Schur positivity of Macdonald Cumulants

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Schur positivity

☆ Representation theory

☆ Algebraic geometry

☆ COMBINATORICS

Examples:

☆ Hall-Littlewood functions $\longmapsto$ $H_\mu(x;t)$

$$[s_\lambda] H_\mu(x;t) \in \mathbb{N}[t].$$

☆ Macdonald polynomials $\longmapsto$ $H_\mu(x;q,t)$

$$[s_\lambda] H_\mu(x;q,t) \in \mathbb{N}[q,t]$$

Examples:

★ Hall-Littlewood functions $\longmapsto H_\mu(x;t)$

$$[s_\lambda] H_\mu(x;t) \in \mathbb{N}[t].$$

Reason:	Combinatorial understanding:
1 Algebraic geometry Lusztig '81	✓ Lascoux-Schützenberger '79

★ Macdonald polynomials $\longmapsto H_\mu(x;q,t)$

$$[s_\lambda] H_\mu(x;q,t) \in \mathbb{N}[q,t]$$

Reason:	Combinatorial understanding:
1 Algebraic geometry + 2 Representation theory Haiman '01	✗ Only some special cases.

HOWEVER...

Monomial positivity

Theorem (Haiman, Haglund, Loehr '05)

$$H_\mu(x;q,t) = \sum_{\sigma:\mu\to\mathbb{N}_+} t^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)} x^\sigma, \quad \text{where}$$

☆ $x^\sigma := \prod_{\square\in\mu} x_{\sigma(\square)}$

☆ maj, inv are some nice, explicit statistics on the set of fillings of Young diagrams.

Corollary: $[m_\lambda] H_\mu(x;q,t) \in \mathbb{N}[q,t]$ admits explicit, combinatorial interpretation

m_λ: monomial symmetric function

Remark: Monomial positivity is **weaker** than Schur positivity.

Cumulants

$X_1, \dots, X_r$ – random variables, $\mathbb{E}$ – expectation

$$\boxed{k(X_1, \dots, X_r)} := [t_1 \cdots t_r] \log \mathbb{E} \exp(t_1 X_1 + \dots + t_r X_r)$$

cumulant

Examples:

- $k(X_1) = \mathbb{E}(X_1)$
- $k(X_1, X_2) = \mathbb{E}(X_1 \cdot X_2) - \mathbb{E}(X_1) \cdot \mathbb{E}(X_2)$
  $= \mathrm{Var}(X_1, X_2)$
- $k(X_1, X_2, X_3) = \mathbb{E}(X_1 \cdot X_2 \cdot X_3) - \mathbb{E}(X_1) \cdot \mathbb{E}(X_2 \cdot X_3)$
  $- \mathbb{E}(X_2) \cdot \mathbb{E}(X_1 \cdot X_3) - \mathbb{E}(X_3) \cdot \mathbb{E}(X_1 \cdot X_2) + 2\mathbb{E}(X_1)\mathbb{E}(X_2)\mathbb{E}(X_3)$

Cumulants „measure" dependencies between random variables.

ALGEBRAIC SETTING:

$\mathcal{A}$ - group $\quad (\mathcal{A}, \cdot), (\mathcal{A}, *)$ - two multiplicative structures on $\mathcal{A}$.

GOAL: We want to understand the discrepancy between these two multiplicative structures.

$$X_1, \dots, X_r \in \mathcal{A}, \quad \log_{\bullet}(X+1) := \sum_{n \geq 1} \frac{(-1)^{n-1} X^{\bullet n}}{n}, \quad \exp_*(X) := \sum_{n \geq 0} \frac{X^{*n}}{n!}$$

$$\boxed{k(X_1, \dots, X_r) := [t_1 \cdots t_r] \log_{\bullet}\left(\exp_*\left(t_1 X_1 + \cdots + t_r X_r\right)\right).} \in \mathcal{A}$$

Examples:

- $k(X_1) = X_1$
- $k(X_1, X_2) = X_1 * X_2 - X_1 \bullet X_2$
- $k(X_1, X_2, X_3) = X_1 * X_2 * X_3 - X_1 \bullet (X_2 * X_3) - X_2 \bullet (X_1 * X_3) - X_3 \bullet (X_1 * X_2) + 2 X_1 \bullet X_2 \bullet X_3$

CUMULANTS - COMBINATORIAL FORMULA

$$\boxed{k(X_1,\dots,X_r) := \sum_{\pi\in P([r])} (-1)^{\#\pi-1}(\#\pi-1)! \bullet_{B\in\pi}\left(\ast_{b\in B} X_b\right),}$$

where

☆ $P([r])$ - set of set-partitions of $[r] := \{1,2,\dots,r\}$

☆ $\#\pi$ - numer of elements (blocks) in π

$\downarrow$

$$(-1)^{\#\pi-1}(\#\pi-1)! = \mu(\pi,[r]) \text{ - Möbius function on } P([r]).$$

MACDONALD CUMULANTS

Observations (Macdonald):

☆ $\{H_\mu(x;q,t)\}_\mu$ – basis of symmetric functions $/\mathbb{Q}(q,t) =: \Lambda$

☆ $H_\mu(x;1,t)\cdot H_\nu(x;1,t) = H_{\mu\oplus\nu}(x;1,t)$, where

$$\boxed{\underset{(\mu_1,\dots,\mu_k)}{\overset{\mu}{\shortparallel}} \oplus \underset{(\nu_1,\dots,\nu_k)}{\overset{\nu}{\shortparallel}} := (\mu_1+\nu_1, \mu_2+\nu_2, \dots, \mu_k+\nu_k).}$$

We define $H_\mu(x;q,t) \oplus H_\nu(x;q,t) := H_{\mu\oplus\nu}(x;q,t)$.

Remark: For any $f,g\in\Lambda$ $f\cdot g = f\oplus g$ as $q\to 1$

Question: What is the difference between $(\Lambda,\cdot)$ and $(\Lambda,\oplus)$?

$\star$ $$\frac{k(H_\mu, H_\nu)}{q-1} = \frac{H_\mu \oplus H_\nu - H_\mu \bullet H_\nu}{q-1} \in \mathbb{Z}[q,t]\{m_\lambda\}_\lambda$$

[HHL '05] + Observation

$\star$ $$\frac{k(H_\mu, H_\nu, H_\rho)}{q-1} \in \mathbb{Z}[q,t]\{m_\lambda\}_\lambda$$ BUT

$$\frac{k(H_\mu, H_\nu, H_\rho)}{(q-1)^2} \in \mathbb{Z}[q,t]\{m_\lambda\}_\lambda$$!! (NON TRIVIAL)

$\lambda^1, \ldots, \lambda^r$ - partitions.

$$\boxed{k(\lambda^1, \ldots, \lambda^r) := \frac{k(H_{\lambda^1}, \ldots, H_{\lambda^r})}{(q-1)^{r-1}}}$$

MACDONALD CUMULANT

PROBLEM 1:

$$k(\lambda^1, \dots, \lambda^r) \in \mathbb{Z}[q,t]\{m_\lambda\}_\lambda \ ???$$

PROBLEM 2:

$$k(\lambda^1, \dots, \lambda^r) \in \mathbb{N}[q,t]\{m_\lambda\}_\lambda \ ???$$

PROBLEM 3:

$$k(\lambda^1, \dots, \lambda^r) \in \mathbb{N}[q,t]\{s_\lambda\}_\lambda \ ???$$

$P3 \implies P2 \implies P1.$

Ad. PROBLEM 1

Theorem (D. '17): The answer is affirmative also when replacing Macdonald polynomials by interpolation Macdonald polynomials.

P-f: Complicated inductive analysis of some differential operators action $\Rightarrow$

We cannot solve PROBLEM 2

BUT ...

...COEFFICIENTS SEEM TO BE FAMILIAR

Ad. PROBLEM 2

Theorem (D. '19)

$$k(\lambda^1,\dots,\lambda^r) = \sum_{\sigma:\ \bigoplus_{i=1} \lambda^i \to \mathbb{N}_+} t^{\mathrm{maj}(\sigma)} J_{G_\sigma^{\lambda^1,\dots,\lambda^r}}(q)\, x^\sigma,$$

where ☆ $G_\sigma^{\lambda^1,\dots,\lambda^r}$ – certain (multi)-graph constructed from $\sigma, \lambda^1,\dots,\lambda^r$.

☆ $J_G(q)$ – generating function of G-parking functions

COROLLARIES:

☆ HHL formula as a special case $r=1$

☆ Macdonald cumulants are positive in
- monomial basis
- fundamental quasi-symmetric basis

PROBLEM 3: Still a conjecture!!!

We need your HELP!!!

WWW.IMPAN.PL/~MDOLEGA/PHD

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KRAKÓW

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