The generalised Delta square conjecture

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joint work with Michele D’Adderio and Alessandro Iraci

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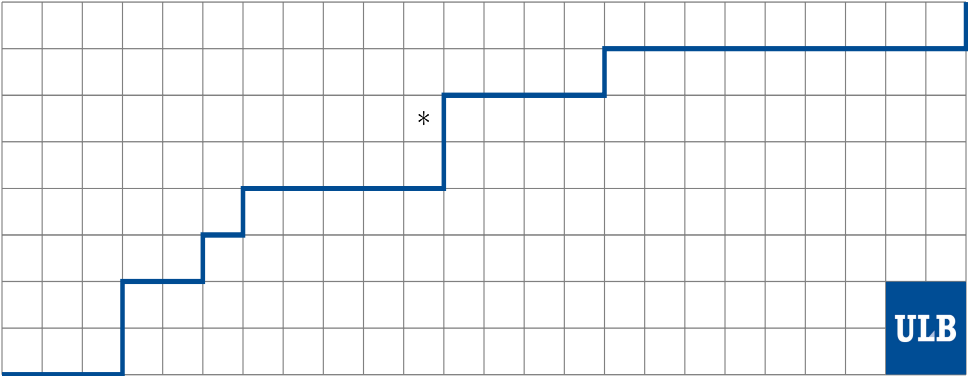

MacDonald Polynomials

$\Lambda_{\mathbb{C}(q,t)} := \mathbb{C}(q,t)[X_1, ..., X_N]^{\mathfrak{S}_N} = \bigoplus_{i=1}^{\infty} \Lambda^n_{\mathbb{C}(q,t)}$

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Macdonald Positivity Conjecture

$\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$, i.e. the Macdonald polynomials are *Schur positive*

$n!$ conjecture

Strategy to prove Schur positivity of Macdonald Polynomials

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- Proved by Haiman in 2001, using tools from Algebraic Geometry

The Delta operators

Working on the Macdonald positivity conjecture, Garsia and Haiman introduced the $\mathfrak{S}_N$-module DH_n of diagonal harmonics. It turns out that

$$\mathcal{F}(DH_n; q, t) = \nabla e_n$$

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- The Delta operators, for some $f \in \Lambda$ are defined by

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- Just a few weeks ago, Zabrocki found a module extending the diagonal harmonics, whose bi-graded Frobenius characteristic he conjectured to be $\Delta'_{e_{n-k-1}} e_n$.

Combinatorial interpretations

Combinatorial interpretations

Function	*Conjecture*	*Proof*
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$\frac{[n-k]_t}{[n]_t} \Delta_{h_m} \Delta_{e_{n-k}} (-1)^{n-1} p_n$	Generalised Delta square conjecture D-I-VW	

The Delta conjecture

$$\Delta'_{e_{n-k-1}} e_n = \sum_{D \in \mathsf{LD}(n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$

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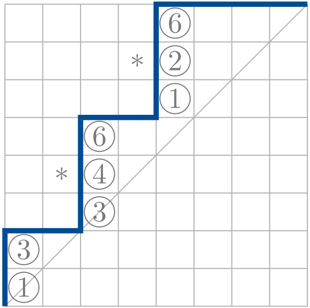

$\mathsf{LD}(n)^{*k}$: labelled decorated Dyck paths

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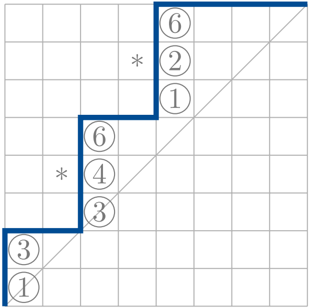

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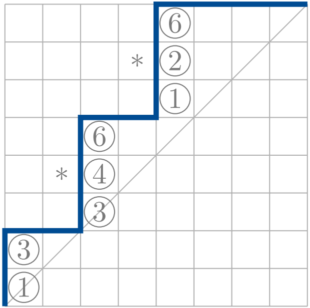

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- k decorations on rises (i.e. vertical steps preceded by another vertical step).

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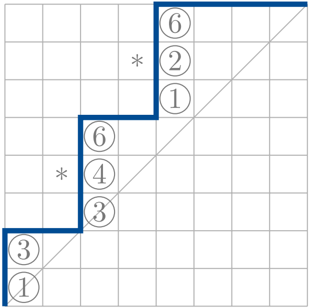

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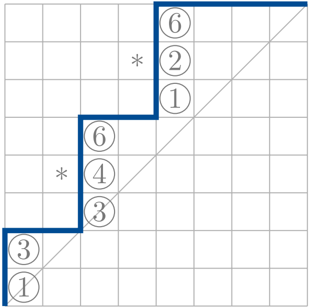

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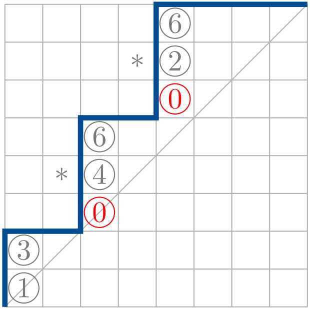
- vertical steps labelled with nonzero, positive integers
- labels strictly increasing in columns

The generalised Delta conjecture

$$\Delta_{h_m} \Delta'_{e_{n-k-1}} e_n = \sum_{D \in \mathsf{PLD}(m,n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$



$\mathsf{PLD}(m,n)^{*k}$: *partially* labelled decorated Dyck paths

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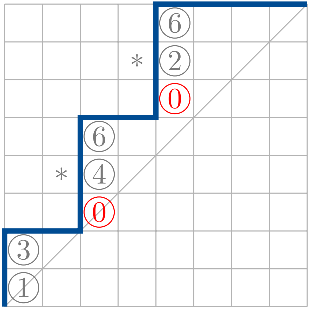

$\mathsf{PLD}(m,n)^{*k}$: *partially* labelled decorated Dyck paths

- m zero labels, n nonzero labels
- first label cannot be zero

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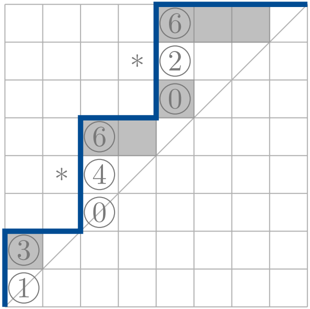

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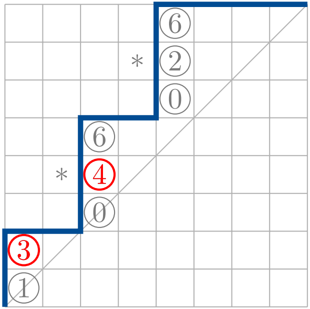

Dinv: count the number of pairs

- same diagonal,
  lower label < upper label
  (primary dinv)

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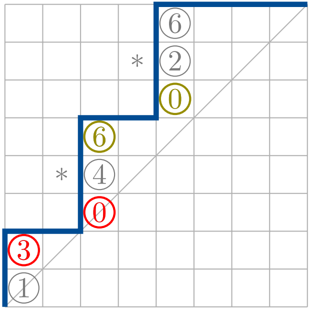

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- lower step one diagonal above upper step
  lower label > upper label
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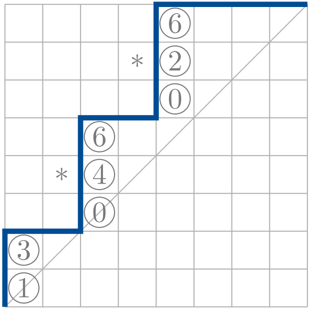

$$x^D := \prod_{i=1}^{m+n} x_{l_i(D)}$$

where $l_i(D)$ is the label of the i-th vertical step of D and we set $x_0 = 1$.

Generalised Delta conjecture: state of the art

Conditions	Reference
$m = 0$ and $k = 0$	Carlsson-Mellit
$m = 0$ and $q = 0$	Garsia-Haglund-Remmel-Yoo
$m = 0$ and $q = 1$	Romero
$m = 0$ and $\langle \cdot, h_{n-d} h_d \rangle$	D'Adderio-Iraci
$\langle \cdot, e_{n-d} h_d \rangle$	D-I-VW
$t = 0$ or $q = 0$	D-I-VW

The generalised Delta square conjecture

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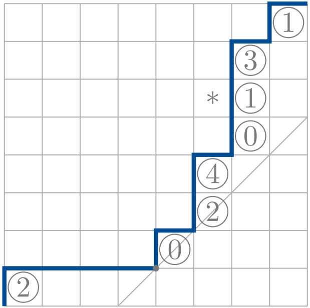

$\mathsf{PLSQ}^{\mathsf{E}}(m,n)^{*k}$: partially labelled, decorated square paths ending east

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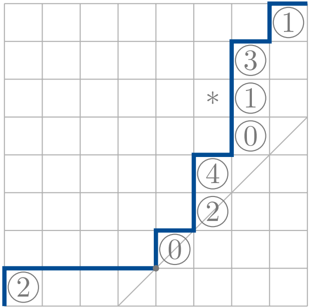

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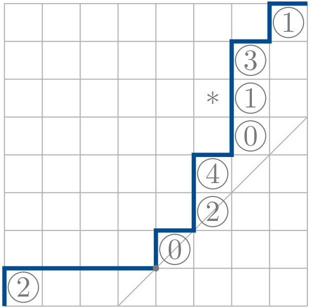

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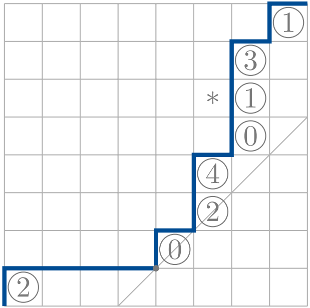

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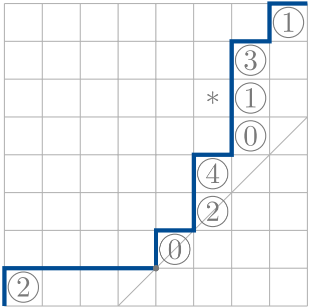

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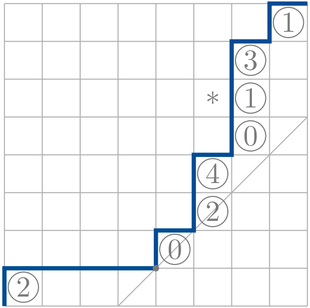

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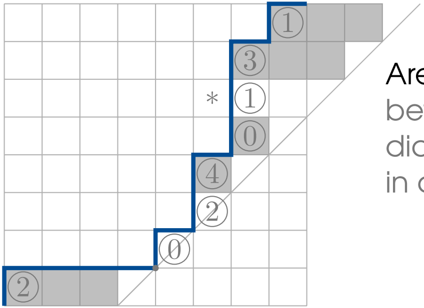

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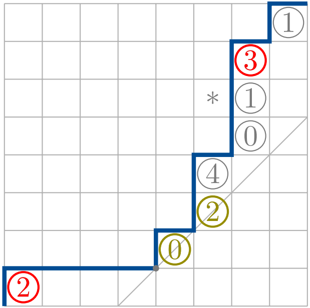

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The generalised Delta square conjecture

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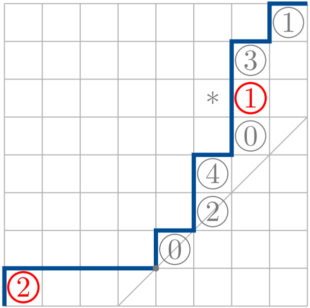

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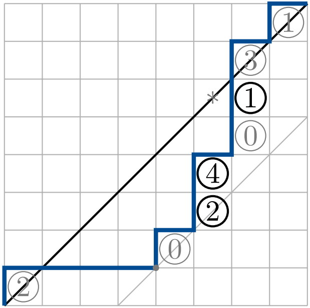

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- Bonus: +1 for every nonzero label under the line $x = y$

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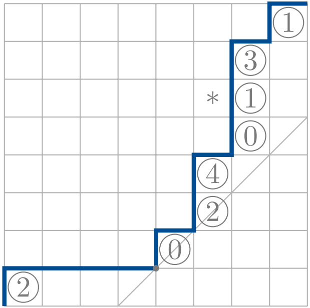

$$x^D := \prod_{i=1}^{m+n} x_{l_i(D)}$$

where $l_i(D)$ is the label of the i-th vertical step of D and we set $x_0 = 1$.

Support for our square conjecture

- $k = m = 0$ is the square conjecture made by Loehr and Warrington, proven by Sergel.

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  - The Schröder case, i.e.

$$\frac{[n-k]_t}{[n]_t}\langle \Delta_{h_m}\Delta_{e_{n-k}}(-1)^{n-1}p_n, e_{n-d}h_d\rangle$$

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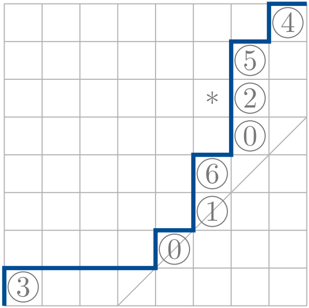

reading word

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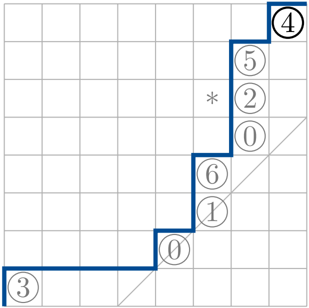

reading word
4

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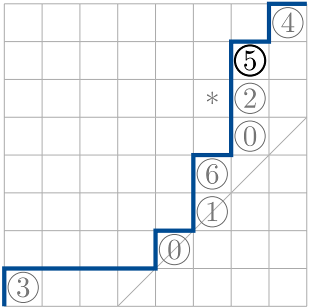

reading word

4 5

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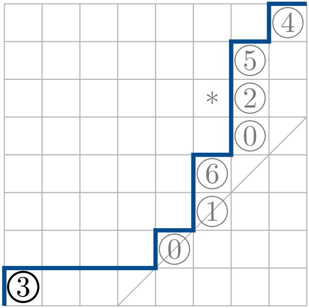

reading word

4 5 3

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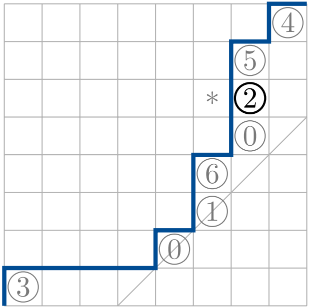

reading word
4 5 3 2

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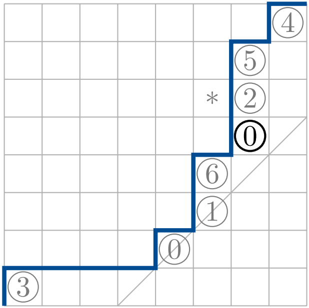

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4 5 3 2 0

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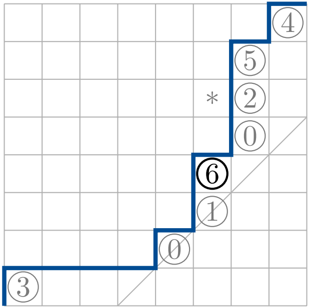

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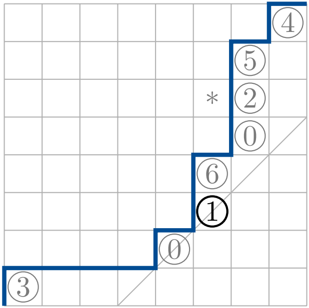

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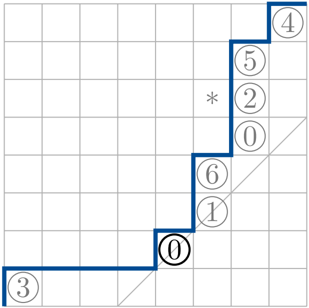

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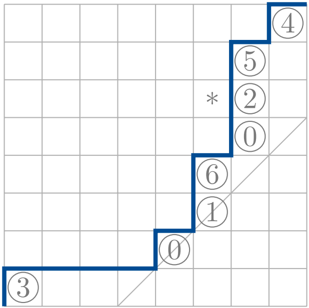

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5 6

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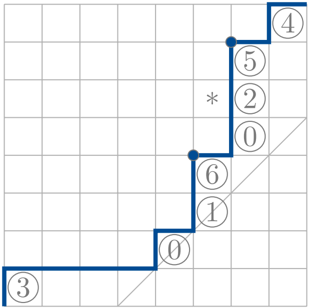

reading word

4 5 3 2 0 6 1 0

				0			0
4		3	2			1	
	5				6		

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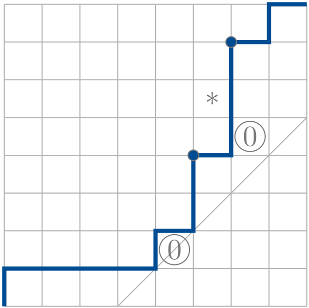

reading word

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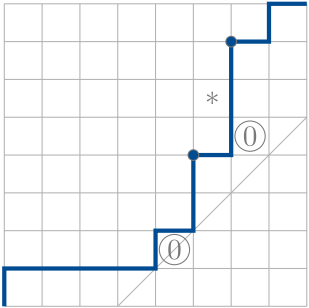

$\mathsf{SQ}^{\mathsf{E}}(m,n)^{*k,\circ d}$

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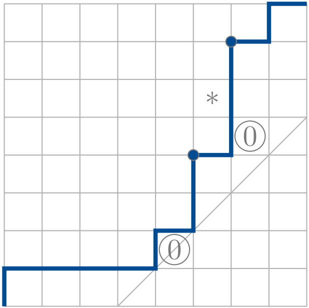

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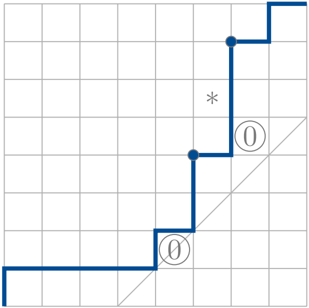

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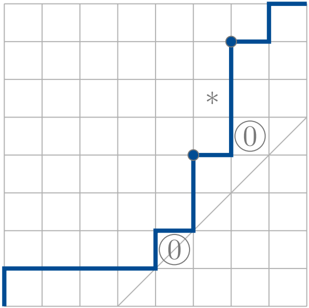

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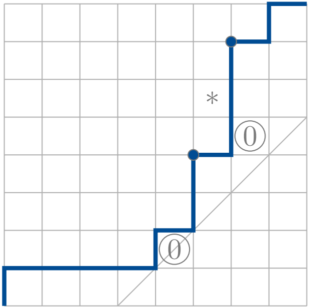

$\mathsf{SQ}^{\mathsf{E}}(m,n)^{*k,\circ d}$

- square paths of ending east of size $m+n$
- m zero labels in valleys
- d decorations on peaks
- dinv induced by the implied labelling.

Schröder case: sketch of the proof

Two families of functions

- $F^{(d,\ell)}_{n,k;p}$ such that

$$\sum_{k=1}^{n-\ell} F^{(d,\ell)}_{n,k;p} = \langle \Delta_{h_p} \Delta'_{e_{n-\ell-1}} e_n, e_{n-d} h_d \rangle.$$

- $S^{(d,\ell)}_{n,k;p}$ such that

$$\sum_{k=1}^{n-\ell} S^{(d,\ell)}_{n,k;p} = \frac{[n-\ell]_t}{[n]_t} \langle \Delta_{h_p} \Delta_{e_{n-\ell}} (-1)^{n-1} p_n, e_{n-d} h_d \rangle.$$

Schröder case: sketch of the proof

Theorem (D-I-VW)

$$F_{n,k;p}^{(d,\ell)} = \sum_{P \in F} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)} \qquad S_{n,k;p}^{(d,\ell)} = \sum_{P \in S} q^{\mathsf{dinv}(P)} t^{\mathsf{area}(P)}$$

Where $F \subseteq S \subseteq \mathsf{SQ}^{\mathsf{E}}(p,n)^{*\ell,\circ d}$ such that

- $P \in F \Rightarrow P$ is a Dyck path
- $P \in S \Leftrightarrow$ the number of vertical steps starting from the lowest diagonal and that are not a zero valley equals k.

To prove this, we show that both sides satisfy the same recursion.

Theorem: recursion for F (D-I-VW)

For $k, \ell, d, p \geq 0$, $n \geq k + \ell$ and $n + p \geq d$, the $F_{n,k;p}^{(d,\ell)}$ satisfy the following recursion: for $n \geq 1$

$$F_{n,n;p}^{(d,\ell)} = \delta_{\ell,0} q^{\binom{n-d}{2}} \begin{bmatrix} n \\ n-d \end{bmatrix} \begin{bmatrix} n+p-1 \\ p \end{bmatrix}$$

and, for $n \geq 1$ and $1 \leq k < n$,

$$\begin{aligned} F_{n,k;p}^{(d,\ell)} = t^{n-k-\ell} \sum_{j=0}^{p} \sum_{s=0}^{k} q^{\binom{s}{2}} \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} k+j-1 \\ j \end{bmatrix}_q \\ \times t^{p-j} \sum_{u=0}^{n-k-\ell} \sum_{v=0}^{s+j} q^{\binom{v}{2}} \begin{bmatrix} s+j \\ v \end{bmatrix}_q \begin{bmatrix} s+j+u-1 \\ u \end{bmatrix}_q F_{n-k,u+v;p-j}^{(d-k+s,\ell-v)}, \end{aligned}$$

with initial conditions

$$F_{0,k;p}^{(d,\ell)} = \delta_{k,0}\delta_{p,0}\delta_{d,0}\delta_{\ell,0} \qquad \text{and} \qquad F_{n,0;p}^{(d,\ell)} = \delta_{n,0}\delta_{p,0}\delta_{d,0}\delta_{\ell,0}.$$

Theorem: recursion for S (D-I-VW)

For $k, \ell, d, p \geq 0$, $n \geq k + \ell$ and $n \geq d$, the $S^{(d,\ell)}_{n,k;p}$ satisfy the following recursion: for $n \geq 1$

$$S^{(d,\ell)}_{n,n;p} = \delta_{\ell,0} q^{\binom{n-d}{2}} \begin{bmatrix} n \\ n-d \end{bmatrix} \begin{bmatrix} n+p-1 \\ p \end{bmatrix}$$

and, for $n \geq 1$ and $1 \leq k < n$,

$$S^{(d,\ell)}_{n,k;p} = F^{(d,\ell)}_{n,k;p} + q^k t^{n-\ell-k} \sum_{j=0}^{p} \sum_{s=0}^{k} q^{\binom{s}{2}} \begin{bmatrix} s+j \\ s \end{bmatrix}_q \begin{bmatrix} k+j-1 \\ s+j-1 \end{bmatrix}_q \times$$

$$\times t^{p-j} \sum_{u=0}^{n-\ell-k} \sum_{v=0}^{s+j} q^{\binom{v}{2}} \begin{bmatrix} u+v \\ v \end{bmatrix}_q \begin{bmatrix} s+j+u-1 \\ s+j-v \end{bmatrix}_q S^{(d-k+s,\ell-v)}_{n-k,u+v;p-j},$$

with initial conditions

$$S^{(d,\ell)}_{0,k;p} = \delta_{k,0}\delta_{p,0}\delta_{d,0}\delta_{\ell,0} \qquad \text{and} \qquad S^{(d,\ell)}_{n,0;p} = \delta_{n,0}\delta_{p,0}\delta_{d,0}\delta_{\ell,0}.$$

Thank you for you attention

Variables for the S recursion

- p is the number of zero valleys.
- n is the number of vertical steps that are not zero valleys.
- k is the number of minima in the area word whose index is not a zero valley.
- ℓ is the number of decorated rises.
- d is the number of decorated peak
- $k - s$ is the number of decorated peaks at height 0.
- s is the number of minima in the area word whose index is not a decorated peak nor a zero valley.
- j is the number of zero valleys at height 0.
- v is the number of decorated rises at height 1.
- $u + v$ is the number of $m + 1$'s in the area word whose index is not a zero valley.

Strategy for the S-recursion

- Start from a path P in $S = \mathsf{SQ}^{\mathsf{E}}(p, n\backslash k)^{*\ell,\circ d}$.
- If it is a Dyck path, thanks to the F recursion it is counted by $F^{(d,\ell)}_{n,k;p}$.
- Otherwise, remove all the minima from the area word, and then remove both the corresponding decoration on peaks, and decorations on rises at height one (which are not rises any more).
- In this way we obtain a path in

$$\mathsf{SQ}^{\mathsf{E}}(p-j, n-k\backslash u+v)^{*\ell-v,\circ d-(k-s)}.$$

Variables for the F recursion

- k is the number of zeroes in the area word whose index is not a zero valley.
- $k - s$ is the number of decorated peaks at height 0.
- The previous two imply that s is the number of zeroes in the area word whose index in not a decorated peak nor a zero valley.
- j is the number of zero valleys at height 0.
- v is the number of decorated rises at height 1.
- $u + v$ is the number of $1'$s in the area word whose index is not a zero valley.

Representation theory

$$\rho : \mathfrak{S}_n \longrightarrow \mathsf{GL}\left(\bigoplus_{(i,j)\in\mathbb{N}\times\mathbb{N}} V^{(j,j)}\right)$$

- $V^{(i,j)}$ are ρ invariant
- Character

$$\chi_\rho = \mathsf{tr} \circ \rho : \mathfrak{S}_n \to \mathbb{C}$$

- We can decompose $\chi_\rho = \sum_{(i,j)} \chi_\rho^{(i,j)}$ and $\chi_\rho^{(i,j)} = \sum c_\lambda \chi_\lambda$ where $c_\lambda \in \mathbb{N}$ (multiplicity) and χ_λ are the irreducible characters of $(\rho_{|V^{(i,j)}}, V^{(i,j)})$ (one per conjugacy class)

Frobenius Characteristic map

$$\mathcal{F} : \text{Class}(\mathfrak{S}_n) \to \Lambda^n_{\mathbb{C}}$$
$$f \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) p_{\lambda(\sigma)}$$

- Irreducible characters get sent to Schur functions
- If a symmetric function is the image of the character of a representation by the Frobenius map then is must be Schur positive because $\mathcal{F}$ is linear
- Bi-graded Frobenius characteristic map

$$\mathcal{F} : \chi_\rho \mapsto \sum_{(i,j)} q^i t^j \mathcal{F}(\chi_\rho^{(i,j)})$$

Symmetric functions

$\Lambda_K := K[X_1, ..., X_N]^{\mathfrak{S}_N}$ space of symmetric functions.

$$\Lambda_K = \bigoplus_{i=1}^{\infty} \Lambda_K^n$$

where Λ_K^n is the space of homogeneous symmetric functions of degree n.

- A lot of different basis for Λ_K^n, indexed by partitions of n: elementary e_λ, homogeneous h_λ, power symmetric p_λ.
- Link with representation theory of $\mathfrak{S}_n$: the Frobenius characteristic map:

$$\mathcal{F} : \text{Class}(\mathfrak{S}_n) \to \Lambda_K^n$$

- Schur functions s_λ form another basis and are the image of the irreducible characters by the Frobenius map.
- Scalar product $\langle , \rangle$ on Λ_K^n such that s_λ are orthonormal $\to \mathcal{F}$ is an isometry