Chains in non-crossing partition posets

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work in progress
with Matthieu Josuat-Vergès (LIGM)
and Lucas Randazzo (LIGM)

SLC 82 Curia & 9th combinatorics days
April 2019

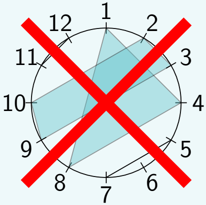
Outline

Saturated chains in non-crossing partition posets

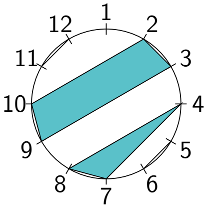
Poset of non-crossing partitions

Definition (Kreweras 1972)

non-crossing partition on n elements$=$ set partition of $\{1, \ldots, n\}$ which parts do not intersect



Ordered by refinement.



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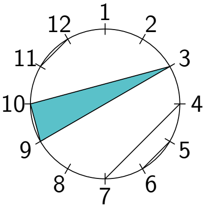
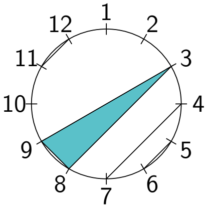


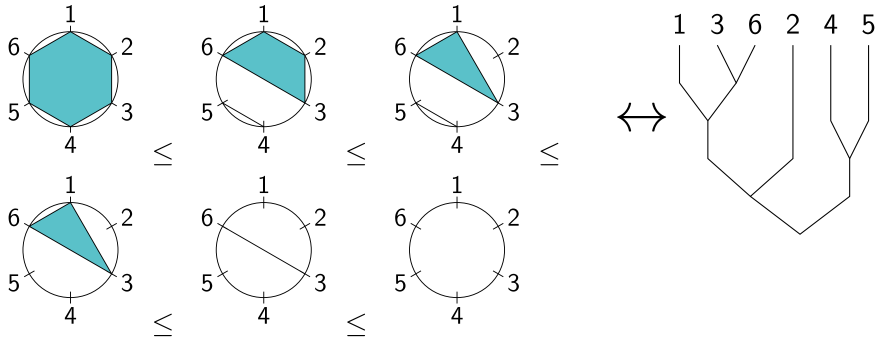
, not comp.



Saturated chains in NCP Poset [Stanley 1996]

Definition

A **saturated chain** in NC_n is a maximal chain $\pi_1 < \ldots < \pi_{n-2}$.



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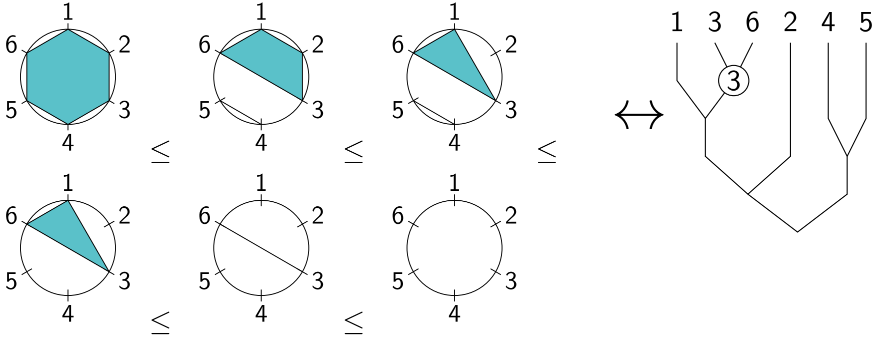

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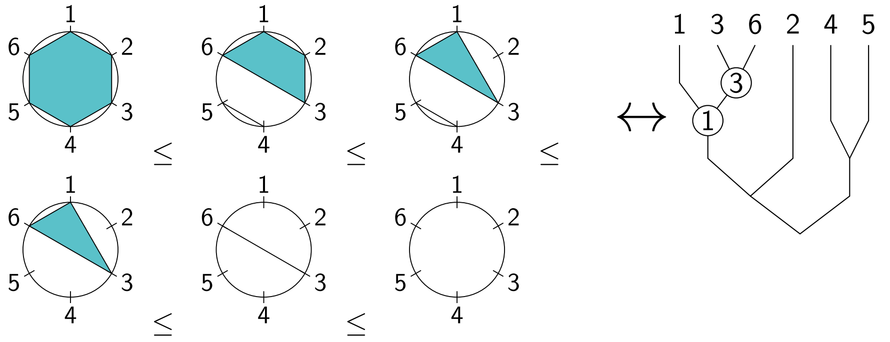

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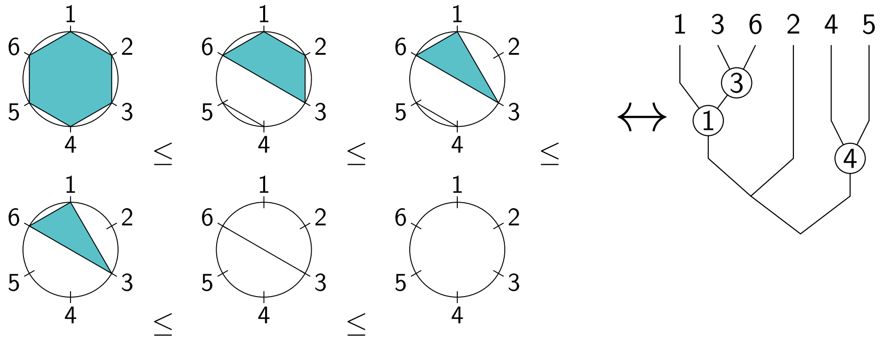

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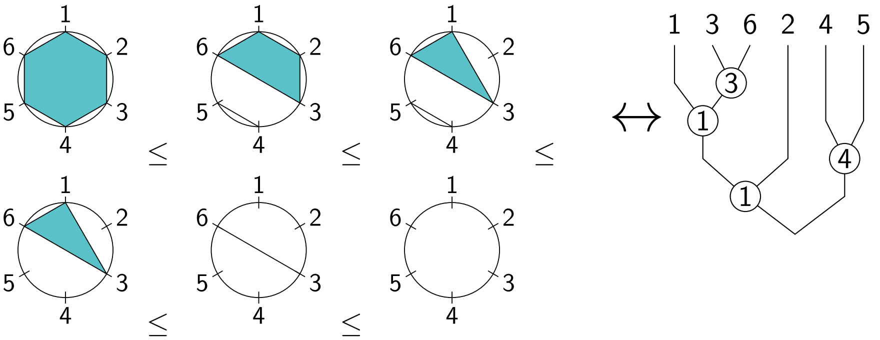

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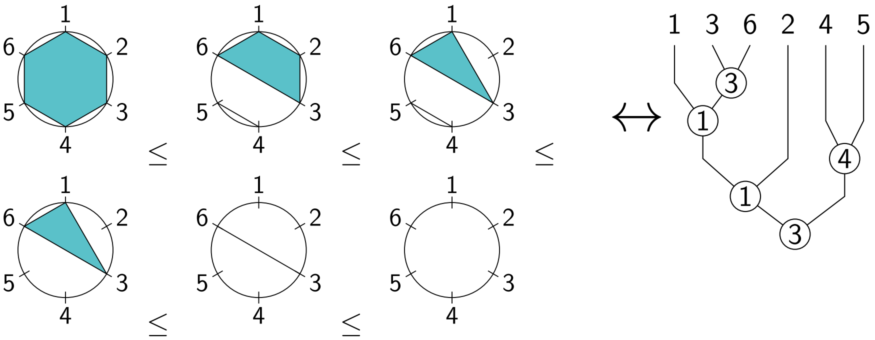

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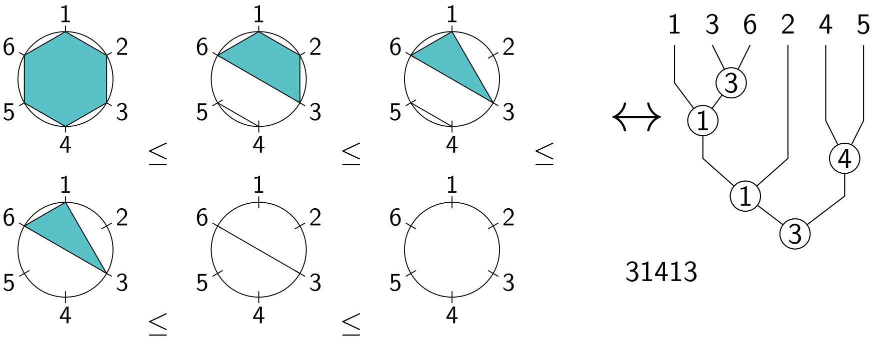

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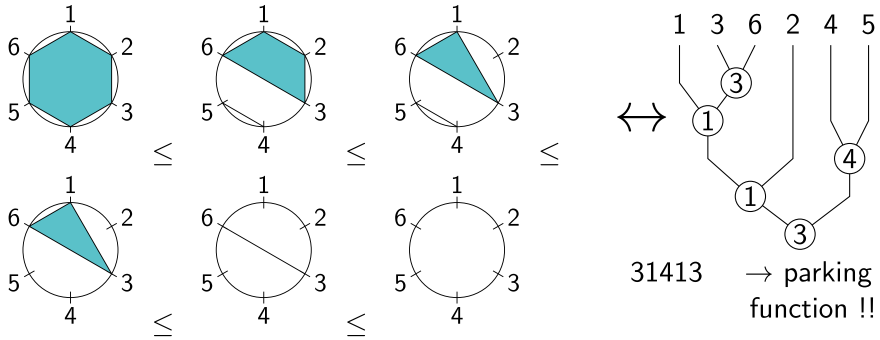

From parking functions to planar binary trees

Chains in NCPP	Parking functions	Decorated Dyck paths

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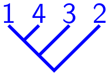

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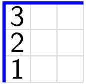

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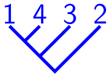
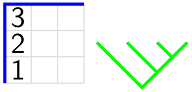
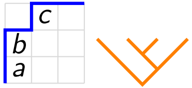
	111	
	112 121 211	

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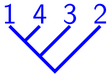
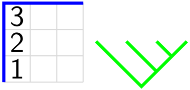
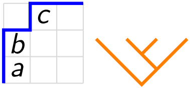
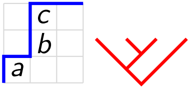
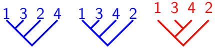
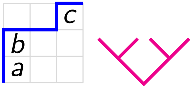
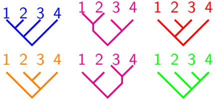
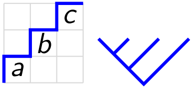
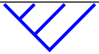
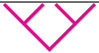
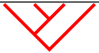
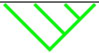
	111	
	112 121 211	
	122 212 221	
	113 131 311	
	123 132 213 231 312 321	

π_1 : Chains in NCPP $\rightarrow$ PBT

π_2 : Decorated Dyck paths $\rightarrow$ PBT

Shape					
π_1	6	4	3	2	1
π_2	6	3	3	3	1

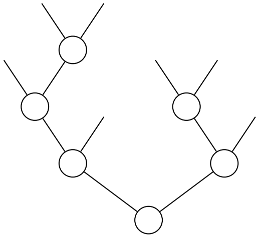
Hook formula(s)

Proposition (DO, Josuat-Vergès)

The multiplicity of a given leveled tree T

$$W(T) = \prod_{v \in L(T)} (h_v + 1),$$

where $L(T) = \{$*left children in* $T\}$
and $h_v =$ *nbr of inner vertices in the subtree of* T *rooted in* v.



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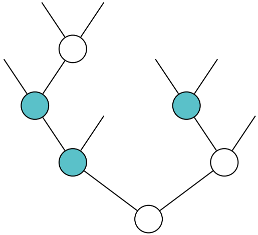

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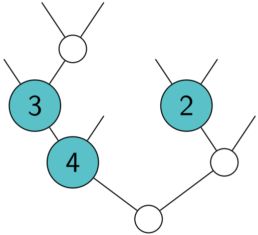

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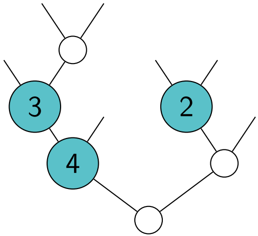

Corollary (Hook formula)

$$(n+1)^{n-1} = \sum_T \frac{n!}{\prod_{v \in V(T)} h_v} \times \prod_{v \in L(T)} (h_v + 1)$$

where the sum runs over any complete binary tree T.

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Proposition (Postnikov 2005)

$$(n+1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{v \in V(T)} \left(\frac{1}{h_v} + 1\right)$$

Rk: $\sum_T W(T)$ given by A0088716

Minimal factorisations of a cycle

Minimal factorisations of a cycle

Definition

A factorisation of $(1\ 2\ \ldots\ n)$, $\sigma_1 \cdots \sigma_j$, is minimal if $\sum_i l(\sigma_i) = n + j - 1$.

Proposition (Biane 1997)

Minimal factorizations

$$\Leftrightarrow$$

Chains of NCP $\hat{0} \leq \pi_1 \leq \cdots \leq \hat{1}$

s.t. $l(\sigma_i)$ *blocks are merged between* π_i *and* π_{i-1}.

Enumeration

The number of factorisations $(1, 2, \ldots, kn+1) = \sigma_1 \cdots \sigma_n$ where σ_i is a cycle on $k+1$ elements is

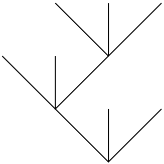
$$(kn+1)^{n-1}.$$

Proposition (DO - Josuat-Vergès)

We have with T running over plane $(k+1)$-ary trees with n increasingly-labeled internal vertices:

$$(kn+1)^{n-1} = \sum_T \prod_{v \text{ left vertex}} h_v,$$

where h_v is equal to the number of leaves above v.



$\rightarrow 5$

YAHF

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$$(kn+1)^{n-1} = \sum_T \prod_{v \text{ left vertex}} h_v,$$

Corollary

Considering planar $(k+1)$-ary trees T, we get:

$$(kn+1)^{n-1} = \sum_T \left(\prod_{v \text{ left vertex}} h_v \right) \times \frac{n!}{\prod_v \frac{h_v - 1}{k}}$$

2-partition posets

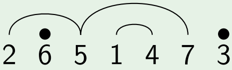
2-partitions [Edelmann 1980]

Definition

A 2-partition is a couple $(P, Q)_\phi$, where

- P is a non-crossing partition,
- Q is set-partition
- and ϕ is a bijections between parts of P and parts of Q s.t. $|P_i| = |\phi(P_i)|$.

Example:

( ,$\{14\}\{3\}\{257\}\{6\})_{3412} \leftrightarrow$ 

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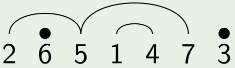
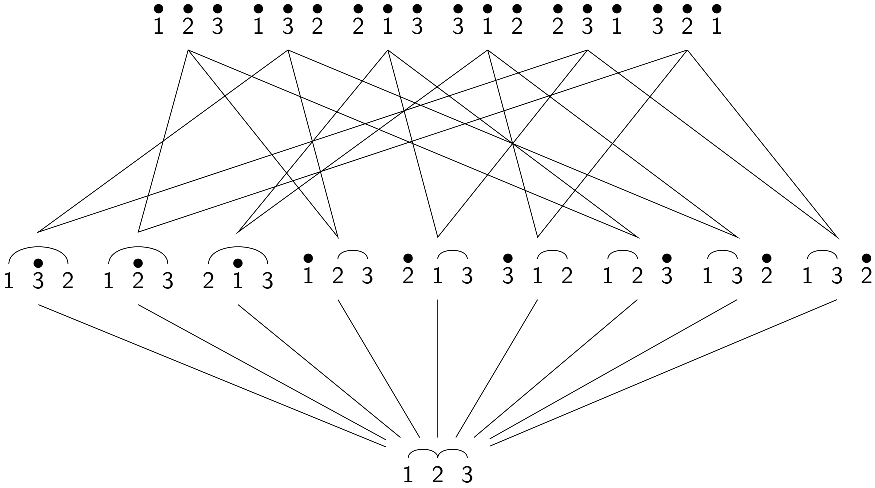
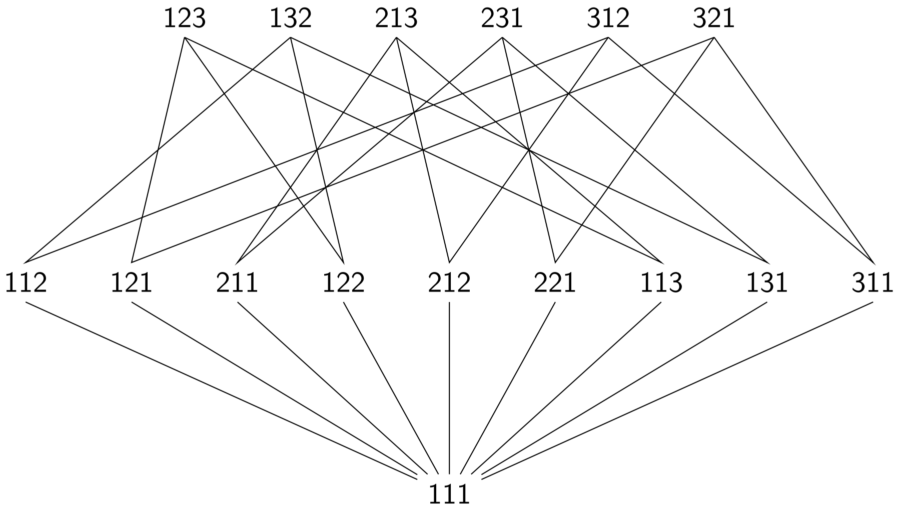
( ,$\{14\}\{3\}\{257\}\{6\})_{3412} \leftrightarrow$ 

Order : $(P, Q)_\phi \leq (P', Q')_\psi$ iff $P \leq P'$, $Q \leq Q'$ and the bijections commute with the order

2-partition poset on 3 vertices [Edelman]



2-partition poset on 3 vertices $P2_3$[Edelman]



Zeta polynomial

Definition (Stanley 1974)

The Zeta polynomial of the posets of 2-partitions is defined by:

$$\zeta(l, k, n) = |\{\pi_1 \leq \ldots \pi_k | \pi_i \in P2_n, \mathrm{rk}(\pi_k) = l\}|$$

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Proposition (DO - Josuat-Vergès - Randazzo)

$$\zeta(l,k,n) = l! \binom{kn}{l} S(n, l+1)$$

In particular, $\zeta(l,-1,n)$ is given by A198204.

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Corollary (Edelman)

$$\sum_{l} \zeta(l, k, n) = (nk+1)^{n-1}$$

Some species

Proposition

The species of weak k-chains satisfies the following relation:

$$\mathcal{C}^{l}_{k,t} = \sum_{p\geq 1} \mathcal{C}^{l,p}_{k-1,t} \times \left(\mathcal{C}^{l}_{k,t} + 1\right)^{p},$$

where $\mathcal{C}^{l,p}_{k-1,t}$ *is the species which coincides with* $\mathcal{C}^{l}_{k-1,t}$ *on any set of size* p *and send any other set to the empty set.*

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Obrigada pela vossa atenção !