Positivity for Symplectic Q-Functions

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Plan:

- Symplectic Q-functions
- Tableau description
- Pieri-type rule
- Other positivity conjectures

Related papers:

- S. Okada, Pfaffian formulas and Schur Q-function identities, arXiv:1706.01029.
- S. Okada, A generalization of Schur's P- and Q-functions, arXiv:1904.03386.

Motivation

Hall–Littlewood functions $P_\lambda(\boldsymbol{x};t)$ interpolate between Schur functions and Schur's P-functions:

Hall–Littlewood functions $P_\lambda(\boldsymbol{x};t)$

$t=0$ $\swarrow$ $\qquad$ $\searrow$ $t=-1$

Schur functions $s_\lambda(\boldsymbol{x})$ $\qquad$ Schur's P-functions $P_\lambda(\boldsymbol{x})$

Macdonald extended a definition of Hall–Littlewood functions to any root systems Δ:

Hall–Littlewood functions

$t=0$ $\swarrow$ $\qquad$ $\searrow$ $t=-1$

irred. char. of $\mathfrak{g}$ $\qquad$???

where $\mathfrak{g}$ is the semi-simple Lie algebra with root system Δ.

Symplectic Hall–Littlewood Functions

The symplectic Hall–Littlewood functions (Hall–Littlewood functions associated to the root system of type C_n) are defined by

$$P_\lambda^C(\boldsymbol{x};t) = \frac{1}{W_\lambda(t)} \sum_{w \in W} w \left(\boldsymbol{x}^\lambda \prod_{\alpha \in \Delta^+} \frac{1 - t\boldsymbol{x}^{-\alpha}}{1 - \boldsymbol{x}^{-\alpha}} \right)$$

where $\lambda = \sum_{i=1}^n \lambda_i e_i$ is a dominant weight (identified with a partition of length $\leq n$), W is the Weyl group of type C_n, and

$$W_\lambda(t) = \sum_{w \in W,\ w\lambda = \lambda} t^{l(\lambda)} = \prod_{j=1}^{m_0} \frac{1 - t^{2j}}{1 - t} \cdot \prod_{k \geq 1} \prod_{j=1}^{m_k} \frac{1 - t^j}{1 - t},$$

$$\Delta^+ = \left\{ e_i \pm e_j : 1 \leq i < j \leq n \right\} \cup \{ 2e_i : 1 \leq i \leq n \}.$$

It can be shown that

$$P_\lambda^C(\boldsymbol{x};t) \in \mathbb{Z}[t][x_1^{\pm 1}, \dots, x_n^{\pm 1}]^W.$$

Symplectic Schur functions

For a partition λ of length $\leq n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$), we define the symplectic Schur function $s^C_\lambda(\boldsymbol{x})$ by

$$s^C_\lambda(\boldsymbol{x}) = P^C_\lambda(\boldsymbol{x}; 0).$$

Then $s^C_\lambda(\boldsymbol{x})$ gives the irreducible character of the symplectic group $\mathbf{Sp}_{2n}$ with highest weight λ.

Symplectic Q-functions

For a strict partition λ of length $l \leq n$ ($\lambda_1 > \cdots > \lambda_l > 0$), we define the symplectic P-function $P^C_\lambda(\boldsymbol{x})$ and the symplectic Q-function $Q^C_\lambda(\boldsymbol{x})$ by

$$P^C_\lambda(\boldsymbol{x}) = P^C_\lambda(\boldsymbol{x}; -1), \quad Q^C_\lambda(\boldsymbol{x}) = 2^l P^C_\lambda(\boldsymbol{x}; -1).$$

Nimmo-type formula

Theorem For a strict partition λ of length l, we have

$$Q^C_\lambda(\boldsymbol{x}) = \frac{1}{D^C(\boldsymbol{x})} \operatorname{Pf} \left(\begin{array}{c|c} A^C(\boldsymbol{x}) & \left(f^C_{\lambda_j}(x_i) \right)_{\substack{1 \le i \le n \\ 1 \le j \le r}} \\ \hline -{}^t\!\left(f^C_{\lambda_j}(x_i) \right)_{\substack{1 \le i \le n \\ 1 \le j \le r}} & O \end{array} \right),$$

where $r = l$ or $l+1$ according to whether $n + l$ is even or odd, and

$$f^C_d(x) = \begin{cases} 2(x^d - x^{-d})(x + x^{-1})/(x - x^{-1}) & \text{if } d \ge 1, \\ 1 & \text{if } d = 0, \end{cases}$$

$$A^C(\boldsymbol{x}) = \left(\frac{(x_j + x_j^{-1}) - (x_i + x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \right)_{1 \le i, j \le n},$$

$$D^C(\boldsymbol{x}) = \prod_{1 \le i < j \le n} \frac{(x_j + x_j^{-1}) - (x_i - x_i^{-1})}{(x_j + x_j^{-1}) + (x_i + x_i^{-1})} \quad (= \operatorname{Pf} A^C(\boldsymbol{x}) \quad \text{if } n \text{ is even}).$$

Schur-type formula

Theorem For a strict partition λ, we have

$$Q^C_\lambda(\boldsymbol{x}) = \mathrm{Pf}\left(Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x})\right)_{1\le i,\, j\le L},$$

where $L = l$ or $l+1$ according to whether l is even or odd, and $Q^C_{(r,0)}(\boldsymbol{x}) = Q^C_{(r)}(\boldsymbol{x})$.

Proposition

$$\sum_{r=0}^{\infty} Q^C_{(r)}(\boldsymbol{x}) z^r = \prod_{i=1}^{n} \frac{(1+x_i z)(1+x_i^{-1}z)}{(1-x_i z)(1-x_i^{-1}z)}.$$

Proposition

$$Q^C_{(r,s)}(\boldsymbol{x}) = Q^C_{(r)}(\boldsymbol{x}) Q^C_{(s)}(\boldsymbol{x}) + 2\sum_{k=1}^{s}(-1)^k \left(Q^C_{(r+k)}(\boldsymbol{x}) + 2\sum_{i=1}^{k-1} Q^C_{(r+k-2i)}(\boldsymbol{x}) + Q^C_{(r-k)}(\boldsymbol{x}) \right) Q^C_{(s-k)}(\boldsymbol{x}).$$

Józefiak–Pragacz–Nimmo-type formula

Theorem For strict partitions λ of length l and μ of length m, we put

$$Q^C_{\lambda/\mu}(\boldsymbol{x}) = \mathrm{Pf}\left(\begin{array}{c|c} \left(Q^C_{(\lambda_i,\lambda_j)}(\boldsymbol{x})\right)_{1\le i,j\le l} & \left(Q^C_{(\lambda_i-\mu_{r+1-j})}(\boldsymbol{x})\right)_{\substack{1\le i\le l\\ 1\le j\le r}} \\ \hline -{}^t\!\left(Q^C_{(\lambda_i-\mu_{r+1-j})}(\boldsymbol{x})\right)_{\substack{1\le i\le l\\ 1\le j\le r}} & O \end{array}\right),$$

where $r = m$ or $m+1$ according to whether $l+m$ is even or odd. Then we have

$$Q^C_\lambda(\boldsymbol{x},\boldsymbol{y}) = \sum_\mu Q^C_{\lambda/\mu}(\boldsymbol{x}) Q^C_\mu(\boldsymbol{y}),$$

where μ runs over all strict partitions.

Symplectic Primed Shifted Tableau

Definition (King–Hamel) A symplectic primed shifted tableau of shape λ is a filling of the boxes in the shifted diagram $S(\lambda)$ with entries from

$$1' < 1 < \overline{1}' < \overline{1} < 2' < 2 < \overline{2}' < \overline{2} < \cdots < n' < n < \overline{n}' < \overline{n}$$

satisfying the following conditions:

- the entries in each row and in each column are weakly increasing;
- each unprimed entry appears at most once in every column;
- each primed entry appears at most once in every row;
- at most one element from $\{k', k, \overline{k}', \overline{k}\}$ appears on the main diagonal.

Example

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & \overline{2}' & 3' \\ \hline \multicolumn{1}{c|}{} & 2' & \overline{2}' & 3 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}, \quad \boldsymbol{x}^T = x_1^2 x_2^{-1} x_3^2 x_4.$$

Tableau Description of Symplectic Q-Functions

Theorem (Conjectured by King–Hamel) For a strict partition λ, we have

$$Q^C_\lambda(\boldsymbol{x}) = \sum_T \boldsymbol{x}^T,$$

where T runs over all symplectic primed shifted tableaux of shape λ.

Idea of Proof Both sides satisfy

- $Q^C_\lambda(x_1, \dots, x_{n-1}, x_n) = \sum_\mu Q^C_\mu(x_1, \dots, x_{n-1}) Q^C_{\lambda/\mu}(x_n)$,
- $Q^C_{\lambda/\mu}(x_n) = 0$ unless $\lambda \supset \mu$ and $l(\lambda) - l(\mu) \leq 1$,
- $Q^C_{\lambda/\mu}(x_n) = \det \left(Q^C_{(\lambda_i - \mu_j)}(x_n) \right)_{1 \leq i, j \leq l(\lambda)}$ if $l(\lambda) - l(\mu) \leq 1$.

Hence the proof is reduced to the case where $\lambda = (r)$ and $\boldsymbol{x} = (x_n)$.

Structure Constants for Symplectic P-Functions

The symplectic P-functions $\{P^C_\lambda(\boldsymbol{x})\}_{\lambda\text{:strict partition of length } \le n}$ form a basis of the ring

$$\Gamma^C_n = \left\{ f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^W : f(t, -t, x_3, \dots, x_n) \text{ is independent of } t \right\}.$$

Conjecture 1 Given two strict partitions μ and ν of length $\le n$, we can expand

$$P^C_\mu(\boldsymbol{x}) \cdot P^C_\nu(\boldsymbol{x}) = \sum_\lambda \widetilde{f}^\lambda_{\mu,\nu} P^C_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions of length $\le n$. Then the structure constants $\widetilde{f}^\lambda_{\mu,\nu}$ are nonnegative integers.

It can be proved that Conjecture 1 is true if $l(\nu) = 1$ (Pieri-type rule).

Pieri-type Rule for Symplectic P-functions

Theorem Let μ and λ be strict partitions of length $\leq n$ and let r be a positive integer. Then we have

(1) $\widetilde{f}^{\lambda}_{\mu,(r)} = 0$ unless $l(\lambda) = l(\mu)$ or $l(\mu) + 1$.

(2) If $l(\lambda) = l(\mu)$ or $l(\mu) + 1$, then

$$\widetilde{f}^{\lambda}_{\mu,(r)} = \sum_{\kappa} 2^{a(\mu,\kappa)+a(\lambda,\kappa)-\chi[l(\mu)>l(\kappa)]-1},$$

where κ runs over all strict partitions satisfying

$$\mu_1 \geq \kappa_1 \geq \mu_2 \geq \kappa_2 \geq \dots, \quad \lambda_1 \geq \kappa_1 \geq \lambda_2 \geq \kappa_2 \geq \dots,$$
$$(|\mu| - |\kappa|) + (|\lambda| - |\kappa|) = r,$$

and

$$a(\mu,\kappa) = \#\{i : \mu_i > \kappa_i > \mu_{i+1}\}, \quad a(\lambda,\kappa) = \#\{i : \lambda_i > \kappa_i > \lambda_{i+1}\},$$

$$\chi[l(\mu) > l(\kappa)] = \begin{cases} 1 & \text{if } l(\mu) > l(\kappa), \\ 0 & \text{otherwise.} \end{cases}$$

Outline of Proof

Step 1 By using Nimmo-type Pfaffian formula for $P^C_\lambda(\boldsymbol{x})$ and

$$1 + 2\sum_{r=1}^{\infty} P^C_{(r)}(\boldsymbol{x}) z^r = \prod_{i=1}^{n} \frac{(1+x_i z)(1+x_i^{-1} z)}{(1-x_i z)(1-x_i^{-1} z)},$$

we can show that

$$P^C_\mu(\boldsymbol{x}) \cdot \left(1+2\sum_{r=1}^{\infty} P^C_{(r)}(\boldsymbol{x}) z^r \right) = \sum_\lambda \det\left(a_{\lambda_i,\mu_j}(z) \right)_{1\le i,\, j\le l(\lambda)} P^C_\lambda(\boldsymbol{x}),$$

where λ runs over all strict partitions with $l(\lambda) = l(\mu)$ or $l(\mu)+1$, and

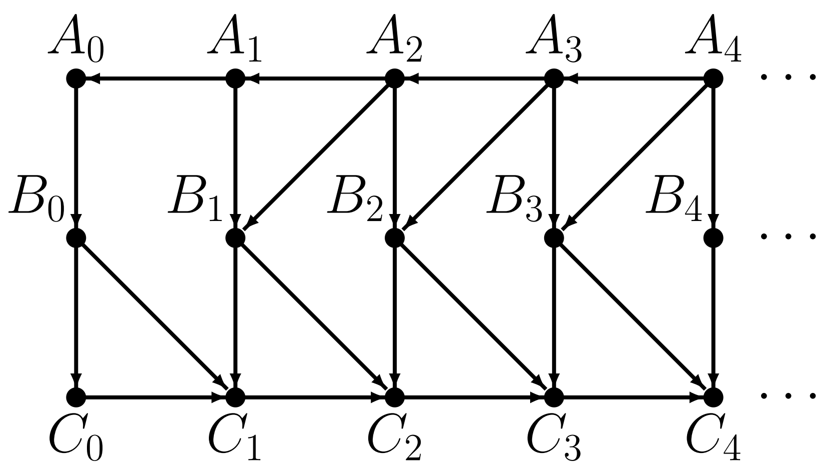
$$(t^l - t^{-l}) \cdot \frac{(1+tz)(1+t^{-1}z)}{(1-tz)(1-t^{-1}z)} = \sum_{k=0}^{\infty} a_{k,l}(z)(t^k - t^{-k}).$$

Outline of Proof

Step 2 By Lindström–Gessel–Vienno lemma, we can show that

$$1+2\sum_{r=1}^{\infty}\widetilde{f}^{\lambda}_{\mu,(r)}z^r=\det\left(a_{\lambda_i,\mu_j}(z)\right)_{1\leq i,j\leq l(\lambda)}$$

is equal to the weighted generating function of non-intersecting lattice paths with starting points $(A_{\mu_1},\ldots,A_{\mu_l})$ and ending points $(C_{\lambda_1},\ldots,C_{\lambda_l})$ on the following directed graph:



where the vertical edges have weight 1 and the other edges have weight z.

Positivity Conjectures for symplectic P-functions

Conjecture 2 For a strict partition λ of length $\leq n$, we can expand

$$P_\lambda(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}) = \sum_\mu c_{\lambda,\mu} P^C_\mu(x_1, \dots, x_n),$$

where μ runs over all strict partitions of length $\leq n$. Then the coefficients $c_{\lambda,\mu}$ are nonnegative integers.

Known Case If $l(\lambda) \leq 2$, then Conjecture 2 is true.

Remark For a partition λ of length $\leq n$, we have

$$s_\lambda(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}) = \sum_\mu b_{\lambda,\mu} s^C_\mu(x_1, \dots, x_n), \quad b_{\lambda,\mu} \geq 0.$$

Positivity Conjectures for symplectic P-functions

Conjecture 3 For a strict partition λ of length $\leq n$, we can expand

$$P^C_\lambda(\boldsymbol{x}) = \sum_\mu \widetilde{g}_{\lambda,\mu} s^C_\mu(\boldsymbol{x}),$$

where μ runs over all partitions of length $\leq n$. Then the coefficients $\widetilde{g}_{\lambda,\mu}$ are nonnegative integers.

Known Case If $l(\lambda) = 1$ or n, then Conjecture 3 is true.

Remark For a strict partition λ of length $\leq n$, we have

$$P_\lambda(\boldsymbol{x}) = \sum_\mu g_{\lambda,\mu} s_\mu(\boldsymbol{x}), \quad g_{\lambda,\mu} \geq 0.$$