The t-term rank of a $(0,1)$-matrix

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Funded by UID/MAT/00297/2019

April 15, 2019

Fundação para a Ciência e a Tecnologia

Associação para a Inovação e Desenvolvimento da FCT

- A $(0,1)$-matrix is a matrix whose entries are the integers 0 and 1.
- $$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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An m-by-n $(0, 1)$-matrix can be regarded as distributing n elements into m sets: the 1's in row i designate the elements that occur in the ith set, and the 1's in column j designate the sets that contain the jth element. Such matrices are thus of fundamental importance in combinatorial investigations. (Ford and Fulkerson 1962)

For a concrete example, consider 5 families, F_1, F_2, F_3, F_4, F_5, to be seated at 4 tables, T_1, T_2, T_3, T_4, where F_1 and F_5 have 2 members and the others have one, and T_1 has 2 seats, T_2 has 3 seats, T_3 and T_4 have 1 seat.

We consider another condition: no two members of the same family are seated at the same table. A possible distribution is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Let $X = [x_{ij}]$ be m-by-n, $(0,1)$-matrix

$$R_i(X) = \sum_{j=1}^{n} x_{ij}, \text{ for } 1 \leq i \leq m,$$

$$S_j(X) = \sum_{i=1}^{m} x_{ij}, \text{ for } 1 \leq j \leq n,$$

  the row sum and column sum of X.
- $(R_1(X), \ldots, R_m(X))$ and $(S_1(X), \ldots, S_n(X))$ are the row sum vector and column sum vector of X.

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- Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $R_1(A) = 2$ (number of seats in T_1)
  $R_2(A) = 3,\ R_3(A) = 1,\ R_4 = 1.$
- $S_1(A) = 2$ (number of elements in F_1)
  $S_2(A) = 1,\ S_3(A) = 1,\ S_4 = 1,\ S_5 = 2.$
- $(R_1(A), R_2(A), R_3(A), R_4(A)) = (2, 3, 1, 1)$
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Gale-Ryser Theorem

- Let $R = (R_1, \ldots, R_m)$ and $S = (S_1, \ldots, S_n)$ be vectors such that $R_1 \geq R_2 \geq \ldots \geq R_m > 0$, $S_1 \geq S_2 \geq \ldots \geq S_n > 0$ and $R_1 + R_2 + \ldots + R_m = S_1 + S_2 + \ldots + S_n$. We say that S is majorized by R when

$$R_1 + R_2 + \ldots + R_i \geq S_1 + S_2 + \ldots + S_i, \text{ for } i = 1, \ldots, n.$$

- $R^* = (R_1^*, \ldots, R_{R_1}^*)$, the conjugate vector of R, defined by $R_j^* = |\{i : m \geq i \geq 1, R_i \geq j\}|$, for $j = 1, \ldots, R_1$

Theorem

There is a matrix with row sum vector R and column sum vector S if and only if S is majorized by R^*.

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There is a matrix with row sum vector R and column sum vector S if and only if S is majorized by R^*.

- Let t be a positive integer. The t-term rank of a $(0,1)$-matrix A, denoted by $\rho_t(A)$, is the maximum number of 1's in A with at most one 1 in each column and at most t 1´s in each row.

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Remarks

- $\rho_1(A) = 3 \leq \rho_2(A) = 4 \leq \rho_3(A) = 5 \leq \ldots\ldots$
- $\rho_1(A) = 3 \geq (\rho_2(A) - \rho_1(A)) = 1 \geq (\rho_3(A) - \rho_2(A)) = 1$
- $(3, 1, 1)$ vector with sum 5 (number of families) the term rank partition of A

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König-Egerváry Theorem

- $$\rho_1(A) = \min\{e+f : \text{there is a cover of A with } e \text{ rows and } f \text{ columns}\}$$

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König-Egerváry Theorem (generalization)

- $$A^{(t)} = \begin{bmatrix} A \\ \hline \vdots \\ \hline A \end{bmatrix} \Big\} \; t \text{ times}$$

- Example:

$$A^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Interchange in A

- An interchange replaces one of the 2-by-2 submatrices of A,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

by the other.

- Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & {\color{red}1} & 1 & 1 & {\color{red}0} \\ 0 & {\color{red}0} & 0 & 0 & {\color{red}1} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} {\color{blue}\leftrightarrow} A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & {\color{red}0} & 1 & 1 & {\color{red}1} \\ 0 & {\color{red}1} & 0 & 0 & {\color{red}0} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Interchange in A

- Let A' be a matrix obtained from A by a single interchange. What is the relation between

$$\rho_t(A) \text{ and } \rho_t(A')?$$

Interchange in em A

- Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$(A')^{(2)} = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

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Interchange in A

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Let A' be a matrix obtained from A by a single interchange. Then

$$\rho_t(A) - 1 \leq \rho_t(A') \leq \rho_t(A) + 1.$$

Interchange in A

Theorem

Let A' be a matrix obtained from A by a single interchange. If there is a positive integer t such that $\rho_t(A') = \rho_t(A) + 1$ and $\rho_{t+1}(A') = \rho_{t+1}(A)$. Then, for all $r > t$,

$$\rho_r(A') = \rho_r(A).$$

Interchange in A

Theorem

Let A' be a matrix obtained from A by a single interchange. Assume there is a positive integer t such that $\rho_t(A') = \rho_t(A) + 1$. Let

$$r = \min\{i : \; \rho_i(A') = \rho_i(A) + 1\},$$

$$s = \max\{i : \; \rho_i(A') = \rho_i(A) + 1\}.$$

Then,

$$\rho_i(A') = \begin{cases} \rho_i(A) & \text{if } 1 \leq i \leq r-1 \\ \rho_i(A) + 1 & \text{if } r \leq i \leq s \\ \rho_i(A) & \text{if } i \geq s+1 \end{cases}$$

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\rho_1(A) = 3$, $\rho_1(A') = 4$
- $\rho_2(A) = 4$, $\rho_2(A') = 5$
- $\rho_3(A) = 5$, $\rho_3(A') = 5$
- $(3, 1, 1)$ is majorized by $(4, 1)$
- Is there a distribution B families versus tables such that $\rho_1(B) = 3$ and $\rho_2(B) = 5$?
  $(3, 1, 1)$ is majorized by $(3, 2)$ is majorized by $(4, 1)$

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Interchange in em A

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow B = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Open Question

Let R and S be vectors. Let A and B be matrices with row sum vector R and column sum vector S.

Let P and E be the term rank partitions of A and B.

Let H be a vector such that
P is majorized by H is majorized by E.

Is there a matrix C with row sum vector R, column sum vector S and term rank partition H?

Thank you!

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