Queens' Graph

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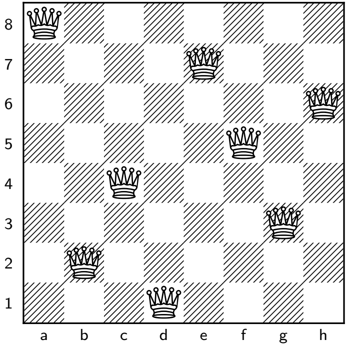
A joint work with Domingos Cardoso and Rui Duarte

Outline

The n-Queens Problem



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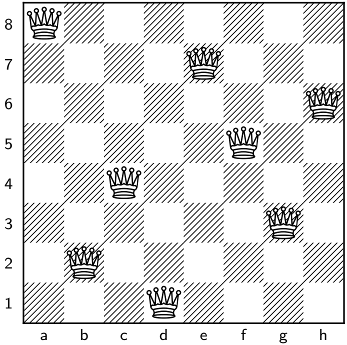
The n-Queens Problem



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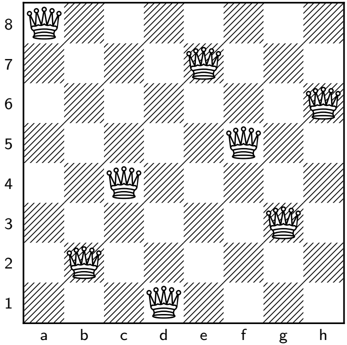

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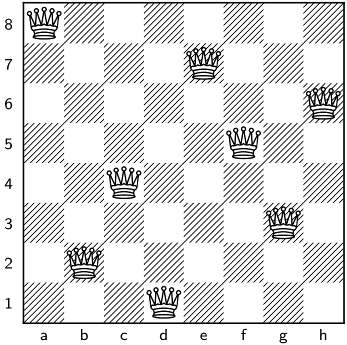

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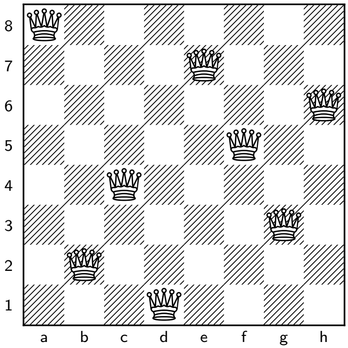

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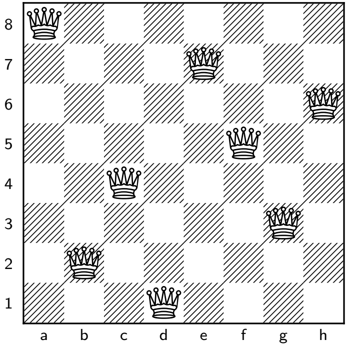

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The n-queens problem is a generalization of the above problem, consisting of placing n non attacking queens on $n \times n$ chessboard.

E. Pauls also proved in 1874 that the n-queens problem has a solution for every $n \geq 4$.

Chessboard and Queens' Graph

$\mathcal{Q}(n)$ and $\mathcal{T}_n$

Queen's Graph, $\mathcal{Q}(n)$, associated to $n \times n$ chessboard $\mathcal{T}_n$ has $n \times n$ vertices, corresponding to each square of the $n \times n$ chessboard.
Two vertices of $\mathcal{Q}(n)$ are adjacent if and only if they are in the same row or column or diagonal of the chessboard.

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The squares of $\mathcal{T}_n$ and the corresponding vertices in $\mathcal{Q}(n)$ are labeled from the left to the right and from the top to the bottom. For instance, $\mathcal{T}_4$ is labelled as in the figure.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Queens' Graph

1	2	3
4	5	6
7	8	9

Table: $\mathcal{T}_n$ - Chessboard for $n = 3$.

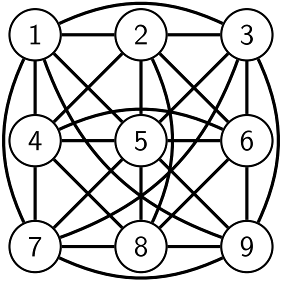

Figure: $\mathcal{Q}(3)$ - Queen's Graph for $n = 3$.

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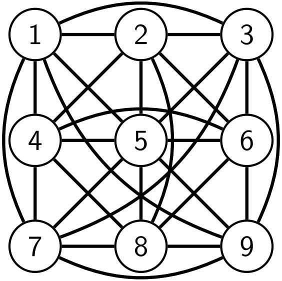

Figure: $\mathcal{Q}(3)$ - Queen's Graph for $n = 3$.

Since two vertices are connected by an edge if and only if they are in the same row, column or diagonal, we have

$$e(\mathcal{Q}(n)) = 2(n+1)\binom{n}{2} + 4\left(\binom{2}{2} + \cdots + \binom{n-1}{2}\right) = \frac{n(n-1)(5n-1)}{3}.$$

Combinatorial Properties

A closed formula, in terms of n, for the degrees of the vertices of $\mathcal{Q}(n)$ can be obtained from its structure.

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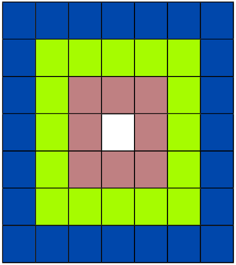
Let $P = \{V_i : i \in \{1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor\}\}$ be a partition of $V(\mathcal{Q}(n))$, such that

- V_1 is the subset of vertices corresponding to the more peripheral squares of $\mathcal{T}_n$;



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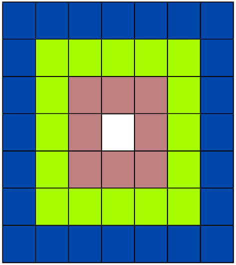
- V_2 is the subset of vertices corresponding to the more peripheral squares of $\mathcal{T}_n$ without V_1;



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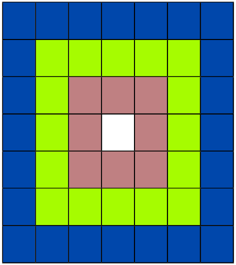
  ...
- $V_{\lfloor \frac{n+1}{2} \rfloor}$ is the subset of vertices corresponding to the more peripheral squares of $\mathcal{T}_n$ without $V_1 \cup V_2 \cup \cdots \cup V_{\lfloor \frac{n+1}{2} \rfloor - 1}$.

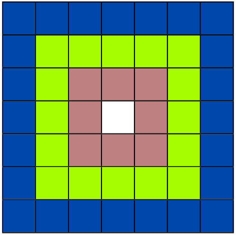


Combinatorial Properties

Theorem

Considering the above partition of the vertices of $\mathcal{Q}(n)$ *into,* $V_1, V_2, \ldots, V_{\lfloor \frac{n+1}{2} \rfloor}$*, the degrees of the vertices are*

$$d(v) = 3(n-1) + 2(i-1), \quad \forall v \in V_i, \forall i = 1, 2, \ldots, \lfloor \frac{n+1}{2} \rfloor. \quad (1)$$



Combinatorial Properties

For all vertices v of $\mathcal{Q}(n)$,

$$3n-3=\delta(\mathcal{Q}(n))\leq d(v)\leq \begin{cases} 4n-5 & \text{if } n \text{ is even,} \\ 4n-4 & \text{otherwise.} \end{cases}$$

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Since $e(\mathcal{Q}(n))=\frac{n(n-1)(5n-1)}{3}$, it follows that the average degree of $\mathcal{Q}(n)$ is

$$\overline{d_{\mathcal{Q}(n)}}=\frac{2e(\mathcal{Q}(n))}{n^2}=\frac{2(n-1)(5n-1)}{3n}.$$

Combinatorial Properties

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The diameter of any $\mathcal{Q}(n)$ with $n \geq 3$ is 2. Any square of the $n \times n$ chessboard is achieved from any other square with a row movement followed by a column movement.

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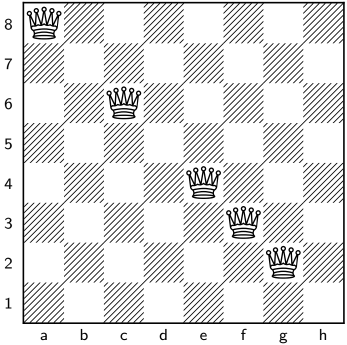
$\omega(\mathcal{Q}(n)) = n, n \geq 5$

Since all the vertices of a row (column or any of the two larger diagonals) produce a maximum clique with size n, for $n \geq 5$.

Combinatorial Properties

The domination number of Queens' Graph, $\gamma(\mathcal{Q}(n))$, is the most studied problem about combinatorial properties of this graph.

Some values of $\gamma(\mathcal{Q}(n))$ are already known but the problem remains open.



n	1	2	3	4	5	6	7	8	9	10	11	12	13
$\gamma(\mathcal{Q}(n))$	1	1	1	2	3	3	4	5	5	5	5	6	7

Spectral Properties

The spectrum of the adjacency matrix of $\mathcal{Q}(n)$ is the multiset $\sigma(\mathcal{Q}(n)) = \{\mu_1^{[m_1]}, \ldots, \mu_p^{[m_p]}\}$, where $\mu_1 > \cdots > \mu_p$ are the p distinct eigenvalues and m_i is the multiplicity of the eigenvalues μ_i for $i = 1, \ldots, p$. When necessary these eigenvalues are also denote by $\mu_1(\mathcal{Q}(n)), \ldots, \mu_p(\mathcal{Q}(n))$.

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Therefore, we may conclude

$$\frac{2(n-1)(5n-1)}{3n} = \overline{d_{\mathcal{Q}(n)}} \leq \mu_1(\mathcal{Q}(n)) \leq \Delta(\mathcal{Q}(n)) = \begin{cases} 4n-5, & \text{if } n \text{ is even,} \\ 4n-4, & \text{otherwise.} \end{cases}$$

In this section, the n^2 entries of vectors are displayed in the $n \times n$ chessboard in the same sequence as the labelling of the vertices in the last section.
Therefore an entry of a vector is referenced by the chessboard coordinates, i.e., $v(i,j)$ with $(i,j) \in [n]^2$.

$$v = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \\ 12 \\ 14 \\ 16 \\ 18 \end{bmatrix}$$

	1	2	3
1	2	4	6
2	8	10	12
3	14	16	18

Table: Vector v displayed on 3×3 chessboard with the coordinates indicated on the outside of the chessboard.

Spectrum of Queens' Graph, $\sigma(\mathcal{Q}(n))$.

n	$\sigma(\mathcal{Q}(n))$
2	$\{3, -1^{[3]}\}$
3	$\{\frac{5+\sqrt{57}}{2}, 1, (-1+\sqrt{2})^{[2]}, -1^{[2]}, \frac{5+\sqrt{57}}{2}, (-1-\sqrt{2})^{[2]}\}$
4	$\{9.6, 1.8^{[2]}, 1.7, 1.3, 0.5^{[2]}, 0, -0.4, -0.8, -1.5^{[2]}, -2.8^{[2]}, 3.3, -4\}$

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From the computations, we detected some similarities in the spectrum of $\mathcal{Q}(n)$ for different values of n.

Spectral Properties

In the table below, the distinct integer eigenvalues are presented for $\mathcal{Q}(n)$ when $4 \leq n \leq 11$.

n	Distinct integer eigenvalues
4	-4, 0
5	-4, -3, 0, 1
6	-4, 2
7	-4, -3, -2, 1, 2, 3
8	-4, 4
9	-4, -3, -2, -1, 2, 3, 4, 5
10	-4, 6
11	-4, -3, -2, -1, 0, 3, 4, 5, 6, 7

Conjecture:

if n is even	$-4,\ n-4$
if n is odd	$\{-4, -3, \ldots, \frac{n-11}{2}\} \cup \{\frac{n-5}{2}, \ldots, n-5, n-4\}$

Spectral Properties

Lemma

Let $X = x_{(i,j)} \in \mathbb{R}^{n^2}$ *be an eigenvector of* $A_{\mathcal{Q}(n)}$ *associated with the eigenvalue* μ. *Then*

$$(\mu+4)\,||X||^2 = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} x_{(k,j)}\right)^2 + \sum_{k=1}^{n}\left(\sum_{i=1}^{n} x_{(i,k)}\right)^2 + \\ + \sum_{k=2}^{2n}\left(\sum_{i+j=k} x_{(i,j)}\right)^2 + \sum_{k=-(n-1)}^{n-1}\left(\sum_{i-j=k} x_{(i,j)}\right)^2 .$$

As a corollary of this lemma, we have the following result.

Theorem

If μ is an eigenvalue of $A_{\mathcal{Q}(n)}$, then $\mu \geq -4$.

This lower bound is not attained for $n = 1, 2, 3$ but for $n \geq 4$, -4 is a eigenvalue of $\mathcal{Q}(n)$ with multiplicity $(n-3)^2$, as it will stated later.

Spectral Properties

Let X_4 be the vector represented bellow.

0	**1**	**-1**	0
-1	0	0	**1**
1	0	0	**-1**
0	**-1**	**1**	0

We define a new family of vectors, $\mathcal{F}_n = \{X_n^{(a,b)} \in \mathbb{R}^{n^2} : (a,b) \in [n-3]^2\}$, for $n \geq 4$, where

$$\left[X_n^{(a,b)}\right]_{(i,j)} = \begin{cases} \left[X_4\right]_{(i-a+1, j-b+1)}, & \text{if } (i,j) \in A \times B \\ 0, & \text{otherwise.} \end{cases}$$

where $A = \{a, a+1, a+2, a+3\}$ and $B = \{b, b+1, b+2, b+3\}$.

Spectral Properties

$\mathcal{F}_5$

0	**1**	**-1**	0	0
-1	0	0	**1**	0
1	0	0	**-1**	0
0	**-1**	**1**	0	0
0	0	0	0	0

Table: $X_5^{(1,1)}$

0	0	0	0	0
0	**1**	**-1**	0	0
-1	0	0	**1**	0
1	0	0	**-1**	0
0	**-1**	**1**	0	0

Table: $X_5^{(2,1)}$

0	0	**1**	**-1**	0
0	**-1**	0	0	**1**
0	**1**	0	0	**-1**
0	0	**-1**	**1**	0
0	0	0	0	0

Table: $X_5^{(1,2)}$

0	0	0	0	0
0	0	**1**	**-1**	0
0	**-1**	0	0	**1**
0	**1**	0	0	**-1**
0	0	**-1**	**1**	0

Table: $X_5^{(2,2)}$

Spectral Properties

Theorem

For $n \geq 4$, -4 is an eigenvalue of $\mathcal{Q}(n)$ with multiplicity $(n-3)^2$. Futhermore, $\mathcal{F}_n$ is a basis for $\mathcal{E}_{\mathcal{Q}(n)}(-4)$.

$\mathcal{F}_5$

0	**1**	**-1**	0	0
-1	0	0	**1**	0
1	0	0	**-1**	0
0	**-1**	**1**	0	0
0	0	0	0	0

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0	0	0	0	0
0	**1**	**-1**	0	0
-1	0	0	**1**	0
1	0	0	**-1**	0
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0	0	**1**	**-1**	0
0	**-1**	0	0	**1**
0	**1**	0	0	**-1**
0	0	**-1**	**1**	0
0	0	0	0	0

Table: $X_5^{(1,2)}$

0	0	0	0	0
0	0	**1**	**-1**	0
0	**-1**	0	0	**1**
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0	0	**-1**	**1**	0

Table: $X_5^{(2,2)}$

Spectral Properties

Definition

We define row vector R_i, column vector C_j, sum vector S_a and difference vector D_a of dimension n^2 for some $n \in \mathbb{N}$ as

$$R_i(x,y) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{otherwise.} \end{cases} \quad C_j(x,y) = \begin{cases} 1, & \text{if } y = j \\ 0, & \text{otherwise.} \end{cases}$$
$$S_a(x,y) = \begin{cases} 1, & \text{if } x + y = a \\ 0, & \text{otherwise.} \end{cases} \quad D_a(x,y) = \begin{cases} 1, & \text{if } x - y = a \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

0	0	0
0	0	0
1	1	1

Table: R_3.

0	1	0
0	1	0
0	1	0

Table: C_2.

0	1	0
1	0	0
0	0	0

Table: S_3.

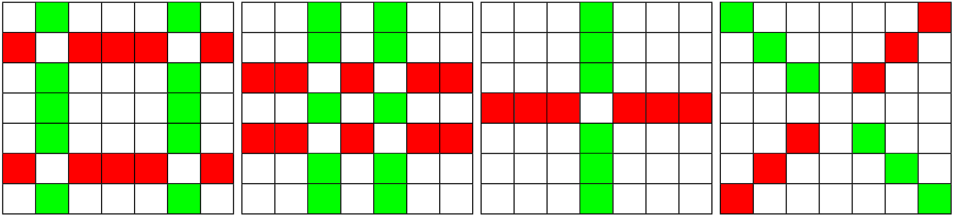
1	0	0
0	1	0
0	0	1

Table: D_0.

Spectral Properties

Theorem

n-4 is eigenvalue of $\mathcal{Q}(n)$, for $n \geq 4$, with multiplicity at least $\frac{n-2}{2}$ if n even and $\frac{n+1}{2}$ if n odd.
Futhermore, $\{Y_i^n = C_i + C_{n-i+1} - R_i - R_{n-i+1} : i \in \{2, \ldots, \frac{n-2}{2}\}\}$ and $\{Y_i^n = C_i + C_{n-i+1} - R_i - R_{n-i+1} : i \in \{2, \ldots, \frac{n+1}{2}\}\} \cup \{Z^n = D_0 - S_{n+1}\}$ are sets of linearly independent vectors of $\mathcal{E}_{\mathcal{Q}(n)}(n-4)$ when n is even and n is odd, respectively.



Equitable partitions

Definition[Equitable partition]

Given a graph G, the partition $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \ldots \dot{\cup} V_k$ is an equitable partition if every vertex in V_i has the same number of neighbours in V_j, for all $i, j \in \{1, 2, \ldots, k\}$. An equitable partition of $V(G)$ is also called equitable partition of G and the vertex subsets $V_1, V_2, \ldots, V_k$ are called the cells of the equitable partition.

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Definition [Divisor (or quociente) matrix]

Considering that π is an equitable partition $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_k$ and that each vertex in V_i has b_{ij} neighbors in V_j (for all $i, j \in \{1, 2, \dots, k\}$), the matrix $B_\pi = (b_{ij})$ is called the divisor (or quociente) matrix of π.

Equitable partitions

Theorem[D. Cvetković, P. Rowlinson, S. Simić, 2010]

Let G be a graph with adjacency matrix A and let π be a partition of $V(G)$ with characteristic matrix C.

1. If π is equitable, with divisor matrix B, then $AC = CB$.
2. The partition π is equitable if and only if the column space of C is A-invariante.
3. The characteristic polynomial of the divisor matrix of any equitable partition of G divides its characteristic polynomial.

Labeling the vertices according to the cell they belong

Considering $n \geq 3$, let us assign to the squares of the chessboard $\mathcal{T}_n$, corresponding to the vertices of $\mathcal{Q}(n)$, the numbers of the cells they belong.

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Labeling procedure (Part I)

We start labeling one square of each cell as follows.

(1) Assign to the first square (the top left square) the number 1;

(2) Assign to the first and second square of the second column (from the top to bottom) the numbers 2 and 3;

$\vdots$

($\lceil \frac{n}{2} \rceil$) Assign to the first $\lceil \frac{n}{2} \rceil$ squares of the $\lceil \frac{n}{2} \rceil$-th column (from top to bottom) the numbers $\sum_{j=1}^{\lceil \frac{n}{2} \rceil - 1} j + 1, \ldots, \frac{(\lceil \frac{n}{2} \rceil + 1)\lceil \frac{n}{2} \rceil}{2}$.

Example

Application of the procedure (Part I) to the 6×6 chessboard

1	2	4			
	3	5			
		6			

From the abobe assignment, we get a right triangle of squares assigned to the numbers $1, 2, \ldots, \frac{(\lceil n/2 \rceil + 1)\lceil n/2 \rceil}{2}$.

Labeling the vertices according to the cell they belong

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Labeling procedure (Part II)

The remainder vertices of each cell are obtained by reflections, as follows.

1. We reflect the obtained triangle using the vertical cathetus of the triangle as the mirror line and after this reflection we have two right triangles sharing the same vertical line.
2. Then we reflect both triangles each one using its hypotenuse as the mirror line.
3. After the above reflections all the squares in the top $\lceil \frac{n}{2} \rceil$ lines are assigned with the numbers of the cells they belong.
4. Finally we reflect the rectangle formed by the the upper $\lfloor \frac{n}{2} \rfloor$ lines taking as the mirror line the horizontal middle line of the chessboard and after that all the squares become assigned to the numbers of their cells.

Example

Application of the procedure (Part I and Part II) to the 6×6 chessboard

1	2	4	4	2	1
2	3	5	5	3	2
4	5	6	6	5	4
4	5	6	6	5	4
2	3	5	5	3	2
1	2	4	4	2	1

A couple of consequences

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Every queens graphs $\mathcal{Q}(n)$, with $n \geq 3$, has an equitable partition with

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Considering the divisor matrix B of the obtained equitable partition and applying Theorem[D. Cvetković, P. Rowlinson, S. Simić, 2010] it follows that the eigenvalues of B with its respective multiplicities are eigenvalues of the adjacency matrix of $\mathcal{Q}(n)$.

Example

Application of Theorem[D. Cvetković, P. Rowlinson, S. Simić, 2010] to the above example

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Divisor matrix B of the obtained equitable partition for $\mathcal{Q}(6)$

$$B = \begin{pmatrix} 3 & 4 & 1 & 4 & 0 & 2 \\ 2 & 4 & 2 & 2 & 4 & 1 \\ 2 & 4 & 3 & 2 & 4 & 2 \\ 2 & 2 & 1 & 4 & 4 & 2 \\ 0 & 4 & 2 & 4 & 4 & 3 \\ 2 & 2 & 1 & 4 & 6 & 3 \end{pmatrix}$$

Example

Application of Theorem[D. Cvetković, P. Rowlinson, S. Simić, 2010] to the above example

Divisor matrix B of the obtained equitable partition for $\mathcal{Q}(6)$

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Characteristic polynomial of the divisor matrix B

$$p(x) = x^6 - 21x^5 + 77x^4 + 89x^3 - 690x^2 + 720x - 245.$$

Open Problems

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