Pak-Stanley labeling of the Catalan arrangement of hyperplanes

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Joint work with António Guedes de Oliveira (CMUP & University of Porto)

Outline

In the 90's Pak and Stanley introduced a bijection between the regions of the Shi arrangement and parking functions which triggered a variety of research projects in several dimensions.

Among these projects is the determination of the sets of labels of the regions of other arrangements labeled using the same rules. (For hyperplane arrangements such as the m-Shi arrangement, the Ish arrangment and a family of hyperplane arrangments "between" the Shi arrangement and the Ish arrangment.)

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In this talk we characterize the labels of the Catalan arrangement and present an algorithm to return a region from its label.

n-dimensional Coxeter arrangement: $\mathrm{Cox}_n = \bigcup_{1\le i<j\le n} \{x_i - x_j = 0\}$

n-dimensional Shi arrangement:

$$\mathrm{Shi}_n = \bigcup_{1\le i<j\le n} \{x_i - x_j = k \mid k \in \{0,1\}\}$$

n-dimensional m-Shi arrangement:

$$m\text{-}\,\mathrm{Shi}_n = \bigcup_{1\le i<j\le n} \{x_i - x_j = k \mid k \in \{-(m-1),\dots,0,1,\dots,m\}\}$$

n-dimensional Catalan arrangement:

$$\mathrm{Cat}_n = \bigcup_{1\le i<j\le n} \{x_i - x_j = k \mid k \in \{-1,0,1\}\}$$

n-dimensional m-Catalan arrangement:

$$m\text{-}\,\mathrm{Cat}_n = \bigcup_{1\le i<j\le n} \{x_i - x_j = k \mid k \in \{-m,\dots,0,1,\dots,m\}\}$$

The n-dimensional Coxeter arrangement has $n!$ regions.

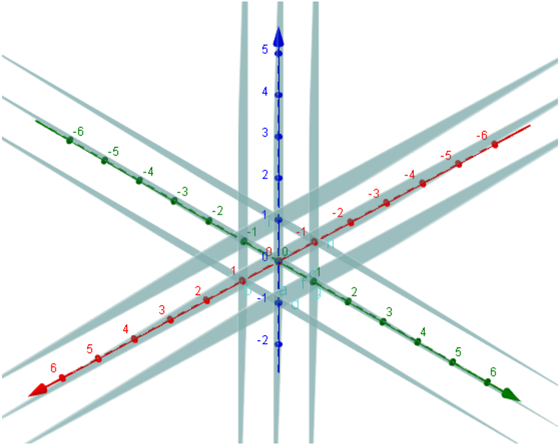
The n-dimensional Shi arrangement has $(n+1)^{n-1}$ regions.

The n-dimensional m-Shi arrangement has $(mn+1)^{n-1}$ regions.

The n-dimensional Catalan arrangement has $\frac{(2n)!}{(n+1)!} = n!C_n$ regions.

The n-dim. m-Catalan arrang. has $\binom{(mn+n)!}{(mn+1)!} = n!F(n,m)$ regions.
($F(n,m) = \frac{1}{mn+1}\binom{mn+n}{mn}$ is a Fuss-Catalan number.)

The 3-dimensional Catalan arrangement



Parking functions

Definition

$\mathbf{a} \in \mathbb{N}^n$ is a parking function of length n iff the unique weakly increasing rearrangement $\mathbf{b} = (b_1, \ldots, b_n)$ of $\mathbf{a}$ satisfies

$$b_i \leq i, \text{ for every } i \in [n].$$

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Pak-Stanley labeling

Let R_0 be the region bounded by the hyperplanes of equation $x_i - x_{i+1} = 0$ for $i = 1, \ldots, n-1$, and by the hyperplane of equation $x_1 - x_n = 1$.
$\ell(R_0) := \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^n$.
Let R_1 and R_2 be two regions separated by a unique hyperplane H such that R_0 and R_1 are on the same side of H, if the equation of H is of form $x_i - x_j = 0$ [resp. of form $x_i - x_j = 1$] then $\ell(R_2) = \ell(R_1) + e_i$ [resp. $\ell(R_2) = \ell(R_1) + e_j$].

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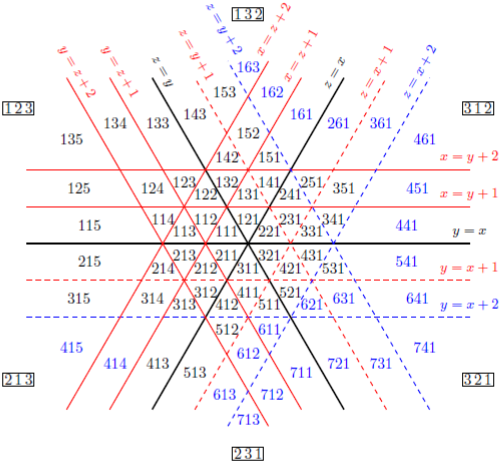

Figure: Pak-Stanley labeling of the 3-dimensional 2-Catalan.

Definition

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $p \in \mathbb{Z}$. The p-center of $\mathbf{a}$, $Z_p(\mathbf{a})$, is the largest subset $X = \{x_1, \ldots, x_q\}$ of $[n]$ such that if $1 \leq x_q < x_{q-1} < \cdots < x_1 \leq n$ then

$$a_{x_j} \leq p + j, \text{ for every } j \in [q].$$

Let $z_p(\mathbf{a}) = |Z_p(\mathbf{a})|$. If $\mathbf{a}$ is a label of the m-Catalan arrangement in dimension n, we call center of $\mathbf{a}$ to

$$\mathbf{z}(\mathbf{a}) := \left(z_0(\mathbf{a}), z_1(\mathbf{a}), \ldots, z_{(n-1)m}(\mathbf{a})\right).$$

Notes: $Z_p(\mathbf{a}) = \emptyset$ if $p < 0$,
$Z_0(\mathbf{a}) \subseteq Z_1(\mathbf{a}) \subseteq \cdots \subseteq Z_p(\mathbf{a})$ and $z_0(\mathbf{a}) \leq z_1(\mathbf{a}) \leq \cdots \leq z_p(\mathbf{a})$.

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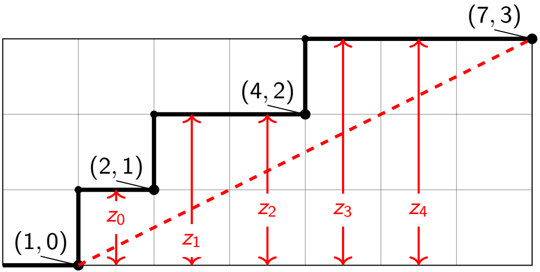

Figure: Dyck path representation of 124. ($\mathbf{z}(124) = (1, 2, 2, 3, 3)$.)

The chamber of $\pi = (\pi_1, \dots, \pi_n) \in \mathcal{S}_n$ is the set $\{\mathbf{x} \in \mathbb{R}^n \mid x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_n}\}$ (a region of the Coxeter arrangement).

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Theorem

Let $\pi, \rho = (i,i+1) \circ \pi \in \mathcal{S}_n$ differ by an adjacent transposition, let R be a region on the fundamental chamber of the m-Catalan arrangement, and **b** *and* **c** *be the Pak-Stanley labels of $\pi(R)$ and $\rho(R)$. Then*

$$\mathbf{c} = \mathbf{b} \circ (i,i+1) + \begin{cases} e_i, & \text{if } b_i \leq b_{i+1} \\ (-e_i), & \text{otherwise} \end{cases} .$$

Definition

For every $\mathbf{a} \in \mathbb{N}^n$, let $\mathbf{p}(\mathbf{a}) \in \mathbb{N}^n$ be defined by $\mathbf{p}(\mathbf{a})_i = j$ if $i \in Z_j \setminus Z_{j-1}$.

Theorem

Let $\pi \in \mathcal{S}_n$, R be a region of the m-Catalan arrangement, and $\mathbf{a}$ and $\mathbf{b}$ be the labels of R and $\pi(R)$. Then, for all $p \in [0, (m-1)n]$,

$$Z_p(\mathbf{b}) = \pi(Z_p(\mathbf{a})) .$$

In addition, $\mathbf{b} - \mathbf{p}(\mathbf{b}) = \mathbf{1} + I(\pi)$.

Note: $\mathbf{1} + I(\pi)$ is minimal among the labels of the regions contained in the same chamber as R.

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Theorem

$\mathbf{z}(\mathbf{a}) = \mathbf{z}(\mathbf{b})$ *if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *label two regions that belong to the same orbit under the action of* $\mathcal{S}_n$.
(In the m-Catalan arrangement, the center is $\mathcal{S}_n$*-invariant.)*

Labels of the regions of the m-Catalan arrangement

Theorem

The Pak-Stanley labeling bijectively labels the regions of the m-Catalan arrangement with sequences $\mathbf{a} \in \mathbb{N}^n$ such that

$$z_{(i-1)m}(\mathbf{a}) \geq i, \text{ for every } i \in [n].$$

Figure: Labeled Dyck path representation of 612 which is on the orbit of 124 and belongs to the chamber of 2 3 1.

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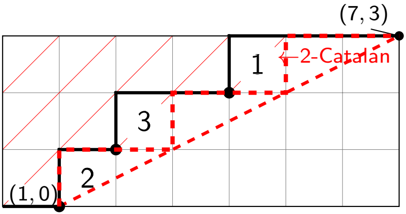

Figure: Labeled Dyck path representation of 612 which is on the orbit of 124 and belongs to the chamber of 2 3 1.

Labels of the relatively bounded regions

Theorem

The Pak-Stanley labeling bijectively labels the relatively bounded regions of the m-Catalan arrangement with sequences $\mathbf{a} \in \mathbb{N}^n$ *s.t.*

$$z_{(i-1)m-1}(\mathbf{a}) \geq i, \text{ for every } i \in [2, n].$$

Figure: Labeled Dyck path representation of 612 and of 513. 612 and 513 are on the orbits of 124 and 133. Both belong to the chamber of 2 3 1.

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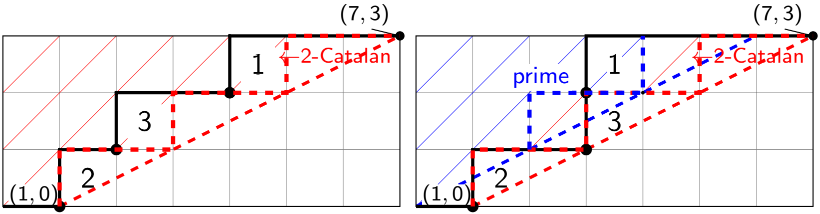

Figure: Labeled Dyck path representation of 612 and of 513. 612 and 513 are on the orbits of 124 and 133. Both belong to the chamber of 2 3 1.

Thank you for your attention!