have* $C_{x+\varepsilon}C_{y-\varepsilon} < C_xC_y$ *whenever* $0 < x < y$ *and* $0 < \varepsilon \leq \dfrac{y-x}{2}$.

Proof. We need to show that $C_{x+\varepsilon}/C_x$ is increasing in x. Let $\mathbf{C}(x) = C_x$. We wish to show that $\dfrac{d}{dx}\dfrac{\mathbf{C}(x+\varepsilon)}{\mathbf{C}(x)} > 0$, which amounts to showing that $\dfrac{\mathbf{C}'(x)}{\mathbf{C}(x)} < \dfrac{\mathbf{C}'(x+\varepsilon)}{\mathbf{C}(x+\varepsilon)}$. We are left to show that $\mathbf{C}'(x)/\mathbf{C}(x)$ is increasing. We can write

$$\frac{\mathbf{C}'(x)}{\mathbf{C}(x)} = 2\psi_0(2x+1) - 2\psi_0(x+1) - \frac{1}{x+1},$$

where ψ_ℓ is the ℓ^{th} polygamma function (meaning ψ_0 is the digamma function). Differentiating, we see that we need to show that

$$4\psi_1(2x+1) - 2\psi_1(x+1) + \frac{1}{(x+1)^2} > 0.$$

Using well known formulas for the polygamma functions, we can write

$$\frac{1}{4}(\psi_1(x+1) + \psi_1(x+3/2)) = \psi_1(2x+2) = \psi_1(2x+1) - \frac{1}{(2x+1)^2}.$$

Rearranging, we find that we are left to show that

$$\psi_1(x+1) - \psi_1(x+3/2) < \frac{1}{(x+1)^2} + \frac{1}{(x+1/2)^2}.$$

The polygamma function ψ_1 can be rewritten using a Hurwitz zeta function, so we can rewrite this last inequality as

$$\sum_{k=0}^{\infty}\left(\frac{1}{(k+x+1)^2} - \frac{1}{(k+x+3/2)^2}\right) < \frac{1}{(x+1)^2} + \frac{1}{(x+1/2)^2}.$$

It suffices to show that

$$\sum_{k=1}^{\infty}\left(\frac{1}{(k+x+1)^2} - \frac{1}{(k+x+3/2)^2}\right) < \frac{1}{(x+1/2)^2},$$

which is straightforward to verify. $\square$

Theorem 30. *We have*

$$8.4199 \leq \lim_{n\to\infty} |s^{-1}(\mathrm{Av}_n(321))|^{1/n} \leq 11.6569.$$

Proof. Throughout the proof, we assume $n \to \infty$. Since we are only interested in exponential growth rates, we often ignore subexponential factors. We also omit floor and ceiling signs when they are not crucial. For example, $\binom{n/2}{k}$ can either be interpreted as $\binom{\lfloor n/2 \rfloor}{k}$ or could be defined exactly using the Gamma function (the specific choice will not matter for the asymptotics).

By reversing permutations, we see that $|\operatorname{Av}_{n,k}(321)| = |\operatorname{Av}_{n,n-1-k}(123)|$. The authors of [2] have computed $|\operatorname{Av}_{n,k}(123)|$; one can easily use their results to see that the values of $|\operatorname{Av}_{n,k}(321)|$ are given by sequence[10] A091156 in [37]. A precise formula is given by

$$|\operatorname{Av}_{n,k}(321)| = \frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n+1-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}. \tag{16}$$

To prove the desired upper bound, we combine (16) with a result of [22], which we recalled as Theorem 22 in the current article, to find that

$$|s^{-1}(\operatorname{Av}_n(321))| = \sum_{k=0}^{n-1}|s^{-1}(\operatorname{Av}_{n,k}(321))| \leq \sum_{k=0}^{n-1}\frac{2k+2}{n+1}\binom{2n-2k-1}{n}|\operatorname{Av}_{n,k}(321)|$$

$$= \sum_{k=0}^{n-1}\frac{2k+2}{n+1}\binom{2n-2k-1}{n}\frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n+1-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}.$$

Up to a subexponential factor, this upper bound is

$$\sum_{k=0}^{n-1}\binom{2n-2k}{n}\binom{n}{k}\sum_{j=0}^{n-2k}\binom{k+j}{k}\binom{n-k}{n-2k-j}. \tag{17}$$

The ratio of consecutive terms in the sum over j in (17) is

$$\frac{\binom{k+j+1}{k}\binom{n-k}{n-2k-j-1}}{\binom{k+j}{k}\binom{n-k}{n-2k-j}} = \frac{n-2k-j}{j+1},$$

so this sum is maximized when $n-2k-j = j+1+O(1)$. That is, $j = n/2-k+O(1)$. Furthermore, the value of the maximum term in this sum is the same as the value of the full sum up to a subexponential factor (since the total number of terms is at most $n+1$). Therefore, our upper bound is (up to a subexponential factor)

$$\sum_{k=0}^{n-1}\binom{2n-2k}{n}\binom{n}{k}\binom{n/2}{k}\binom{n-k}{n/2-k}.$$

Let $K(n)$ denote the value of k for which the term in this last summation is maximized, and put $c(n) = K(n)/n$. Note that $c(n) \in [0,1/2]$. A straightforward application of Stirling's formula shows that, up to a subexponential factor, the n^{th} root of this last upper bound is

[10] We are not the first to observe this interpretation of the sequence A091156; it is mentioned in a comment by Andrew Baxter in [37].

at most

$$\frac{(2-2c(n))^{2-2c(n)}}{(1-2c(n))^{1-2c(n)}}\frac{1}{c(n)^{c(n)}(1-c(n))^{1-c(n)}}$$
$$\times\frac{(1/2)^{1/2}}{c(n)^{c(n)}(1/2-c(n))^{1/2-c(n)}}\frac{(1-c(n))^{1-c(n)}}{(1/2-c(n))^{1/2-c(n)}(1/2)^{1/2}}=f(c(n)),$$

where

$$f(x)=\frac{(2-2x)^{2-2x}}{(1-2x)^{1-2x}x^{2x}(1/2-x)^{1-2x}}$$

(we make the convention that $0^0=1$). One can easily verify that $f(x)\leq 11.6569$ whenever $x\in[0,1/2]$. This proves the desired upper bound.

The *reverse complement* of a permutation $\pi_1\cdots\pi_n\in S_n$ is the permutation whose i^{th} entry is $n+1-\pi_{n+1-i}$. The proof of the desired lower bound requires two crucial observations. The first is that the set $\text{Av}_{n,k}(321)$ is closed under taking reverse complements. If $\pi\in S_n$ has ℓ left-to-right maxima, then the reverse complement of π has $n-\ell$ left-to-right maxima. It follows that at least half of the permutations in $\text{Av}_{n,k}(321)$ have at least $n/2$ left-to-right maxima.

The second observation is that if $\pi\in\text{Av}_{n,k}(321)$ has last entry $\pi_n=n$, then π has a valid hook configuration (i.e., π is sorted). Indeed, let $d_1<\cdots<d_k$ denote the descents of π. For each $i\in\{1,\ldots,k\}$, let π_{b_i} be the leftmost left-to-right maximum of π that lies to the right of π_{d_i}. The condition $\pi_n=n$ guarantees that π_{b_k} exists. The assumption that π avoids 321 forces the entries $\pi_{d_1},\ldots,\pi_{d_k}$ to be left-to-right maxima of π, and this implies that $d_i<b_i\leq d_{i+1}$ for all $i\in\{1,\ldots,k-1\}$. In particular, the entries π_{b_i} are distinct. The canonical hook configuration $(H_1^*,\ldots,H_k^*)$ of π (described in Section 2) is formed by declaring that H_i^* has southwest endpoint (d_i,π_{d_i}) and northeast endpoint (b_i,π_{b_i}) for all i.

There are exactly $|\text{Av}_{n-1,k}(321)|$ permutations in $\text{Av}_{n,k}(321)$ with last entry n (we obtain a bijection by adding the entry n to the end of a permutation in $\text{Av}_{n-1,k}(321)$). According to the discussion above, there are at least $\frac{1}{2}|\text{Av}_{n-1,k}(321)|$ permutations in $\text{Av}_{n,k}(321)$ with at least $n/2$ left-to-right maxima. Choose one such permutation π, and let ℓ be the number of left-to-right maxima in π. The canonical hook configuration of π (described in the previous paragraph) induces a valid composition $(q_0^*,\ldots,q_k^*)\in\mathcal{V}(\pi)$. In the coloring of the plot of π induced by the canonical hook configuration, there are exactly $\ell-k$ points colored blue (sky-colored). Indeed, these points are precisely the left-to-right maxima of the plot of π that are not northeast endpoints of hooks (see Figure 11). This tells us that $q_0^*=\ell-k$. According to Theorem 7,

$$|s^{-1}(\pi)|\geq C_{(q_0^*,\ldots,q_k^*)}=C_{\ell-k}C_{q_1^*}\cdots C_{q_k^*}. \tag{18}$$

As in Lemma 29, we define $C_x=\dfrac{\Gamma(2x+1)}{\Gamma(x+2)\Gamma(x+1)}$. The lemma tells us that a product of (generalized) Catalan numbers decreases when we make the indices closer while preserving the sum of the indices. Let us assume that n is sufficiently large, that $5<k<0.4n$, and that π is chosen as in the previous paragraph. By the properties of valid compositions, we

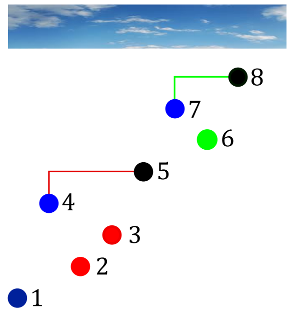


FIGURE 10. The coloring induced by the canonical hook configuration of a 321-avoiding permutation.

know that

$$\ell - k + q_1^* + \cdots + q_k^* = n - k. \tag{19}$$

It follows from Lemma 29 and (19) that

$$C_{\ell-k} C_{q_1^*} \cdots C_{q_k^*} \geq C_{\ell-k} C_{(n-\ell)/k}^k. \tag{20}$$

The assumption $5 < k < 0.4n$ and the fact that $\ell \geq n/2$ guarantee that

$$\frac{n-\ell}{k} \leq \frac{n}{2k} < \frac{n}{2} - k \leq \ell - k.$$

Because $(\ell - k) + k \cdot ((n-\ell)/k) = (n/2 - k) + k \cdot (n/(2k))$, it follows from Lemma 29 that

$$C_{\ell-k} C_{(n-\ell)/k}^k \geq C_{n/2-k} C_{n/(2k)}^k. \tag{21}$$

When we combine (18), (20), and (21) with the discussion above, we find that there are at least $\frac{1}{2}|\operatorname{Av}_{n-1,k}(321)|$ permutations in $\operatorname{Av}_{n,k}(321)$ that each have at least $C_{n/2-k} C_{n/(2k)}^k$ preimages under s (for $5 < k < 0.4n$ and n large enough). We now use (16) to see that

$$\begin{aligned}
|s^{-1}(\operatorname{Av}_n(321))| &\geq \frac{1}{2}|\operatorname{Av}_{n-1,k}(321)| C_{n/2-k} C_{n/(2k)}^k \\
&\geq \frac{1}{2n}\binom{n}{k} \sum_{j=0}^{n-2k} \binom{k+j-1}{k-1}\binom{n-k}{n-2k-j-1} C_{n/2-k} C_{n/(2k)}^k \\
&\geq \frac{1}{2n}\binom{n}{k} \binom{k+(\lfloor n/2 \rfloor - k)-1}{k-1}\binom{n-k}{n-2k-(\lfloor n/2 \rfloor - k)-1} C_{n/2-k} C_{n/(2k)}^k \\
&= \frac{1}{2n}\binom{n}{k} \binom{\lfloor n/2 \rfloor - 1}{k-1}\binom{n-k}{\lceil n/2 \rceil - k - 1} C_{n/2-k} C_{n/(2k)}^k.
\end{aligned}$$

This holds whenever $5 < k < 0.4n$. In particular, we can put $k = \lfloor 0.06582n \rfloor$ (this value is chosen to maximize the lower bound). With this choice of k, we can use Stirling's formula to see that our lower bound is at least 8.4199^n for sufficiently large n. $\square$

12. Concluding Remarks and Further Directions

Let us collect some open problems and conjectures arising from and related to the topics studied in this article.

Recall that a sequence $a_1, \ldots, a_m$ is called *unimodal* if there exists $j \in \{1, \ldots, m\}$ such that $a_1 \leq \cdots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \cdots \geq a_m$ and is called *log-concave* if $a_j^2 \geq a_{j-1}a_{j+1}$ for all $j \in \{2, \ldots, m-1\}$. This sequence is called *real-rooted* if all of the complex roots of the polynomial $\sum_{k=1}^m a_k x^k$ are real. It is well known that a real-rooted sequence of nonnegative numbers is log-concave and that a log-concave sequence of nonnegative numbers is unimodal (see [14]).

The notions of unimodality, log-concavity, and real-rootedness are prominent in the study of the stack-sorting map. For example, let $f_k(\pi)$ denote the number of elements of $s^{-1}(\pi)$ with k descents. Bóna proved [7] that the sequence $f_0(\pi), \ldots, f_{n-1}(\pi)$ is symmetric and unimodal for each $\pi \in S_n$. We conjecture the following much stronger result, which we have verified for all permutations of size at most 8.

Conjecture 31. *For each permutation $\pi \in S_n$, the sequence $f_0(\pi), \ldots, f_{n-1}(\pi)$ is real-rooted.*

Of course, even if it is too difficult to prove that $f_0(\pi), \ldots, f_{n-1}(\pi)$ is always real-rooted, it would be very interesting to prove the weaker statement that this sequence is always log-concave. One could also attempt to find large classes of permutations for which Conjecture 31 holds.

A consequence of Bóna's result is that

$$W_t(n, 0), \ldots, W_t(n, n-1) \tag{22}$$

is symmetric and unimodal for all $t, n \geq 1$, where $W_t(n, k)$ is the number of t-stack-sortable permutations in S_n with k descents. Knowing that the sequence in (22) is real-rooted when $t = 1$ and when $t = n$, Bóna [7] conjectured that the sequence is real-rooted in general. Brändén [13] later proved this conjecture in the cases $t = 2$ and $t = n - 2$. This leads us to the following much more general problem.

Question 32. *Given a set $\mathcal{U}$ of permutations, let $f_k(\mathcal{U} \cap S_n)$ denote the number of permutations in $s^{-1}(\mathcal{U} \cap S_n)$ with exactly k descents. Can we find interesting examples of sets $\mathcal{U}$ (such as permutation classes) with the property that $f_0(\mathcal{U} \cap S_n), \ldots, f_{n-1}(\mathcal{U} \cap S_n)$ is a real-rooted sequence for every $n \geq 1$? Is this sequence always real-rooted?*

Recall from Section 3 that the set $s^{-1}(\text{Av}_n(123))$ is empty when $n \geq 4$. In general, $s^{-1}(\text{Av}_n(123 \cdots m))$ is empty if $n \geq 2^{m-1}$. This is certainly true for $m \leq 3$. To see that this is true for $m \geq 4$, suppose $n \geq 2^{m-1}$ and $\pi \in S_n$. Write $\pi = LnR$ so that $s(\pi) = s(L)s(R)n$. One of L and R has size at least 2^{m-2}, so it follows by induction on m that either $s(L)$ or $s(R)$ contains an increasing subsequence of size $m - 1$. Therefore, $s(\pi)$ contains the pattern $123 \cdots m$.

Despite the uninteresting behavior of the sequence $(|s^{-1}(\mathrm{Av}_n(123\cdots m))|)_{n\geq 1}$ for large values of n, it could still be interesting to study the initial terms in this sequence. For example, the nonzero terms of the sequence are $1, 2, 6, 10, 13, 10, 3$ when $m = 4$.

Conjecture 33. *For each integer $m \geq 2$, the sequence $(|s^{-1}(\mathrm{Av}_n(123\cdots m))|)_{n=1}^{2^{m-1}-1}$ is unimodal.*

We next consider some natural questions that we have not attempted to answer.

Question 34. *Can we obtain interesting results by enumerating sets of the form $s^{-1}(\mathrm{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ when the patterns $\tau^{(1)}, \ldots, \tau^{(r)}$ are not all of size 3?*

Question 35. *Let $S_{n,k}$ denote the set of permutations in S_n with exactly k descents. Can we find formulas for $|s^{-1}(S_{n,1})|$, $|s^{-1}(S_{n,2})|$, or $|s^{-1}(S_{n,3})|$?*

The article [21] provides a general method for computing the number of decreasing plane trees of various types that have a specified permutation as their postorder reading (see the article for the relevant definitions). In the special case in which the trees are decreasing binary plane trees, this is equivalent to computing fertilities of permutations. This is also very similar to the approach taken in [20, 35, 41], where one considers trees endowed with a canonical ordering of the vertices and asks how many linear extensions of the trees (viewed as posets) avoid certain patterns. In our case, the trees are binary plane trees and the canonical ordering of the vertices is given by the postorder reading. This suggests that one could obtain enumerative results analogous to those from this paper by replacing decreasing binary plane trees with other types of trees. Thus, we have a very general new collection of enumerative problems. Namely, we want to count the decreasing plane trees of a certain type whose postorders lie in some permutation class. Two very specific examples of this type of problem are the following. Preserve the notation from the article [21].

Question 36. *For a fixed sequence of permutation patterns $\tau^{(1)}, \ldots, \tau^{(r)}$, how many decreasing $\mathbb{N}$-trees have postorders that lie in the set $\mathrm{Av}_n(\tau^{(1)}, \ldots, \tau^{(r)})$? How many unary-binary trees have postorders that lie in the set $\mathrm{Av}_n(\tau^{(1)}, \ldots, \tau^{(r)})$?*

We now collect the open problems and conjectures that arose throughout Sections 4–10. First, recall the following conjecture from Section 6.

Conjecture 21. *In the notation of Theorem 19, we have*

$$\sum_{q\,\in\,\mathrm{Comp}_{k+1}(n-k)} C_q D_q = \frac{1}{n+1}\binom{n-k-1}{k}\binom{2n-2k}{n}$$

for all nonnegative integers n and k.

We stated the following intriguing conjecture in Section 10.

Conjecture 28. *We have*

$$\sum_{n\geq 1} |s^{-1}(\mathrm{Av}_n(132, 312))|x^n = \sum_{n\geq 1} |s^{-1}(\mathrm{Av}_n(132, 231))|x^n = \frac{1-2x-\sqrt{1-4x-4x^2}}{4x}.$$

Of course, our results in Section 11 are far from perfect.

Question 37. *Can we enumerate the permutations in* $s^{-1}(\text{Av}(321))$ *exactly? Can we at least improve the estimates from Theorem 30?*

We have focused primarily on preimage sets of the form $s^{-1}(\text{Av}(\tau^{(1)}, \ldots, \tau^{(r)}))$ when $\emptyset \neq \{\tau^{(1)}, \ldots, \tau^{(r)}\} \subseteq \{132, 231, 312, 321\}$. The astute reader may have realized that the only preimage set of this form that we have not mentioned is $s^{-1}(\text{Av}(231, 321))$. This set appears to be enumerated by the OEIS sequence A165543 [37]. More precisely, we have the following conjecture.

Conjecture 38. *We have*

$$\sum_{n \geq 0} |s^{-1}(\text{Av}_n(231, 321))| x^n = \frac{1}{1 - xC(xC(x))},$$

where $C(x) = \dfrac{1 - \sqrt{1 - 4x}}{2x}$ *is the generating function of the sequence of Catalan numbers.*

It is not clear to the author how to explicitly describe the set of valid hook configurations of an arbitrary permutation in $\text{Av}(231, 321)$, so proving Conjecture 38 might require a new technique.

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