Solving a linear system

(a tribute to the memory of A.L.)

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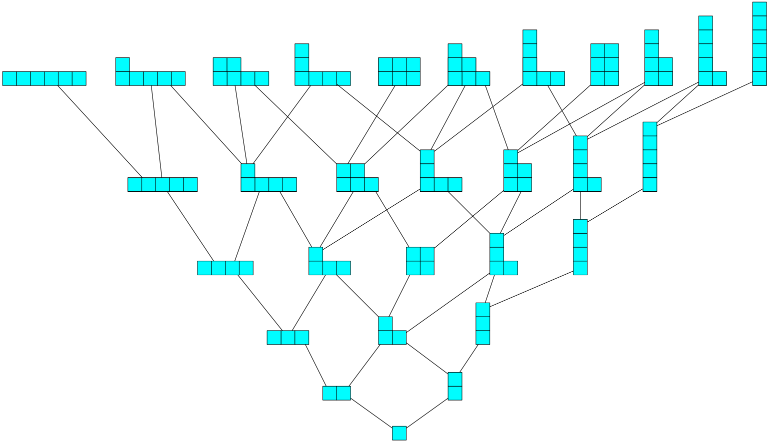
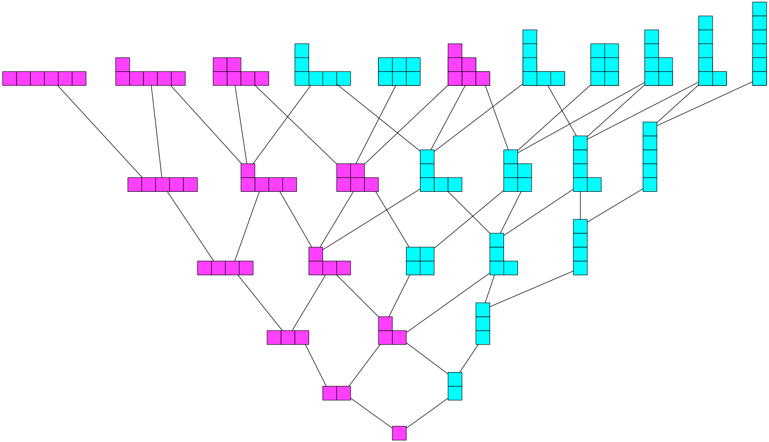
82nd Séminaire Lotharingien de Combinatoire,
Curia (Portugal), April 2019

What?

the partition lattice

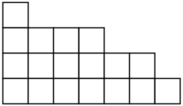
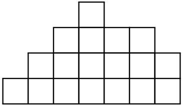


the sublattice $\mathcal{S}$ of strict partitions



strict partitions

- Strict partitions can be denoted/illustrated as

partitions	:	$(7, 6, 4, 1)$
binary sequences	:	1101001
decimal integers	:	105
diagrams	:	
shifted diagrams	:	

- The covering relation of $\mathcal{S}$ in binary notation (switching 01 to 10 in position r)

$$b_\ell \ldots b_{r+2}\,\underline{0\,1}\,b_{r-1} \ldots b_2\,b_1 \quad \lessdot_r \quad b_\ell \ldots b_{r+2}\,\underline{1\,0}\,b_{r-1} \ldots b_2\,b_1$$

strict partitions

Strict partitions written as partitions

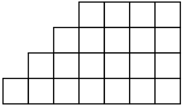
$$
\begin{array}{ccccccccccccccc}
 & & & & & & (3,2,1) & \rightarrow & (4,2,1) & \rightarrow & (4,3,1) & \rightarrow & (4,3,2) & \rightarrow & (4,3,2,1) \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & & & & & (3,2) & \rightarrow & (4,2) & \rightarrow & (4,3) & & & & \\
 & & & & & & \uparrow & & \uparrow & & & & & & \\
 & & & & (2,1) & \rightarrow & (3,1) & \rightarrow & (4,1) & & & & & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & & & & & \\
\varnothing & \rightarrow & (1) & \rightarrow & (2) & \rightarrow & (3) & \rightarrow & (4) & & & & & &
\end{array}
$$

join-irreducible strict partitions

- The *join-irreducible* strict partitions are precisely the partitions

$$\boldsymbol{\lambda} = (n, n-1, \ldots, n-k+1, n-k)$$

$$\leftrightarrow \quad 00\ldots0\underbrace{11\ldots1}_{k+1}\underbrace{00\ldots0}_{n-k-1}$$



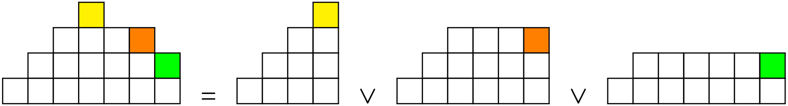
$<7, 3>$

- Writing $<n, k>$ for the partition $\boldsymbol{\lambda}$, the poset of join-irreducibles displays as

$$\begin{array}{ccccccc}
 & & & & & & <4,3> \\
 & & & & & & \uparrow \\
 & & & & <3,2> & \rightarrow & <4,2> \\
 & & & & \uparrow & & \uparrow \\
 & & <2,1> & \rightarrow & <3,1> & \rightarrow & <4,1> \\
 & & \uparrow & & \uparrow & & \uparrow \\
<1,0> & \rightarrow & <2,0> & \rightarrow & <3,0> & \rightarrow & <4,0>
\end{array}$$

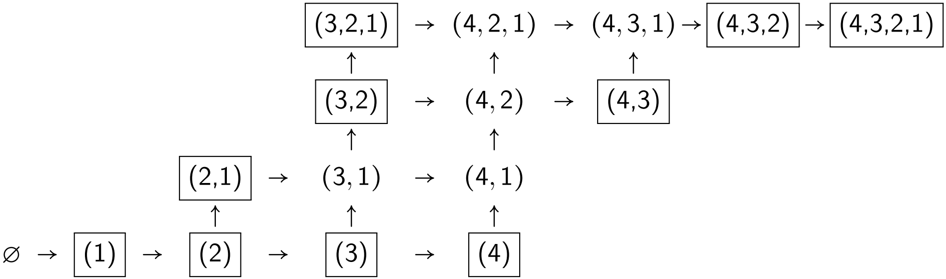
join-irreducible strict partitions

- Join-irreducibles are the "backbone" of the (distributive) lattice of strict partitions
- Strict partitions are (bijectively) joins over antichains (or lower ideals) of the poset of join-irreducibles



join-irreducible strict partitions

Join-irreducibles in the lattice of strict partitions



join-irreducible strict partitions

Poset of join-irreducibles

$$
\begin{array}{ccccccc}
 & & & & & & (4,3,2,1) \\
 & & & & & & \uparrow \\
 & & & & (3,2,1) & \to & (4,3,2) \\
 & & & & \uparrow & & \uparrow \\
 & & (2,1) & \to & (3,2) & \to & (4,3) \\
 & & \uparrow & & \uparrow & & \uparrow \\
(1) & \to & (2) & \to & (3) & \to & (4)
\end{array}
$$

shifted tableaux

- A *shifted standard (Young) tableau* (sSYT) for a shifted diagram of a strict partition $\boldsymbol{\lambda}$ of size n is
  - a filling of the n boxes with $1, 2, \ldots, n$ (bijectively) that is strictly increasing along rows and columns, e.g.,

			12		
		7	11		
	3	5	10	13	
1	2	4	6	8	9

shifted tableaux

- For $\boldsymbol{\lambda} = (4,2)$ the set of *sSYT*s of shape $\boldsymbol{\lambda}$ consists of

	5	6	
1	2	3	4

	4	6	
1	2	3	5

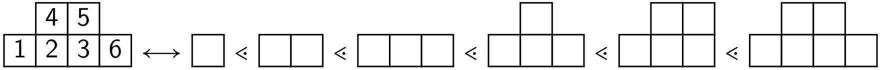
	4	5	
1	2	3	6

	3	6	
1	2	4	5

	3	5	
1	2	4	6

- Such a sSYT t is nothing but a covering sequence of strict partitions

$$t : \varnothing \lessdot \boldsymbol{\lambda}^{(1)} \lessdot \boldsymbol{\lambda}^{(2)} \lessdot \boldsymbol{\lambda}^{(3)} \lessdot \ldots \lessdot \boldsymbol{\lambda}^{(s)} \lessdot \boldsymbol{\lambda}^{(s+1)} \lessdot \ldots \lessdot \boldsymbol{\lambda}^{|\boldsymbol{\lambda}|} = \boldsymbol{\lambda}$$



weighted shifted tableaux

- $X = \{x_0, x_1, x_2, \ldots\}$: set of variables, $X_{a,b} = \{x_a, x_{a+1}, \ldots, x_{b-1}, x_b\}$
- The *weight* of a strict partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is

$$w(\boldsymbol{\lambda}) = x_{\lambda_1} + x_{\lambda_2} + \cdots + x_{\lambda_k} (+x_0).$$

  x_0 is taken or not so as to make the total number of summands even
- The *total weight of a tableau t* of (shifted) shape $\boldsymbol{\lambda}$ is then

$$w(t) = \prod_{1 \leq s \leq |\boldsymbol{\lambda}} w(\boldsymbol{\lambda}^{(s)})$$

- As an abbreviation

$$x_{abc\ldots} = x_{a,b,c,\ldots} = x_a + x_b + x_c \cdots$$

weighted shifted tableaux

- a shifted tableau t for $\boldsymbol{\lambda} = (4,2,1)$

		7	
	4	6	
1	2	3	5

- as a weighted sequence of strict partitions

i	1	2	3	4	5	6	7
$\boldsymbol{\lambda}^{(i)}$	(1)	(2)	(3)	$(3,1)$	$(4,1)$	$(4,2)$	$(4,2,1)$
$w(\boldsymbol{\lambda}^{(i)})$	x_0+x_1	x_0+x_2	x_0+x_3	x_1+x_3	x_1+x_4	x_2+x_4	$x_0+x_1+x_2+x_4$

- and with total weight

$$w(t) = (x_0+x_1)(x_0+x_2)(x_0+x_3)(x_1+x_3)(x_1+x_4)(x_2+x_4)(x_0+x_1+x_2+x_4)$$

$$= x_{01}x_{02}x_{03}x_{13}x_{14}x_{24}x_{0124}$$

the problem

- Problem: For the strict partitions $\boldsymbol{\lambda} \in \mathcal{S}$ compute

$$Y_{\boldsymbol{\lambda}} = \sum \left\{ \frac{1}{w(t)} \,;\, t \in sSYT(\boldsymbol{\lambda}) \right\}$$

- Equivalently: Solve the linear system

$$w(\boldsymbol{\lambda}) \cdot Y_{\boldsymbol{\lambda}} = \sum \left\{ Y_{\boldsymbol{\mu}} \,;\, \boldsymbol{\mu} \lessdot \boldsymbol{\lambda} \right\} \quad (\boldsymbol{\lambda} \in \mathcal{S}), \qquad Y_{\varnothing} = 1$$

  where $\lessdot$ is the covering relation in the lattice of strict partitions
- The $Y_{\boldsymbol{\lambda}}$ are rational functions in the $x_0, x_1, x_2, \ldots$.
- What about a solution in "closed form" ?

the linear system

$$
\begin{aligned}
(x_0 + x_1)\, Y_{(1)} &= Y_{\varnothing} \\
(x_0 + x_2)\, Y_{(2)} &= Y_{(1)} \\
(x_1 + x_2)\, Y_{(2,1)} &= Y_{(2)} \\
(x_0 + x_3)\, Y_{(3)} &= Y_{(2)} \\
(x_1 + x_3)\, Y_{(3,1)} &= Y_{(2,1)} + Y_{(3)} \\
(x_2 + x_3)\, Y_{(3,2)} &= Y_{(3,1)} \\
(x_0 + x_1 + x_2 + x_3)\, Y_{(3,2,1)} &= Y_{(3,2)} \\
(x_0 + x_4)\, Y_{(4)} &= Y_{(3)} \\
(x_1 + x_4)\, Y_{(4,1)} &= Y_{(3,1)} + Y_{(4)} \\
(x_2 + x_4)\, Y_{(4,2)} &= Y_{(3,2)} + Y_{(4,1)} \\
(x_0 + x_1 + x_2 + x_4)\, Y_{4,2,1)} &= Y_{(3,2,1)} + \\
\vdots &= \vdots
\end{aligned}
$$

first values

$$Y_{(1)} \to \frac{1}{x_0 + x_1}$$

$$Y_{(2)} \to \frac{1}{(x_0 + x_1)(x_0 + x_2)}$$

$$Y_{(2,1)} \to \frac{1}{(x_1 + x_2)(x_0 + x_1)(x_0 + x_2)}$$

$$Y_{(3)} \to \frac{1}{(x_0 + x_1)(x_0 + x_2)(x_0 + x_3)}$$

$$Y_{(3,1)} \to \frac{x_0 + x_1 + x_2 + x_3}{(x_0 + x_1)(x_0 + x_2)(x_0 + x_3)(x_1 + x_2)(x_1 + x_3)}$$

$$Y_{(3,2)} \to \frac{x_0 + x_1 + x_2 + x_3}{(x_0 + x_1)(x_0 + x_2)(x_0 + x_3)(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)}$$

$$Y_{(3,2,1)} \to \frac{1}{(x_0 + x_1)(x_0 + x_2)(x_0 + x_3)(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)}$$

- $Y_{(4,2,1)}$:

$$\frac{x_3^2 + x_2x_3 + x_4x_3 + x_4^2 + x_2x_4 + x_1\left(x_2 + x_3 + x_4\right) + x_0\left(x_1 + x_2 + x_3 + x_4\right)}{\prod\left\{x_i + x_j\,;\, 0 \leq i < j \leq 4; i < 3\right\}}$$

- $Y_{(4,3)}$:

$$\frac{1}{\prod\left\{x_i + x_j\,;\, 0 \leq i < j \leq 4\right\}}\Big(\left(x_1 + x_2 + x_3 + x_4\right)x_0^2 + \left(x_1 + x_2 + x_3 + x_4\right)^2 x_0 + \\ + x_1\left(x_2 + x_3 + x_4\right)^2 + \left(x_2 + x_3\right)\left(x_2 + x_4\right)\left(x_3 + x_4\right) + x_1^2\left(x_2 + x_3 + x_4\right)\Big)$$

- $Y_{(4,3,1)}$ defined:

$$Y_{13} = \frac{1}{x_0 + x_1 + x_3 + x_4}\left(Y_{(4,2,1)} + Y_{(4,3)}\right)$$

- $Y_{(4,3,1)}$ computed:

$$\frac{\left(x_0 + x_2 + x_3 + x_4\right)\left(x_1 + x_2 + x_3 + x_4\right)}{\prod\left\{x_i + x_j\,;\, 0 \leq i < j \leq 4\right\}}$$

– $Y_{(4,3,2)}$ defined:

$$Y_{(4,3,2)} = \frac{1}{x_0 + x_2 + x_3 + x_4} Y_{(4,3,1)}$$

– $Y_{(4,3,2)}$ computed:

$$Y_{(4,3,2)} = \frac{x_1 + x_2 + x_3 + x_4}{\prod \left\{ x_i + x_j \,;\, 0 \le i < j \le 4 \right\}}$$

– $Y_{(4,3,2,1)}$ defined:

$$Y_{(4,3,2,1)} = \frac{1}{x_1 + x_2 + x_3 + x_4} Y_{(4,3,2)}$$

– $Y_{(4,3,2,1)}$ computed:

$$Y_{(4,3,2,1)} = \frac{1}{\prod \left\{ x_i + x_j \,;\, 0 \le i < j \le 4 \right\}}$$

numerator of $Y_{(5,2)}$

$$
\begin{aligned}
&\Big(x_1^2 + (x_2 + x_3 + x_4 + x_5)\, x_1 + x_2^2 + x_3 x_4 + x_3 x_5 + x_4 x_5 + x_2\, (x_3 + x_4 + x_5)\Big)\, x_0^3 \\
&\quad + \Big(x_1^3 + 2\, (x_2 + x_3 + x_4 + x_5)\, x_1^2 + \Big(2x_2^2 + 3\, (x_3 + x_4 + x_5)\, x_2 + x_3^2 + x_4^2 + x_5^2 + 3x_4 x_5 + 3x_3\, (x_4 + x_5)\Big)\, x_1 \\
&\qquad + x_2^3 + x_3 x_4^2 + x_3 x_5^2 + x_4 x_5^2 + x_3^2 x_4 + x_3^2 x_5 + x_4^2 x_5 + 3x_3 x_4 x_5 + 2x_2^2\, (x_3 + x_4 + x_5) \\
&\qquad + x_2\, \Big(x_3^2 + 3\, (x_4 + x_5)\, x_3 + x_4^2 + x_5^2 + 3x_4 x_5\Big)\Big)\, x_0^2 \\
&\quad + \Big(\big((x_2 + x_3 + x_4 + x_5)\, x_1^3 + \Big(2x_2^2 + 3\, (x_3 + x_4 + x_5)\, x_2 + x_3^2 + x_4^2 + x_5^2 + 3x_4 x_5 + 3x_3\, (x_4 + x_5)\Big)\, x_1^2 \\
&\quad + \Big(x_2^3 + 3\, (x_3 + x_4 + x_5)\, x_2^2 + \Big(2x_3^2 + 5\, (x_4 + x_5)\, x_3 + 2x_4^2 + 2x_5^2 + 5x_4 x_5\Big)\, x_2 + 2x_3^2\, (x_4 + x_5) + 2x_4 x_5\, (x_4 + x_5) \\
&\qquad + x_3\, \Big(2x_4^2 + 5x_5 x_4 + 2x_5^2\Big)\Big)\, x_1 + (x_4 x_5 + x_3\, (x_4 + x_5))^2 + x_2^3\, (x_3 + x_4 + x_5) \\
&\quad + x_2^2\, \Big(x_3^2 + 3\, (x_4 + x_5)\, x_3 + x_4^2 + x_5^2 + 3x_4 x_5\Big) + x_2\, \Big(2\, (x_4 + x_5)\, x_3^2 + \Big(2x_4^2 + 5x_5 x_4 + 2x_5^2\Big)\, x_3 + 2x_4 x_5\, (x_4 + x_5)\Big)\, x_0 \\
&\quad + (x_2 + x_3)\, (x_2 + x_4)\, (x_2 + x_5)\, (x_4 x_5 + x_3\, (x_4 + x_5))\big)\Big) + x_1^3\, \Big(x_2^2 + (x_3 + x_4 + x_5)\, x_2 + x_4 x_5 + x_3\, (x_4 + x_5)\Big) \\
&\quad + x_1^2\, \Big(x_2^3 + 2\, (x_3 + x_4 + x_5)\, x_2^2 + \Big(x_3^2 + 3\, (x_4 + x_5)\, x_3 + x_4^2 + x_5^2 + 3x_4 x_5\Big)\, x_2 + x_3^2\, (x_4 + x_5) + x_4 x_5\, (x_4 + x_5) \\
&\quad + x_3\, \Big(x_4^2 + 3x_5 x_4 + x_5^2\Big)\Big) + x_1\, \Big((x_3 + x_4 + x_5)\, x_2^3 + \Big(x_3^2 + 3\, (x_4 + x_5)\, x_3 + x_4^2 + x_5^2 + 3x_4 x_5\Big)\, x_2^2 + \Big(2\, (x_4 + x_5)\, x_3^2 \\
&\qquad + \Big(2x_4^2 + 5x_5 x_4 + 2x_5^2\Big)\, x_3 + 2x_4 x_5\, (x_4 + x_5)\Big)\, x_2 + (x_4 x_5 + x_3\, (x_4 + x_5))^2
\end{aligned}
$$

denominator of $Y_{(5,2)}$

$$(x_0 + x_1)\, (x_0 + x_2)\, (x_1 + x_2)\, (x_0 + x_3)\, (x_1 + x_3)\, (x_2 + x_3)\, (x_0 + x_4)\, (x_1 + x_4)\, (x_2 + x_4)\, (x_0 + x_5)\, (x_1 + x_5)\, (x_2 + x_5)$$

about denominators

- For $n > k \geq 0$ define *standard denominator polynomials*

$$q_{n,k}(x_0, x_1, x_2, \ldots, x_n) = \prod_{0 \leq i \leq k} \prod_{i < j \leq n} (x_i + x_j) = \prod_{(j,i) \leq (n,k)} (x_i + x_j)$$

- The polynomials $q_{n,k}$ are in 1-1-correspondence with the join irreducibles in the lattice $\mathcal{S}$
- $q_{n,k}(x_0, x_1, x_2, \ldots, x_n)$ is a polynomial that is separately symmetric in $X_{0,k} = \{x_0, x_1, \ldots, x_k\}$ and in $X_{k+1,n} = \{x_{k+1}, x_{k+2}, \ldots, x_n\}$.
- The case $k = n - 1$ is special, however, because $q_{n,n-1}$ is symmetric in all variables $X_{0,n} = \{x_0, x_1, \ldots, x_n\}$.
- The degree of $q_{n,k}$ is

$$\sum_{i=0}^{k} (n - i) = (k + 1)n - \binom{k+1}{2} = \frac{(k + 1)(2n - k)}{2}.$$

about denominators

- From the computed data one is led to conjecture
  - For all $\boldsymbol{\lambda} \in \mathcal{S}$ with $\boldsymbol{\lambda} = (n, k, \ldots)$

$$Y_{\boldsymbol{\lambda}} = \frac{p_{\boldsymbol{\lambda}}}{q_{n,k}},$$

    where $p_{\boldsymbol{\lambda}}$ is a polynomial, $q_{n,k}$ is the standard denominator polynomial, i.e., the denominator of $Y_{\boldsymbol{\lambda}}$
    - depends only on the two largest parts of $\boldsymbol{\lambda}$
    - is a product of binomials
  - This is indeed true!

a special case: partitions $(n,1)$

- Task: compute the rational functions $Y_{\boldsymbol{\lambda}}$ for the very special case of 2-part partitions of type $\boldsymbol{\lambda} = (n,1)$ for all $n \geq 2$.
- One-part partitions are easy: because of

$$Y_{(n)} = \frac{1}{x_0 + x_n} Y_{(n-1)} \quad \text{for} \quad n > 0$$

we have

$$Y_{(n)} = \frac{1}{\prod_{1\leq i\leq n}(x_0 + x_i)} = \frac{1}{q_{n,0}}.$$

- The computation of $Y_{(n,1)}$ in general is not so obvious. One gets as first values (writing $x_{abc\ldots}$ for $x_a + x_b + x_c + \cdots$)

$$Y_{(2,1)} = \frac{1}{x_{12}} Y_{(2)} = \frac{1}{x_{01}x_{02}x_{12}} = \frac{1}{q_{2,1}}$$

$$Y_{(3,1)} = \frac{1}{x_{13}}\left(Y_{(2,1)} + Y_{(3)}\right) = \frac{x_{0123}}{x_{01}x_{02}x_{03}x_{12}x_{13}} = \frac{x_{03} + x_{12}}{q_{3,1}}$$

a special case: partitions $(n,1)$

$$Y_{(4,1)} = \frac{1}{x_{14}}\left(Y_{(3,1)} + Y_{(4)}\right) = \ldots = \frac{x_{03}x_{04} + x_{12}x_{04} + x_{12}x_{13}}{q_{4,1}}$$

$$Y_{(5,1)} = \frac{1}{x_{15}}\left(Y_{(4,1)} + Y_{(5)}\right)$$

$$= \ldots = \frac{x_{05}x_{04}x_{03} + x_{05}x_{04}x_{12} + x_{05}x_{13}x_{12} + x_{14}x_{13}x_{12}}{q_{5,1}}$$

- Examples suggests that in general

$$Y_{(n,1)} = \frac{\sum_{\boldsymbol{\lambda}} \prod_{i=1}^{n-2}(x_{\lambda_i} + x_{\lambda'_i})}{q_{n,1}}$$

where $\sum_{\boldsymbol{\lambda}}$ runs over all $\boldsymbol{\lambda} \in \mathcal{S}$ with

$$\underline{\boldsymbol{\tau}}^{(n)} = (n-1, n-2, \ldots, 3, 2) \leq \boldsymbol{\lambda} \leq \overline{\boldsymbol{\tau}}^{(n)} = (n, n-1, \ldots 4, 3)$$

$$\boldsymbol{\lambda}' = \overline{\boldsymbol{\tau}}^{(n)} - \boldsymbol{\lambda}$$

$\boldsymbol{\lambda}'$ is not a strict partition, but a vector of type $0^\ell 1^{n-2-\ell}$.

a special case: partitions $(n, 1)$

- To illustrate this in the case $n = 5$:
  - The relevant $\boldsymbol{\lambda}$ with $\underline{\boldsymbol{\tau}}^{(5)} = 432 \leq \boldsymbol{\lambda} \leq \overline{\boldsymbol{\tau}}^{(5)} = 543$ are

$\boldsymbol{\lambda}'$	000	001	011	111
$\boldsymbol{\lambda}$	543	542	532	432

  - Compare with

$$Y_{(5,1)} = \frac{x_{05}x_{04}x_{03} + x_{05}x_{04}x_{12} + x_{05}x_{13}x_{12} + x_{14}x_{13}x_{12}}{q_{5,1}}$$

a special case: partitions $(n, 1)$

This is routinely verified by induction:

$$Y_{(n+1,1)} = \frac{1}{x_1 + x_{n+1}} \left(Y_{(n,1)} + Y_{(n+1)} \right)$$

$$= \frac{1}{x_1 + x_{n+1}} \left(\frac{\sum_{\boldsymbol{\lambda}} \prod_{i=1}^{n-2} (x_{\lambda_i} + x_{\lambda'_i})}{q_{n,1}} + \frac{1}{q_{n+1,0}} \right)$$

$$= \frac{1}{q_{n+1,1}} \frac{1}{(x_1 + x_{n+1})} \times$$

$$\times \left(\sum_{\boldsymbol{\lambda}} \prod_{i=1}^{n-2} (x_{\lambda_i} + x_{\lambda'_i}) \cdot (x_0 + x_{n+1})(x_1 + x_{n+1}) + \prod_{1<j\le n+1} (x_1 + x_j) \right)$$

$$= \frac{1}{q_{n+1,1}} \left(\underbrace{\sum_{\boldsymbol{\lambda}} \prod_{i=1}^{n-2} (x_{\lambda_i} + x_{\lambda'_i}) \cdot (x_0 + x_{n+1})}_{\boldsymbol{\mu}=(n+1,\boldsymbol{\lambda})} + \underbrace{\prod_{1<j\le n} (x_1 + x_j)}_{\boldsymbol{\mu}=(n,n-1,\dots,3,2)} \right)$$

a special case: partitions $(n, 1)$

- One can rewrite the formula for $Y_{(n,1)}$ in a much neater way in terms of symmetric functions. Indeed,

$$Y_{(n,1)} = \frac{1}{q_{n,1}} \sum_{k=0}^{n-2} h_k(X_{0,1}) \cdot e_{n-2-k}(X_{2,n}),$$

  where the $h_k(A)$ resp. $e_\ell(B)$ denote the homogeneous resp. elementary symmetric functions over the alphabets A resp. B.

- Once one has made this guess, it is a routine matter to verify that indeed both sides of

$$\sum_{\boldsymbol{\lambda}} \prod_{i=1}^{n-2} (x_{\lambda_i} + x_{\lambda'_i}) = \sum_{k=0}^{n-2} h_k(X_{0,1}) \cdot e_{n-2-k}(X_{2,n})$$

  contain the same monomials.

a special case: partitions $(n, 1)$

As an illustration for $n = 4$:

$$
\begin{aligned}
h_2(X_{0,1})e_0(X_{2,4} + h_1(X_{0,1})e_1(X_{2,4}) + h_0(X_{0,1}e_2(X_{2,4})
&= \dots \\
&= x_1^2 + (x_0 + x_1 + x_2 + x_3)(x_0 + x_4) + x_1(x_2 + x_3) + x_2x_3 \\
&= (x_0 + x_3)(x_0 + x_4) + (x_2 + x_3)(x_0 + x_4) + (x_1 + x_2)(x_2 + x_3)
\end{aligned}
$$

a special case: partitions $(n, 1)$

- This identifies the numerator of $Y_{(n,1)}$ as a Schur polynomial over a pair alphabets, viz.

$$\sum_{k=0}^{n-2} h_k(X_{0,1}) \cdot e_{n-2-k}(X_{2,n}) = S_{n-2}(X_{0,1}|X_{2,n}).$$

- This view is interesting because the denominator polynomials $q_{n,1}$, and the $q_{n,m}$ in general, can be written as Schur polynomials over a pair of alphabets:

$$q_{n,m} = S_{\widetilde{<n,m>}}(X_{0,m}|X_{m+1,n}) = S_{<n,m>}(X_{m+1,n}|X_{0,m}),$$

where $<n,m>= (n, n-1, \ldots, n-m)$ and where $\widetilde{<n,m>}$ is the conjugate partition of $<n,m>$.

How?

Schur functions (1)

- Schur symmetric functions
  - For $A = \{x_1, x_2, \ldots, x_a\}$ and a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ with $n \le a$

$$s_{\boldsymbol{\lambda}}(A) = \frac{\det \left| x_i^{\lambda_j + a - j} \right|_{1 \le i,j \le a}}{\det \left| x_i^{a-j} \right|_{1 \le i,j \le a}}$$

  - Equivalently (Jacobi-Trudi identities)

$$s_{\boldsymbol{\lambda}}(A) = \det \left| h_{\lambda_i - i + j}(A) \right|_{1 \le i,j \le n} = \det \left| e_{\widetilde{\lambda}_i - i + j}(A) \right|_{1 \le i,j \le n}$$

  - In particular

$$s_{(k)}(A) = h_k(A) \qquad s_{(1^k)}(A) = e_k(A)$$

  - We will use

$$s_{\boldsymbol{\lambda}}(A) = \det \begin{bmatrix} h_{\lambda_n} & h_{\lambda_{n-1}+1} & \cdots & h_{\lambda_1+n-1} \\ h_{\lambda_n-1} & h_{\lambda_{n-1}} & \cdots & h_{\lambda_1+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-n+1} & h_{\lambda_{n-1}-n+2} & \cdots & h_{\lambda_1} \end{bmatrix}_A$$

Schur functions (2)

- The determinantal form of $s_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} = (4,2,1,0)$, i.e. $(\lambda_j + a - j)_{j=1..4} = (7,4,2,0)$

$$s_{(4,2,1)} = \frac{\det \begin{vmatrix} x_1^7 & x_1^4 & x_1^2 & x_1^0 \\ x_2^7 & x_2^4 & x_2^2 & x_2^0 \\ x_3^7 & x_3^4 & x_3^2 & x_3^0 \\ x_4^7 & x_4^4 & x_4^2 & x_4^0 \end{vmatrix}}{\det \begin{vmatrix} x_1^3 & x_1^2 & x_1^1 & x_1^0 \\ x_2^3 & x_2^2 & x_2^1 & x_2^0 \\ x_3^3 & x_3^2 & x_3^1 & x_3^0 \\ x_4^3 & x_4^2 & x_4^1 & x_4^0 \end{vmatrix}}$$

- The Schur function $s_{(4,2,1)}$ in terms of monomial functions

$$2m_{3,2,2} + m_{3,3,1} + m_{4,2,1} + 6m_{2,2,2,1} + 4m_{3,2,1,1} + 2m_{4,1,1,1}$$

Schur functions (3)

- Jacobi-Trudi identities

$$s_{(4,2,1)} = \det \begin{vmatrix} h_1 & h_3 & h_6 \\ h_0 & h_2 & h_5 \\ 0 & h_1 & h_4 \end{vmatrix}$$

$$= h_1 h_2 h_4 - h_3 h_4 - h_1^2 h_5 + h_1 h_6$$

$$= \det \begin{vmatrix} e_1 & e_3 & e_4 & e_6 \\ e_0 & e_1 & e_3 & e_5 \\ 0 & e_0 & e_2 & e_4 \\ 0 & 0 & e_1 & e_3 \end{vmatrix}$$

$$= e_1^2 e_2 e_3 - e_2^2 e_3 - e_1 e_3^2 - e_1^3 e_4 + e_1 e_2 e_4 + e_3 e_4$$

Schur functions (4)

- The Schur function $s_{(4,2,1)}$
  - in terms of monomial functions

$$2m_{3,2,2} + m_{3,3,1} + m_{4,2,1} + 6m_{2,2,2,1} + 4m_{3,2,1,1} + 2m_{4,1,1,1}$$

  - in terms of elementary functions

$$e_1^2 e_2 e_3 - e_2^2 e_3 - e_1 e_3^2 - e_1^3 e_4 + e_1 e_2 e_4 + e_3 e_4$$

  - in terms of homogeneous functions

$$h_1 h_2 h_4 - h_3 h_4 - h_1^2 h_5 + h_1 h_6$$

Schur functions (5)

- The Schur function $s_{4,2,1}(\{x_1,\ldots,x_4\})$ fully expanded

$$x_1^4x_2^2x_3 + x_1^3x_2^3x_3 + x_1^2x_2^4x_3 + x_1^4x_2x_3^2 + 2x_1^3x_2^2x_3^2 + 2x_1^2x_2^3x_3^2 + x_1x_2^4x_3^2$$
$$+ x_1^3x_2x_3^3 + 2x_1^2x_2^2x_3^3 + x_1x_2^3x_3^3 + x_1^2x_2x_3^4 + x_1x_2^2x_3^4 + x_1^4x_2^2x_4 + x_1^3x_2^3x_4 + x_1^2x_2^4x_4$$
$$+ 2x_1^4x_2x_3x_4 + 4x_1^3x_2^2x_3x_4 + 4x_1^2x_2^3x_3x_4 + 2x_1x_2^4x_3x_4 + x_1^4x_3^2x_4 + 4x_1^3x_2x_3^2x_4$$
$$+ 6x_1^2x_2^2x_3^2x_4 + 4x_1x_2^3x_3^2x_4 + x_2^4x_3^2x_4 + x_1^3x_3^3x_4 + 4x_1^2x_2x_3^3x_4 + 4x_1x_2^2x_3^3x_4 + x_2^3x_3^3x_4$$
$$+ x_1^2x_3^4x_4 + 2x_1x_2x_3^4x_4 + x_2^2x_3^4x_4 + x_1^4x_2x_4^2 + 2x_1^3x_2^2x_4^2 + 2x_1^2x_2^3x_4^2 + x_1x_2^4x_4^2$$
$$+ x_1^4x_3x_4^2 + 4x_1^3x_2x_3x_4^2 + 6x_1^2x_2^2x_3x_4^2 + 4x_1x_2^3x_3x_4^2 + x_2^4x_3x_4^2 + 2x_1^3x_3^2x_4^2 + 6x_1^2x_2x_3^2x_4^2$$
$$+ 6x_1x_2^2x_3^2x_4^2 + 2x_2^3x_3^2x_4^2 + 2x_1^2x_3^3x_4^2 + 4x_1x_2x_3^3x_4^2 + 2x_2^2x_3^3x_4^2 + x_1x_3^4x_4^2 + x_2x_3^4x_4^2$$
$$+ x_1^3x_2x_4^3 + 2x_1^2x_2^2x_4^3 + x_1x_2^3x_4^3 + x_1^3x_3x_4^3 + 4x_1^2x_2x_3x_4^3 + 4x_1x_2^2x_3x_4^3 + x_2^3x_3x_4^3$$
$$+ 2x_1^2x_3^2x_4^3 + 4x_1x_2x_3^2x_4^3 + 2x_2^2x_3^2x_4^3 + x_1x_3^3x_4^3 + x_2x_3^3x_4^3 + x_1^2x_2x_4^4 + x_1x_2^2x_4^4$$
$$+ x_1^2x_3x_4^4 + 2x_1x_2x_3x_4^4 + x_2^2x_3x_4^4 + x_1x_3^2x_4^4 + x_2x_3^2x_4^4$$

Schur functions (6)

- $\Delta_a = \langle a, a-1 \rangle = (a, a-1, a-2, \ldots, 2, 1)$: staircase of size $\binom{a+1}{2}$.
- alphabet $X_{\ell,m} = \{x_\ell, x_{\ell+1}, \ldots, x_m\}$

(1) If $X_{\ell,m}$ is an alphabet of size a, then

$$s_{\Delta_a}(X_{\ell,m}) = \prod_{\ell \leq i \leq m} x_i \cdot \prod_{\ell \leq j < k \leq m} (x_j + x_k)$$

(2) If $X_{\ell,m}$ is an alphabet of size $a+1$, then

$$s_{\Delta_a}(X_{\ell,m}) = \prod_{\ell \leq j < k \leq m} (x_j + x_k)$$

- If $\sharp A > a+1$ then $s_{\Delta_a}(A)$ does not factor (let alone factor into linear factors)
- Obviously $s_{\Delta_a}(A) = 0$ if $\sharp A < a$.

main results (1): denominators

- For arbitrary strict partitions $\boldsymbol{\lambda} = (n, k, \ldots)$ the denominator of $Y_{\boldsymbol{\lambda}}$ depends only on λ_1 and λ_2 and is

$$q_{n,k} = \prod_{\substack{0 \le i < j \le n \\ i \le k}} (x_i + x_j)$$

- The *lcm* of the denominators of the $Y_{\boldsymbol{\lambda}}$ with $\max \boldsymbol{\lambda} \le n$

$$Y_{\varnothing} = 1,\ Y_{(1)} = \frac{1}{x_0 + x_1}, \ldots, Y_{(n,n-1,\ldots,1)} = \frac{1}{\prod_{0 \le i < j \le n} (x_i + x_j)}$$

is

$$s_{\Delta_n}(X_{0,n}) = \prod_{0 \le i < j \le n} (x_i + x_j)$$

main results (2): join-irreducibles

- For join-irreducible strict partitions
  $\boldsymbol{\lambda} = \langle n, k \rangle = (n, n-1, \ldots, n-k+1, n-k)$

$$Y_{\boldsymbol{\lambda}} = \frac{S_{\Delta_{n-k-1}}\left(X_{k+1 \bmod 2, n}\right)}{S_{\Delta_n}\left(X_{0,n}\right)}$$

main results (2): join-irreducibles

- Important special cases for join-irreducible $\boldsymbol{\lambda}$ are $k \in \{n-2, n-1, 0\}$:

$$Y_{(n,n-1,\ldots,2)} = \begin{cases} \dfrac{s_{\Delta_1}(X_{0,n})}{s_{\Delta_n}(X_{0,n})} = \dfrac{x_0 + x_1 + \cdots + x_n}{\prod_{0 \le i < j \le n}(x_i + x_j)} & \textit{if } n \textit{ is odd} \\[2ex] \dfrac{s_{\Delta_1}(X_{1,n})}{s_{\Delta_n}(X_{0,n})} = \dfrac{x_1 + x_2 + \cdots + x_n}{\prod_{0 \le i < j \le n}(x_i + x_j)} & \textit{if } n \textit{ is even} \end{cases}$$

$$Y_{(n,n-1,\ldots,1)} = \begin{cases} \dfrac{s_{\Delta_0}(X_{0,n})}{s_{\Delta_n}(X_{0,n})} = \dfrac{1}{\prod_{0 \le i < j \le n}(x_i + x_j)} & \textit{if } n \textit{ is odd} \\[2ex] \dfrac{s_{\Delta_0}(X_{1,n})}{s_{\Delta_n}(X_{0,n})} = \dfrac{1}{\prod_{0 \le i < j \le n}(x_i + x_j)} & \textit{if } n \textit{ is even} \end{cases}$$

$$Y_{(n)} = \frac{s_{\Delta_n}(X_{1,n})}{s_{\Delta_n}(X_{0,n})} = \frac{\prod_{1 \le i < j \le n}(x_i + x_j)}{\prod_{0 \le i < j \le n}(x_i + x_j)} = \frac{1}{\prod_{1 \le j \le n}(x_0 + x_j)}$$

- These cases have to be proved independently since they are needed as induction bases for the proof of the main result

Schur functions (7)

- Schur functions over two alphabets A and B
  - Definition of $S_n(A|B)$ for $n \in \mathbb{N}$

$$\sum_{n\geq 0} S_n(A|B)t^n = \frac{\prod_{\beta\in B}(1+\beta t)}{\prod_{\alpha\in A}(1-\alpha t)}$$

$$S_n(A|B) = \sum_{k=0}^{n} h_k(A)\cdot e_{n-k}(B)$$

  - Definition of $S_{\boldsymbol{\lambda}}(A|B)$ for partitions $\boldsymbol{\lambda} = (\lambda_1 \ldots, \lambda_n)$ via an JT determinant

$$S_{\boldsymbol{\lambda}}(A|B) = \det \begin{vmatrix} S_{\lambda_n} & S_{\lambda_{n-1}+1} & \ldots & S_{\lambda_1+n-1} \\ S_{\lambda_n-1} & S_{\lambda_{n-1}} & \ldots & S_{\lambda_1+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{\lambda_n-n+1} & S_{\lambda_{n-1}-n+2} & \ldots & S_{\lambda_1} \end{vmatrix}_{(A|B)}$$

main results (3): two-part partitions

- For two-part strict partitions $\boldsymbol{\lambda} = (n, m)$ with $n > m \geq 1$:

$$Y_{(n,m)} = \frac{S_{(n-2,n-3,\ldots,n-m-1)}(X_{0,m}|X_{m+1,n})}{\prod_{\substack{0 \leq i < j \leq n \\ i \leq m}} (x_i + x_j)}$$

divided differences

- For $f(x_0, x_1, \ldots)$ and $r \geq 0$ let
$$f^{(r)}(x_0, x_1, \ldots, x_r, x_{r+1}, \ldots) = f(x_0, x_1, \ldots, x_{r+1}, x_r, \ldots)$$
and
$$f\,\delta_r = \frac{f^{(r)} - f}{x_r - x_{r+1}}$$
- The δ_r are "derivations" with product rule
$$(f \cdot g)\delta_r = f^{(r)} \cdot (g\delta_r) + (f\delta_r) \cdot g$$
and "Coxeter type relations"
$$\delta_r \circ \delta_r = 0 \qquad \delta_r \circ \delta_{r+1} \neq \delta_{r+1} \circ \delta_r$$
$$\delta_r \circ \delta_s = \delta_s \circ \delta_r \quad \text{if } |r - s| \geq 2 \qquad \delta_r \circ \delta_{r+1} \circ \delta_r = \delta_{r+1} \circ \delta_r \circ \delta_{r+1}$$
- The δ_r are symmetrizing operators: $f\delta_r$ is symmetric w.r.t. $x_r \leftrightarrow x_{r+1}$, but other symmetries f had involving x_r or x_{r+1} may no longer exist in $f\delta_r$

i

covering for strict partitions and divided differences

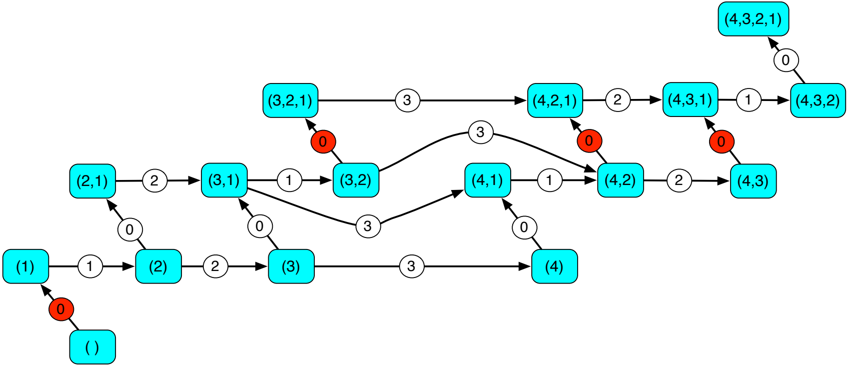
- Covering in the lattice of strict partitions:

$$\boldsymbol{\lambda} \lessdot_r \boldsymbol{\mu} \quad \leftrightarrow \quad \begin{cases} r \in \boldsymbol{\lambda}, \\ r+1 \notin \boldsymbol{\lambda} \\ \boldsymbol{\mu} = \boldsymbol{\lambda}|_{r \to r+1} \end{cases}$$

- The key property:

$$\boldsymbol{\lambda} \lessdot_r \boldsymbol{\mu} \quad \Rightarrow \quad Y_{\boldsymbol{\lambda}}\, \delta_r = \begin{cases} Y_{\boldsymbol{\mu}} & \text{if } r > 0 \text{ or } |\boldsymbol{\lambda}| \text{ odd} \\ 0 & \text{if } r = 0 \text{ and } |\boldsymbol{\lambda}| \text{ even} \end{cases}$$

strict partitions and divided differences



about the 2-part case

- Recall the claim:

$$Y_{(n,m)} = \frac{S_{(n-2,n-3,\ldots,n-m-1)}(X_{0,m}|X_{m+1,n})}{\prod_{\substack{0\le i<j\le n\\ i\le m}}(x_i+x_j)}$$

- Consider partitions $\boldsymbol{\lambda} = (n, m)$ with $n > m \geq 1$ and their extensions $(n+1, m)$ and $(n, m+1)$ (if $m < n-1$):

$$(n, m) \lessdot_n (n+1, m) \qquad (n, m) \lessdot_m (n, m+1)$$

- Hence

$$Y_{n+1,m} = Y_{n,m}\,\delta_n \qquad Y_{n,m+1} = Y_{n,m}\,\delta_m$$

about the 2-part case

- Plugging in the asserted results and clearing denominators one gets equivalent polynomial relations (writing from now on $(a..b)$ for $X_{a,b}$)

$$\left[S_{m+1}(n+1|0..m)\cdot S_{n-2,n-3,\ldots,n-m-1}(0..m|m+1..n)\right]\delta_n = S_{n-1,n-2,\ldots,n-m}(0..m|m+1..n+1)$$

and

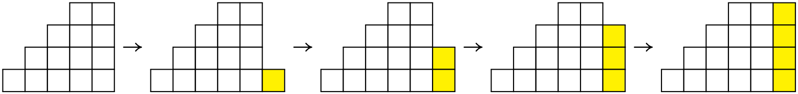
$$\left[S_{n-m+1}(m+1|m+2..n)\cdot S_{n-2,n-3,\ldots,n-m-1}(0..m|m+1..n)\right]\delta_m = S_{n-2,n-3,\ldots,n-m-2}(0..m+1|m+2..n)$$

join-irreducible strict partitions and divided differences

$$\delta_{abc} \quad \equiv \quad \delta_a \delta_b \delta_c$$

$$\begin{array}{ccccccccc}
 & & & & & & & & (5,4,3,2,1) \\
 & & & & & & & & \uparrow \delta_0 \\
 & & & & & & (4,3,2,1) & \xrightarrow{\delta_{4321}} & (5,4,3,2) \\
 & & & & & & \uparrow \delta_0 & & \uparrow \delta_{01} \\
 & & & & (3,2,1) & \xrightarrow{\delta_{321}} & (4,3,2) & \xrightarrow{\delta_{432}} & (5,4,3) \\
 & & & & \uparrow \delta_0 & & \uparrow \delta_{01} & & \uparrow \delta_{012} \\
 & & (2,1) & \xrightarrow{\delta_{21}} & (3,2) & \xrightarrow{\delta_{32}} & (4,3) & \xrightarrow{\delta_{43}} & (5,4) \\
 & & \uparrow \delta_0 & & \uparrow \delta_{01} & & \uparrow \delta_{012} & & \uparrow \delta_{0123} \\
(1) & \xrightarrow{\delta_1} & (2) & \xrightarrow{\delta_2} & (3) & \xrightarrow{\delta_3} & (4) & \xrightarrow{\delta_4} & (5)
\end{array}$$

about the join-irreducible case



$$\langle n,k\rangle = \langle 5,3\rangle \longrightarrow \langle 6,3\rangle = \langle n+1,k\rangle$$

via

$$\begin{array}{rcl}
\langle n,k\rangle & = & (n, n-1, n-2, \ldots, n-k) \\
 & \lessdot_n & (n+1, n-1, n-2, \ldots, n-k) \\
 & \lessdot_{n-1} & (n+1, n, n-2, \ldots, n-k) \\
 & \vdots & \\
 & \lessdot_{n-k} & (n+1, n, n-1, \ldots, n+1-k) \\
 & = & \langle n+1,k\rangle
\end{array}$$

one has

$$Y_{\langle n,k\rangle} \xrightarrow{\delta_{n,n-1,\ldots,n-k}} Y_{\langle n+1,k\rangle}$$

about the join-irreducible case

- Plugging in

$$Y_{\triangleleft n,k\triangleright} \stackrel{\delta_{n,n-1,\dots,n-k}}{\longrightarrow} Y_{\triangleleft n+1,k\triangleright}$$

the asserted results and clearing denominators one gets equivalent <u>polynomial</u> relation

$$\left[S_{n+1}(x_{n+1}|X_{0,n}) \cdot S_{\Delta_{n-k-1}}(X_{\varepsilon,n})\right]\delta_n\delta_{n-1}\dots\delta_{n-k} = S_{\Delta_{n-k}}(X_{\varepsilon,n+1}).$$

about the join-irreducible case

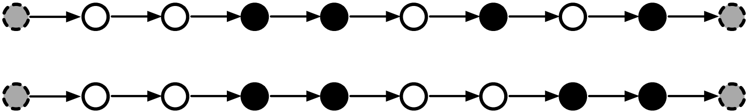
- A crucial step in the proof requires the evaluation of expressions like

$$\sum_{\substack{0 \le k \le 2a+1 \\ k \equiv_2 b-a}} h_k(B) \cdot \det \begin{bmatrix} e_1 & e_3 & \dots & e_{2a-1} & e_{2a+1-k} \\ e_0 & e_2 & \dots & e_{2a-2} & e_{2a-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & \dots & \dots & e_a & e_{a+2-k} \\ \dots & \dots & \dots & e_{a-1} & e_{a+1-k} \end{bmatrix}_{A+B}$$

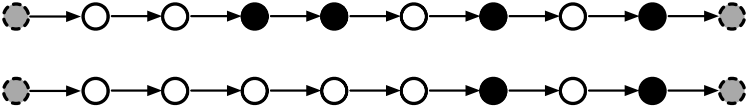
Why?

the asymmetric exclusion process with annihilation (Ayyer, Mallick)

- right shift

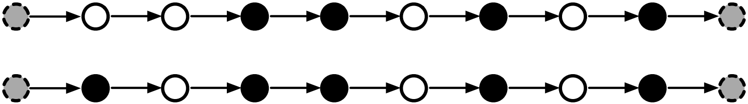


- annihilation

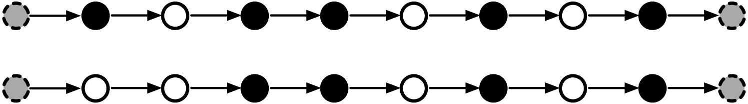


the asymmetric exclusion process with annihilation

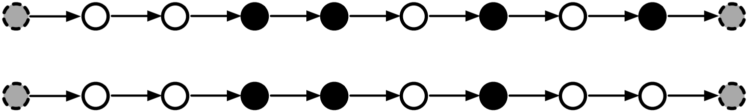
- left creation



- left annihilation



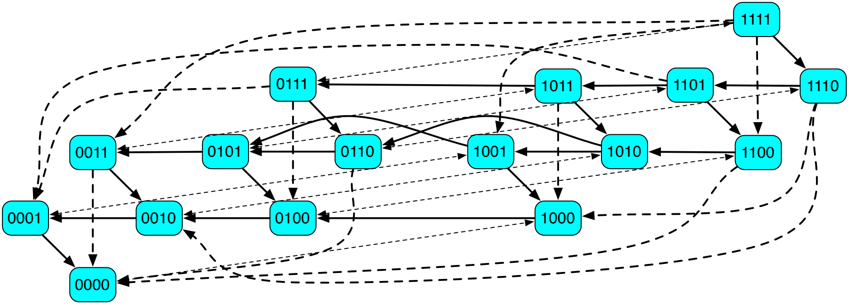
- right annihilation



encoding of the process

- notation
  - $\mathbb{B}^n$: binary vectors of length n
  - vectors and matrices indexed by $\mathbb{B}^n$ (lexicographically)
- the process
  - states encoded as elements of $\mathbb{B}^n$
  - transitions "in the bulk" with rate = 1
    - $\ldots 10 \ldots \mapsto \ldots 01 \ldots$ (right shift)
    - $\ldots 11 \ldots \mapsto \ldots 00 \ldots$ (annihilation)
  - at the left end with rate = α
    - $0 \ldots \mapsto 1 \ldots$ (left creation)
    - $1 \ldots \mapsto 0 \ldots$ (left annihilation)
  - at the right end with rate = β
    - $\ldots 1 \mapsto \ldots 0$ (right annihilation)
- $M_n(\alpha, \beta)$: generator matrix of a continuous-time Markov chain

dominance order with annihilation



generator matrix for $n = 3$

$$M_3 = \begin{bmatrix} \star & \beta & & 1 & \alpha & & 1 & \\ & \star & 1 & & & \alpha & & 1 \\ & & \star & \beta & 1 & & \alpha & \\ & & & \star & & 1 & & \alpha \\ \alpha & & & & \star & \beta & & 1 \\ & \alpha & & & & \star & 1 & \\ & & \alpha & & & & \star & \beta \\ & & & \alpha & & & & \star \end{bmatrix}$$

$\alpha : 0xy \to 1xy,\ 1xy \to 0xy$ — left creation and annihilation

$1 : 11y \to 00y,\ 10y \to 01y$ — right shift and annihilation

$1 : x11 \to x00,\ x10 \to x01$ — right shift and annihilation

$\beta : xy1 \to xy0$ — right annihilation

$\star$: column sums $= 0$ — $\langle 1 \ldots 1|$ as left eigenvector

results for $M_n(\alpha, \beta)$ (Ayyer, Mallick)

- the model admits a "transfer matrix Ansatz":
  there are matrices $T_{n+1,n}$ of size $2^{n+1} \times 2^n$ s.th.

$$T_{n+1,n} \cdot M_n(\alpha,\beta) = M_{n+1}(\alpha,\beta) \cdot T_{n+1,n}$$

- nontrivial kernel vectors v_n of M_n can be computed inductively:

$$M_n \, |v_n\rangle = 0 \quad \Rightarrow \quad M_{n+1} \underbrace{T_{n+1,n} \, |v_n\rangle}_{|v_{n+1}\rangle} = 0$$

  starting with $|v_1\rangle = \begin{bmatrix} \alpha + \beta \\ \alpha \end{bmatrix}$

- the partition function of M_n can be obtained from v_n:

$$Z_n(\alpha,\beta) = \langle 1_n \,|\, v_n \rangle = 2^{\binom{n-1}{2}} (1+2\alpha)^{n-1} (1+\beta)^{n-1} (2\alpha+\beta)$$

transfer matrices by induction

$$
\left(\begin{array}{c|c}
\begin{array}{c} T_1 M_{L-1} + T_2 M_{L-1}/\alpha \\ -\alpha T_1(\sigma \otimes \mathbb{1}) + 2\alpha T_2 \end{array} & \begin{array}{c} \alpha T_1 + 2T_2 M_{L-1} - 2T_2 + T_1(\sigma \otimes \mathbb{1}) \\ -2\alpha T_2(\sigma \otimes \mathbb{1}) + T_2(\sigma \otimes \mathbb{1}) M_{L-1}/\alpha \end{array} \\
\hline
\begin{array}{c} T_2 M_{L-1} + T_2 \\ -\alpha T_2(\sigma \otimes \mathbb{1}) \end{array} & \begin{array}{c} \alpha T_2 - T_2/\alpha \\ +T_2 M_{L-1}/\alpha \end{array} \\
\hline
2T_2 M_{L-1} - \alpha T_2(\sigma \otimes \mathbb{1}) & \alpha T_2 + T_2(\sigma \otimes \mathbb{1}) + T_2(\sigma \otimes \mathbb{1}) M_{L-1} \\
\hline
\alpha T_2 & T_2 M_{L-1} - T_2 - \alpha T_2(\sigma \otimes \mathbb{1})
\end{array}\right),
$$

$$
\left(\begin{array}{c|c}
\begin{array}{c} M_{L-1} T_1 + M_{L-1} T_2/\alpha \\ -\alpha(\sigma \otimes \mathbb{1}) T_1 + 2\alpha T_2 \\ \end{array} & \begin{array}{c} T_2 + 2M_{L-1} T_2 - \alpha(\sigma \otimes \mathbb{1}) T_2 \\ +(\sigma \otimes \mathbb{1}) T_2/\alpha + M_{L-1} T_2(\sigma \otimes \mathbb{1})/\alpha \\ -(\sigma \otimes \mathbb{1}) T_2(\sigma \otimes \mathbb{1}) \end{array} \\
\hline
\begin{array}{c} M_{L-1} T_2 + T_2 \\ -\alpha(\sigma \otimes \mathbb{1}) T_2 \end{array} & \begin{array}{c} \alpha T_2 + M_{L-1} T_2/\alpha \\ -T_2/\alpha - (\sigma \otimes \mathbb{1}) T_2 + T_2(\sigma \otimes \mathbb{1}) \end{array} \\
\hline
\begin{array}{c} \alpha T_1 + 2M_{L-1} T_2 - T_2 \\ -2\alpha(\sigma \otimes \mathbb{1}) T_2 \end{array} & \begin{array}{c} 2\alpha T_2 + M_{L-1} T_2(\sigma \otimes \mathbb{1}) \\ +(\sigma \otimes \mathbb{1}) T_2 - \alpha(\sigma \otimes \mathbb{1}) T_2(\sigma \otimes \mathbb{1}) \end{array} \\
\hline
\alpha T_2 & M_{L-1} T_2 - T_2 - \alpha(\sigma \otimes \mathbb{1}) T_2
\end{array}\right)
$$

the eigenvalue conjecture for $M_n(\alpha, \beta)$ (Ayyer, Mallick)

- the characteristic polynomial of M_n is given by

$$A_n(z)\, A_n(z + 2\alpha + \beta)\, B_n(z + \beta)\, B_n(z + 2\alpha)$$

where

$$A_n(z) = \prod_{k \geq 0} (z + 2k)^{\binom{n-1}{2k}} \qquad B_n(z) = \prod_{k \geq 0} (z + 2k + 1)^{\binom{n-1}{2k+1}}$$

- proved by myself, the proof works in a more general setting
- main ingredient: the M_n are Hadamard-conjugate to triangular matrices

$$H_n \cdot M_n \cdot H_n \simeq \widetilde{M_n}$$

where

$$H_n = \frac{1}{2^{n/2)}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{\otimes n}$$

generalized generator matrix for $n = 3$

$$M_3(\alpha,\beta,\gamma,\delta) = \begin{bmatrix} \star & \delta & & \gamma & \alpha & & \beta & \\ & \star & \gamma & & & \alpha & & \beta \\ & & \star & \delta & \beta & & \alpha & \\ & & & \star & & \beta & & \alpha \\ \alpha & & & & \star & \delta & & \gamma \\ & \alpha & & & & \star & \gamma & \\ & & \alpha & & & & \star & \delta \\ & & & \alpha & & & & \star \end{bmatrix}$$

α : $0\,x\,y \to 1\,x\,y$, $1\,x\,y \to 0\,x\,y$ — left creation and annihilation

β : $1\,1\,y \to 0\,0\,y$, $1\,0\,y \to 0\,1\,y$ — right shift and annihilation

γ : $x\,1\,1 \to x\,0\,0$, $x\,1\,0 \to x\,0\,1$ — right shift and annihilation

δ : $x\,y\,1 \to x\,y\,0$ — right annihilation

$\star$: column sums $= 0$

about transfer matrices for the generalized process

- M_n : generator matrix for the asymmetric annihilation process with n sites and site variables $x_0, x_1, \dots, x_n$
- Hadamard-conjugation works here as well!
- wanted: transfer matrices $T_{n,n-1}$ (format $2^n \times 2^{n-1}$) with

$$M_n \cdot T_{n,n-1} = T_{n,n-1} \cdot M_{n-1}$$

- the partition functions can be obtained inductively once the $T_{n,n-1}$ are known
- I have a created an algebraic setting (skew tensor product) in order to describe and deduce the transfer matrices and the partition functions

deriving the partition functions inductively

- the steady state vectors (= right kernel vectors) of the M_n are given by

$$\left|v_1\right\rangle = \begin{bmatrix} x_0 + x_1 \\ x_0 \end{bmatrix} \qquad \left|v_k\right\rangle = T_{k,k-1}\left|v_{k-1}\right\rangle$$

- and the partition functions are obtained by induction

$$\begin{aligned} Z_n &= \langle 1_n \mid v_n \rangle \\ &= \langle 1_n \mid T_{n,n-1} \mid v_{n-1} \rangle \\ &\vdots \\ &= (2x_0 + x_n)(x_1 + x_n)\cdots(x_{n-1} + x_n) Z_{n-1} \\ &= \prod_{1 \le j \le n} (2x_0 + x_j) \cdot \prod_{1 \le i < j \le n} (x_i + x_j) \end{aligned}$$

compare the matrices

$$M_3 = \begin{bmatrix} -\alpha & \delta & & \gamma & \alpha & & \beta & \\ & -\alpha-\delta & \gamma & & & \alpha & & \beta \\ & & -\alpha-\gamma & \delta & \beta & & \alpha & \\ & & & -\alpha-\gamma-\delta & & \beta & & \alpha \\ \alpha & & & & -\alpha-\beta & \delta & & \gamma \\ & \alpha & & & & -\alpha-\beta-\delta & \gamma & \\ & & \alpha & & & & -\alpha-\beta-\gamma & \delta \\ & & & \alpha & & & & -\alpha-\beta-\gamma-\delta \end{bmatrix}$$

$$\widetilde{M}_3 = \begin{bmatrix} & \beta & & \gamma & & & \delta & \\ & -2\,\alpha-\beta & \gamma & & & & & \delta \\ & & -2\,\alpha-\gamma & \beta & \delta & & & \\ & & & -\gamma-\beta & & \delta & & \\ & & & & -2\,\alpha-\delta & \beta & & \gamma \\ & & & & & -\beta-\delta & \gamma & \\ & & & & & & -\delta-\gamma & \beta \\ & & & & & & & -2\,\alpha-\delta-\gamma-\beta \end{bmatrix}$$

the partition functions as seen via Hadamard transform

- we know from $M_n|v_n\rangle = |0_n\rangle$ that

$$Z_n = \prod_{1\leq j\leq n}(2x_0 + x_j)\cdot\prod_{1\leq i<j\leq n}(x_i + x_j)$$
$$= \langle 1_n|v_n\rangle = \langle 1_n|H_n|H_n|v_n\rangle = \langle \varepsilon_n|H_n|v_n\rangle$$

  where $\langle\varepsilon_n| = \langle 1_n|H_n = 2^{-n/2}(1,0,0,\ldots,0)$
- From

$$\underbrace{H_n\cdot M_n\cdot H_n}_{\sim\widetilde{M}_n^t}\cdot H_n|v_n\rangle = H_n\cdot M_n|v_n\rangle = |0_n\rangle$$

- Z_n is the first component of the left kernel of the triangular(!) matrix $\widetilde{M}_n$ (suitably normalized)
- in other words: the partition functions Z_n are the

$$\text{denominators of the}\quad Y_{2^n-1}\quad : \quad S_{\Delta_n}(X_{0,n}) = \prod_{0\leq i<j\leq n}(x_i + x_j)$$

  with $x_0 \to 2x_0$