Representation type and combinatorics

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1. Incorporation of the topic into general representation theory

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2. Representing gendo-symmetric algebras by quivers and relations
3. Methods applied to decide the representation type

Incorporation of the topic into general representation theory

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Remark

We will be looking at so-called **gendo-symmetric** *algebras and try to determine, which of these algebras are representation-finite.*

Remember that a K-algebra A is called **symmetric**, iff ${}_A A_A \cong {}_A D(A)_A$ as A-bimodules, where $D(A) := \mathrm{Hom}_K(A, K)$.

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Because they contain other intriguing classes of algebras. E. g., every Schur Algebra $S(n, r)$ with $n \geq r$ is gendo-symmetric.

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- ...but it is closed under almost ν-stable derived equivalences. This is a new kind of derived equivalence, which is more restrictive.
- For symmetric algebras both equivalences coincide.
- Therefore, the classifcation of all representation-finite gendo-symmetric algebras - up to almost ν-stable derived quivalence - would generalize Skowronski's result.

Representing gendo-symmetric algebras by quivers and relations

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- Standard and non-standard penny-farthing algebras

Skowronski gave the representants via quivers and relations.

The following theorem helps us during the classification:

Theorem

Every gendo-symmetric algebra originating from a representation-finite symmetric algebra is almost ν-stable derived equivalent to an algebra isomorphic to $End_A(A_A \oplus M)$, where A is in canonical form (see above) and M is an A-module that does not have any projective direct summand. In case the symmetric algebras are standard, two algebras $End_{A_1}(A_{1A_1} \oplus M_1)$ and $End_{A_2}(A_{2A_2} \oplus M_2)$ with the properties mentioned before are almost ν-stable derived equivalent, if and only if $A_1 \cong A_2$ and $M_1 \cong \Omega^i(M_2)$ for some $i \in \mathbb{Z}$.

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Fact

Almost ν-stable derived equivalences preserve the representation type!

Methods applied to decide the representation type

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  This is a fully developed theory that gives us the following combinatorial procedure to determine the representation type of a finite-dimensional K-algebra A:

We need the following

Theorem

Let $A = KQ/I$ be a connected distributive algebra given by a quiver and an admissible ideal. Let $\pi : \widetilde{P} \to P$ be the universal covering of $P := \vec{A}$. Then, A is representation-finite, iff it satisfies the following two conditions:

1. *P is chainless*
2. *$\widetilde{P}$ contains no algebra of the BHV-list as a full convex subcategory.*

This allows us to use the following algorithm in order to decide the representation type of a finite-dimensional connected algebra of the form $A = KQ/I$:

Is A a distributive category? —*No*→ A is rep.-inf.!

↓ *Yes*

Is $P := \vec{A}$ chainless? —*No*→ A is rep.-inf.!

↓ *Yes*

Does $\widetilde{P}$ contain a crit. convex subcat. from BHV? —*Yes*→ A is rep.-inf.!

↓ *No*

A is rep.-fin.!

Examples

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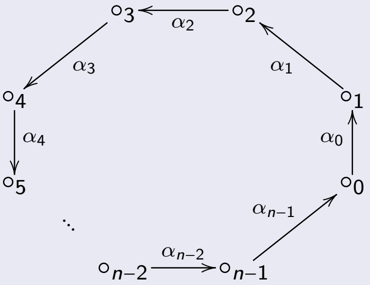
Theorem

A **basic and connected** *algebra $A = KQ/I$ is a* **Nakayama algebra**, *iff its ordinary quiver Q is one of the following quivers:*

a $Q =$



b $Q = \circ_1 \xleftarrow{\alpha_1} \circ_2 \xleftarrow{\alpha_2} \circ_3 \quad \cdots \xleftarrow{\alpha_{n-1}} \circ_{n-1} \xleftarrow{\alpha_n} \circ_n$

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Remark

A Nakayama algebra A is symmetric, iff the Loewy length of all projective indecomposable A-modules is equal to $n \cdot q + 1$*, where n denotes the number of simple A-modules and q is a natural number.*

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Example

Let $A = KG$ be a group algebra with the property that the p-Sylow subgroup of G is cyclic and normal in G. Then, A is a (not necessarily connected) Nakayama algebra.

Summary and results

The case $A = K[x]/(x^w)$

We were able to solve the case $A := K[x]/(x^w)$ completely: Set $B := \text{End}_A(A \oplus A/J^k)$ and assume that $w - k \geq k$. Then we have:

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We were able to solve the case $A := K[x]/(x^w)$ completely: Set $B := \mathrm{End}_A(A \oplus A/J^k)$ and assume that $w - k \geq k$. Then we have:

- $w > 3$ and $k = 1 \rightarrow B$ is rep.-fin.!
- $w > 3$ and $k \neq 1 \rightarrow B$ is rep.-inf.!
- $w > 3$ and $C := \mathrm{End}_A(A \oplus A/J \oplus A/J^{w-1}) \rightarrow C$ is rep.-inf.!
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Theorem (B. '16)

Let A be a symmetric Nakayama algebra with $n \geq 2$ simple modules and let $q \in \mathbb{N}_{\geq 2}$. Set $B := End_A(A_A \oplus e_0 A / e_0 J^k)$ and let $w := nq + 1$. Then:

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- *If $k = w - 1$, then B is again a Nakayama algebra and thus representation-finite.*
- *If $k = 1$, then B is almost ν-stable derived equivalent to the algebra appearing in the case $k = w - 1$ and, therefore, representation-finite.*

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- *If $k = 1$, then B is almost ν-stable derived equivalent to the algebra appearing in the case $k = w - 1$ and, therefore, representation-finite.*
- *If $k \in \{2, \cdots, w - 2\}$, then B is representation-infinite.*

Thus, up to almost ν-stable derived equivalence, the only case until now where B is representation-finite is the case $B := \mathrm{End}_A(A_A \oplus S)$, where $S := e_0 A / e_0 J$.

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Hence, in order to find **all** representation-finite gendo-symmetric algebras in this case, we have to look at the Ω-orbit of S, i.e. we have to compute the endomorphism rings

$$\mathrm{End}_A(A_A \oplus S \oplus \bigoplus_{j=1}^{r} \Omega^{i_j}(S)) \text{ for } i_j \in \{1, \cdots, 2n-1\},$$

because A is a periodic algebra with period $2n$.

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We did this partially. Namely, the following theorem was conjectured by Marczinzik and proved by B.:

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Conjecture

In all other cases (concerning this Ω-orbit) B is representation-finite.

This is the end of my talk, but I don't want to forget to **mention**:

Thank you for your **attention**

☺