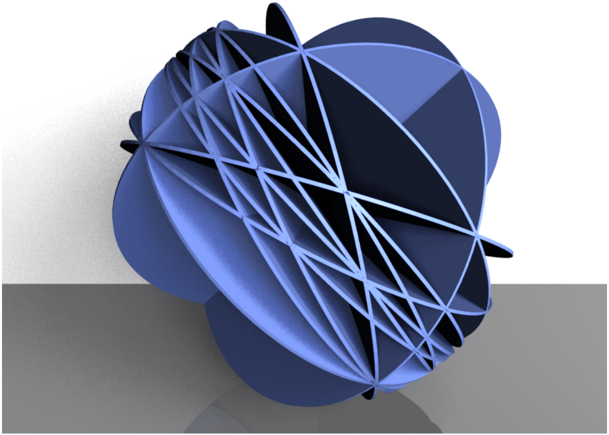
Weyl groupoids



Michael Cuntz, Leibniz Universität Hannover

SLC 83, Bad Boll, September 2019

0	1	1	2	3	4	1	1	1	0								
	0	1	3	5	7	2	3	4	1	0							
		0	1	2	3	1	2	3	1	1	0						
			0	1	2	1	3	5	2	3	1	0					
				0	1	1	4	7	3	5	2	1	0				
					0	1	5	9	4	7	3	2	1	0			
						0	1	2	1	2	1	1	1	1	0		
							0	1	1	3	2	3	4	5	1	0	
								0	1	4	3	5	7	9	2	1	0

Frieze patterns

Frieze patterns

Definition

Let R be a subset of a commutative ring.
A **frieze pattern** over R is an array $\mathcal{F}$ of the form

$$\begin{array}{ccccccccccc}
 & & & & \vdots & & & & & & \\
0 & 1 & c_{i-1,i+1} & c_{i-1,i+2} & \cdots & c_{i-1,n+i} & 1 & 0 & & & \\
 & 0 & 1 & c_{i,i+2} & c_{i,i+3} & \cdots & c_{i,n+i+1} & 1 & 0 & & \\
 & & 0 & 1 & c_{i+1,i+3} & c_{i+1,i+4} & \cdots & c_{i+1,n+i+2} & 1 & 0 & \\
 & & & & & \vdots & & & & &
\end{array}$$

where $c_{i,j}$ are numbers in R, and such that every (complete) adjacent 2×2 submatrix has determinant 1. We call n the **height** of the frieze pattern $\mathcal{F}$. We say that the frieze pattern $\mathcal{F}$ is **periodic** with period $m > 0$ if $c_{i,j} = c_{i+m,j+m}$ for all i, j.

A frieze pattern is called **tame** if every adjacent 3×3-submatrix has determinant 0.

Example

(1) Frieze patterns over $\mathbb{N}$ are called **Conway-Coxeter frieze patterns**.

(2) The array

$$
\begin{array}{cccccccccccc}
 & & & \ddots & & & & & & & & \\
0 & 1 & -i+1 & 1 & i+1 & 1 & 0 & & & & & \\
 & 0 & 1 & i+1 & 2i+1 & 2 & 1 & 0 & & & & \\
 & & 0 & 1 & 2 & -2i+1 & -i+1 & 1 & 0 & & & \\
 & & & 0 & 1 & -i+1 & 1 & i+1 & 1 & 0 & & \\
 & & & & 0 & 1 & i+1 & 2i+1 & 2 & 1 & 0 & \\
 & & & & & 0 & 1 & 2 & -2i+1 & -i+1 & 1 & 0 \\
 & & & & & & & & \ddots & & &
\end{array}
$$

repeated infinitely many times to both sides, is a frieze pattern over the Gaussian integers $\mathbb{Z}[i]$; it is periodic with period 6.

Example

(3) For every sequences $(a_i)_{i\in\mathbb{Z}}$ and $(b_i)_{i\in\mathbb{Z}}$ we have a non-periodic frieze pattern of the form

$$
\begin{array}{cccccccccccc}
 & & & \ddots & & & & & & & & \\
0 & 1 & a_1 & -1 & b_1 & 1 & 0 & & & & & \\
 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & & & & \\
 & & 0 & 1 & a_2 & -1 & b_2 & 1 & 0 & & & \\
 & & & 0 & 1 & 0 & -1 & 0 & 1 & 0 & & \\
 & & & & 0 & 1 & a_3 & -1 & b_3 & 1 & 0 & \\
 & & & & & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
 & & & & & & & & & \ddots & &
\end{array}
$$

η-matrices

Definition

For c in a commutative ring, let

$$\eta(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark

Notice that up to a transposition, $\eta(c)$ may be viewed as a **reflection**:

$$\eta(c)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta(c) = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.$$

Propagation

Let $\mathcal{F} = (c_{i,j})$ be a tame frieze pattern over R.

Consider an adjacent 3×3-submatrix M of $\mathcal{F}$. The first two columns of M cannot be linearly dependent because the upper left 2×2-submatrix has determinant 1.

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$$M = \begin{pmatrix} a & b & sa + tb \\ c & d & sc + td \\ e & f & se + tf \end{pmatrix}$$

for suitable a, b, c, d, e, f, s, t.

Propagation

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$$M = \begin{pmatrix} a & b & sa + tb \\ c & d & sc + td \\ e & f & se + tf \end{pmatrix}$$

for suitable a, b, c, d, e, f, s, t. Now the fact that all adjacent 2×2-determinants are 1 implies

$$1 = b(sc + td) - d(sa + tb) = s(bc - ad) = -s,$$

so $s = -1$.

We see that for fixed i, there is a c_i such that

$$\eta(c_i)\begin{pmatrix} c_{j,i+1} \\ c_{j,i} \end{pmatrix} = \begin{pmatrix} -c_{j,i} + c_i c_{j,i+1} \\ c_{j,i+1} \end{pmatrix} = \begin{pmatrix} c_{j,i+2} \\ c_{j,i+1} \end{pmatrix} \tag{1}$$

for all j.

Extend the frieze:

$$\begin{array}{ccccccccccccc}
 & & & & & \vdots & & & & & & & \\
-1 & 0 & 1 & \mathbf{c_{i-1,i+1}} & \cdots & c_{i-1,n+i} & 1 & 0 & -1 & & & & \\
 & -1 & 0 & 1 & \mathbf{c_{i,i+2}} & \cdots & c_{i,n+i+1} & 1 & 0 & -1 & & & \\
 & & -1 & 0 & 1 & \mathbf{c_{i+1,i+3}} & \cdots & c_{i+1,n+i+2} & 1 & 0 & -1 & & \\
 & & & & & & \vdots & & & & & &
\end{array}$$

So in Fact, $c_i = c_{i,i+2}$.

Propagation

$$\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{and} \qquad c_{i,j+2} = \left(\prod_{k=i}^{j} \eta(c_k) \right)_{1,1}.$$

0	1	**1**	2	3	4	1	1	1	0								
	0	1	**3**	5	7	2	3	4	1	0							
		0	1	**2**	3	1	2	3	1	1	0						
			0	1	**2**	1	3	5	2	3	1	0					
				0	1	**1**	4	7	3	5	2	1	0				
					0	1	**5**	9	4	7	3	2	1	0			
						0	1	**2**	1	2	1	1	1	1	0		
							0	1	**1**	3	2	3	4	5	1	0	
								0	1	**4**	3	5	7	9	2	1	0

Propagation

Proposition

Tame frieze patterns over a commutative ring R correspond bijectively to sequences $(c_1, \ldots, c_m) \in R^m$ with

$$\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Quiddity cycles

Definition

Let R be a subset of a commutative ring and $\lambda \in \{\pm 1\}$.
A **λ-quiddity cycle** over R is a sequence $(c_1, \ldots, c_m) \in R^m$ satisfying

$$\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \mathrm{id}. \quad (2)$$

A (-1)-quiddity cycle is called a **quiddity cycle** for short.

Quiddity cycles

Example

Consider the commutative ring $\mathbb{C}$ and $R = \mathbb{C}$.

- $(0, 0)$ is the only λ-quiddity cycle of length 2, for

$$\eta(a)\eta(b) = \begin{pmatrix} ab - 1 & -a \\ b & -1 \end{pmatrix} = \pm \mathrm{id}$$

  implies $a = b = 0$.

Quiddity cycles

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  implies $a = b = 0$.

- $(1,1,1)$ and $(-1,-1,-1)$ are the only λ-quiddity cycles of length 3 for

$$\eta(a)\eta(b)\eta(c) = \begin{pmatrix} abc-a-c & -ab+1 \\ bc-1 & -b \end{pmatrix} = \pm \mathrm{id}$$

  implies $b = \pm 1$, $a = b = c$.

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  implies $b = \pm 1$, $a = b = c$.

- $(t, 2/t, t, 2/t)$, t a unit and $(a, 0, -a, 0)$, a arbitrary, are the only λ-quiddity cycles of length 4.

Quiddity cycles

Definition

Let D_n be the dihedral group with $2n$ elements acting on $\{1,\dots,n\}$. If $\underline{c}=(c_1,\dots,c_n)$ is a λ-quiddity cycle, then we write

$$\underline{c}^\sigma := (c_1,\dots,c_n)^\sigma := (c_{\sigma(1)},\dots,c_{\sigma(n)})$$

for $\sigma\in D_n$.

Proposition

Let $\underline{c}=(c_1,\dots,c_m)$ be a λ-quiddity cycle. Then for any $\sigma\in D_n$, the cycle $\underline{c}^\sigma$ is a λ-quiddity cycle as well.

Quiddity cycles

Lemma

Let $(a_1, \ldots, a_k)$ *be a* λ'*-quiddity cycle and* $(b_1, \ldots, b_\ell)$ *be a* λ''*-quiddity cycle. Then*

$$(a_1, \ldots, a_k) \oplus (b_1, \ldots, b_\ell) := (a_1 + b_\ell, a_2, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_{\ell-1})$$

is a $(-\lambda'\lambda'')$*-quiddity cycle of length* $k + \ell - 2$ *which we call the* **sum**.

Proof.

We use the identities $\eta(a+b) = -\eta(a)\eta(0)\eta(b)$ and $\eta(0)^2 = -\mathrm{id}$:

$$\begin{aligned} & \eta(a_1 + b_\ell)\eta(a_2)\cdots\eta(a_{k-1})\eta(a_k + b_1)\eta(b_2)\cdots\eta(b_{\ell-1}) \\ = \quad & \eta(b_\ell)\eta(0)\eta(a_1)\eta(a_2)\cdots\eta(a_{k-1})\eta(a_k)\eta(0)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1}) \\ = \quad & \lambda'\eta(b_\ell)\eta(0)\eta(0)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1}) \\ = \quad & -\lambda'\eta(b_\ell)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1}) = -\lambda'\lambda''\mathrm{id}. \quad \square \end{aligned}$$

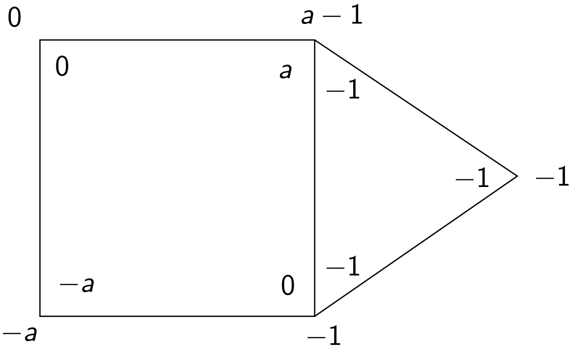


Figure: $(a, 0, -a, 0) \oplus (-1, -1, -1) = (a-1, 0, -a, -1, -1)$.

Irreducibility

Definition (C., 2019)

Let R be a subset of a commutative ring.

A λ-quiddity cycle $(c_1, \ldots, c_m) \in R^m$, $m > 2$ is called **reducible over** R if there exist a λ'-quiddity cycle $(a_1, \ldots, a_k) \in R^k$, a λ''-quiddity cycle $(b_1, \ldots, b_\ell) \in R^\ell$, and $\sigma \in D_m$ such that $\lambda = -\lambda'\lambda''$, $k, \ell > 2$ and

$$\begin{aligned}(c_1, \ldots, c_m)^\sigma &= (a_1 + b_\ell, a_2, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_{\ell-1}) \\ &= (a_1, \ldots, a_k) \oplus (b_1, \ldots, b_\ell).\end{aligned}$$

A λ-quiddity cycle of length $m > 2$ is called **irreducible over** R if it is not reducible.

Tame frieze patterns are **reducible/irreducible** if their quiddity cycles are.

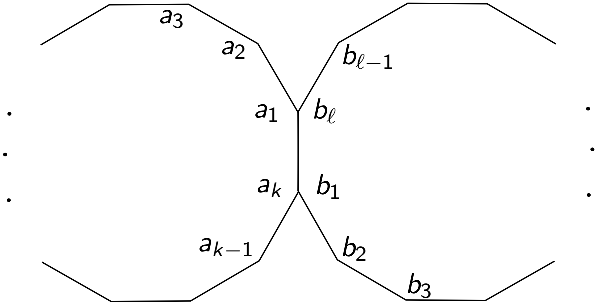
Irreducibility

Lemma

Let R be a commutative ring. A λ-quiddity cycle is reducible over R if and only if the corresponding tame frieze pattern contains an entry 1 or -1.

Combinatorial model

$(a_1, \ldots, a_k)$ a λ'-quiddity cycle, and $(b_1, \ldots, b_\ell)$ a λ''-quiddity cycle.



$$(a_1, \ldots, a_k) \oplus (b_1, \ldots, b_\ell) = (a_1 + b_\ell, a_2, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_{\ell-1})$$

Bounds

Lemma

Let $(c_1, \ldots, c_m) \in \mathbb{C}^m$ *such that* $\prod_{j=1}^{m} \eta(c_j)$ *is a scalar multiple of the identity matrix. Then there is an index* $j \in \{1, \ldots, m\}$ *with* $|c_j| < 2$.

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Proof.

Let $a, b \in \mathbb{C}$ with $|a| \geqslant |b|$ and $|c| \geqslant 2$. Then

$$|ac - b| \geqslant |ac| - |b| = |a|(|c| - 1) + |a| - |b| \geqslant |a|(|c| - 1) \geqslant |a|.$$

Lemma

Let $(c_1, \ldots, c_m) \in \mathbb{C}^m$ *such that* $\prod_{j=1}^{m} \eta(c_j)$ *is a scalar multiple of the identity matrix. Then there is an index* $j \in \{1, \ldots, m\}$ *with* $|c_j| < 2$.

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The claim follows from this inequality and from

$$\eta(c) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ac - b \\ a \end{pmatrix}.$$

□

Theorem

The only irreducible λ-quiddity cycles over $\mathbb{Z}_{\geqslant 0}$ are $(0,0,0,0)$ and $(1,1,1)$.

Theorem

Let $(x_{ij})_{i,j}$ be a (tame) frieze pattern with entries in $\mathbb{N}_{>0}$ and $\underline{c}$ its quiddity cycle. Then (up to a rotation) there exists a quiddity cycle $\underline{c}'$ such that $\underline{c} = (1,1,1) \oplus \underline{c}'$ and such that the frieze pattern of $\underline{c}'$ has entries in $\mathbb{N}_{>0}$.

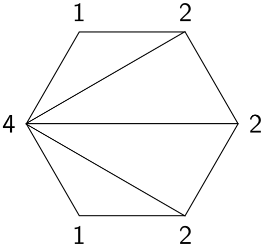
Corollary

The set of frieze patterns with entries in $\mathbb{N}_{>0}$ is in bijection with the set of triangulations of convex polygons by non-intersecting diagonals.

$\ddots$

0	1	1	3	2	1	0					
	0	1	4	3	2	1	0				
		0	1	1	1	1	1	0			
			0	1	2	3	4	1	0		
				0	1	2	3	1	1	0	
					0	1	2	1	2	1	0

$\ddots$



Other domains

Theorem (C., Holm, 2019)

The set of irreducible λ-quiddity cycles over $\mathbb{Z}$ is

$$\{(1,1,1),(-1,-1,-1),(a,0,-a,0),(0,a,0,-a) \mid a \in \mathbb{Z}\backslash\{\pm 1\}\}.$$

Other domains

Proposition

Let $k \in \mathbb{N}_{>0}$ *and* $\mathrm{i} = \sqrt{-1}$. *Then*

$$\underline{c} = (2\mathrm{i}, -\mathrm{i}+1, \underbrace{2, \ldots, 2}_{2k\text{-times}}, \mathrm{i}+1, -2\mathrm{i}, \mathrm{i}-1, \underbrace{-2, \ldots, -2}_{2k\text{-times}}, -\mathrm{i}-1)$$

is an irreducible quiddity cycle over $\mathbb{Z}[\mathrm{i}]$.

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is an irreducible quiddity cycle over $\mathbb{Z}[\mathrm{i}]$.

Corollary

There are infinitely many irreducible λ*-quiddity cycles over the Gaussian integers* $\mathbb{Z}[\mathrm{i}]$.

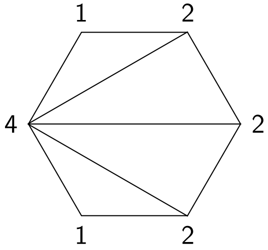
Other domains

Open Problem

Classify irreducible quiddity cycles for "interesting" sets R.

Quiddity cycles over $\mathbb{N}$ and subsequences

Every triangulation of an n-gon by non-intersecting diagonals has an **ear**:



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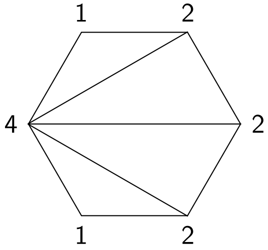

Every quiddity cycle over $\mathbb{N}$ contains an entry 1.

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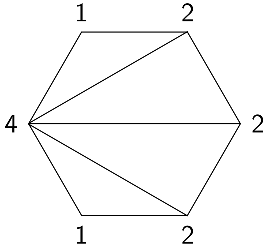
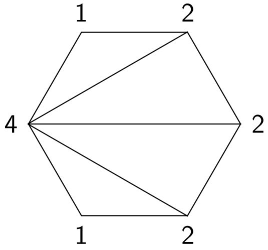


Every quiddity cycle over $\mathbb{N}$ contains an entry 1.

Every quiddity cycle over $\mathbb{N}$ contains a subsequence $(1,1)$, $(1,2)$, $(2,1)$, or $(1,3,1)$.

Quiddity cycles over $\mathbb{N}$ and subsequences

Every triangulation of an *n*-gon by non-intersecting diagonals has an **ear**:



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Every quiddity cycle over $\mathbb{N}$ contains a subsequence $(1,1)$, $(1,2)$, $(2,1)$, or $(1,3,1)$.

Every quiddity cycle over $\mathbb{N}$ except $(1,1,1)$ contains a subsequence $(1,2)$, $(2,1)$, or $(1,3,1)$.

Quiddity cycles over $\mathbb{N}$ and subsequences

Theorem (C., 2018)

For any $\ell \in \mathbb{N}$ we may compute finite sets of sequences E and F, where the elements of F have length at least ℓ, and such that every quiddity cycle over $\mathbb{N}$ not in E has an element of F as a (consecutive) subsequence.

In other words, this theorem gives a local description of quiddity cycles.

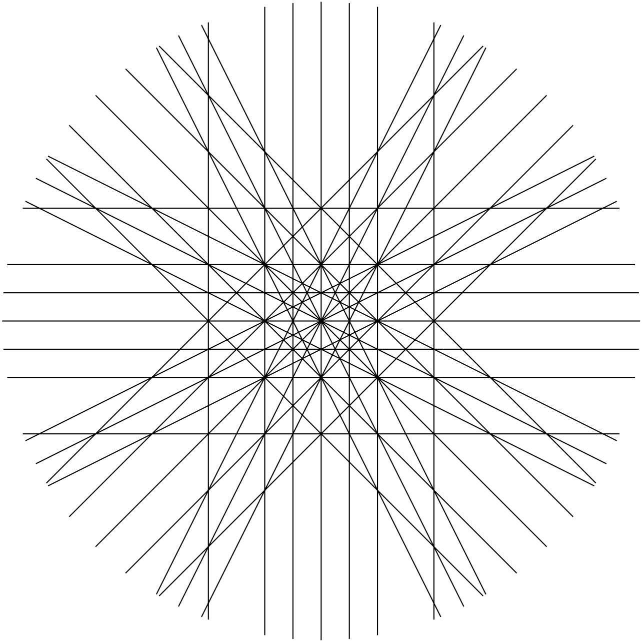
For example if $\ell = 4$:

Corollary

Every quiddity cycle (considered up to the action of the dihedral group) $c \notin \{(0,0),(1,1,1),(1,2,1,2)\}$ *contains at least one of*

$$(1,2,2,1),(1,2,2,2),(1,2,2,3),(1,2,2,4),(1,2,3,1),(1,2,3,2),$$
$$(1,2,3,3),(1,2,4,1),(1,2,4,3),(1,2,5,1),(1,2,5,2),(1,2,6,1),$$
$$(1,3,1,3),(1,3,1,4),(1,3,1,5),(1,3,1,6),(1,3,4,1),(1,4,1,2),$$
$$(1,5,1,2),(1,6,1,2),(1,7,1,2),(2,1,3,2),(2,1,3,3),(2,2,1,4),$$
$$(2,2,1,5),(3,1,2,3),(3,1,2,4).$$

Frieze patterns over $\mathbb{R}$ correspond to arrangements of lines in $\mathbb{R}^2$.



Arrangements of hyperplanes

Arrangements

A finite set $\mathcal{A} := \{H_1, \ldots, H_n\}$ of linear hyperplanes in a vector space $V = K^r$ is called an **arrangement of hyperplanes**.

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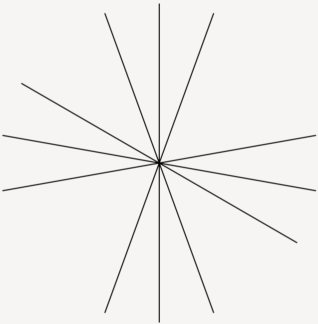
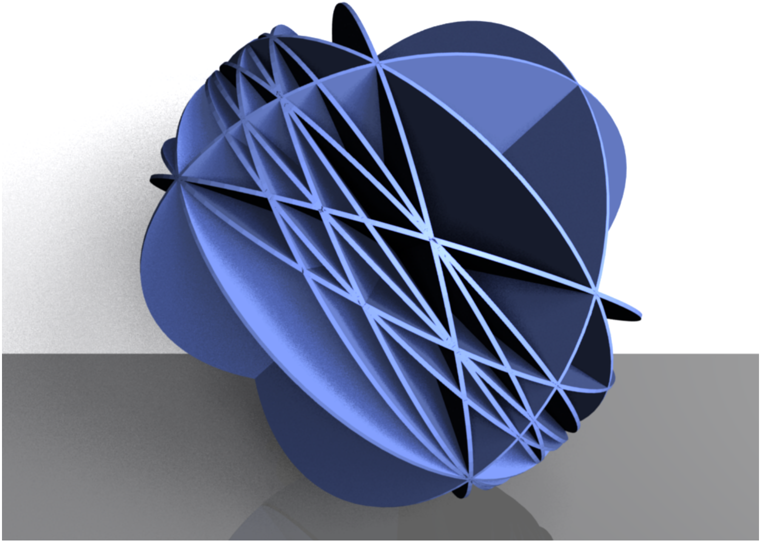
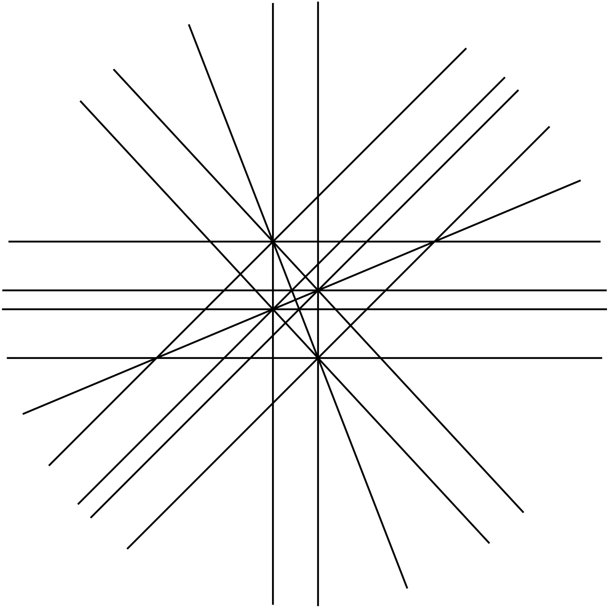
Example

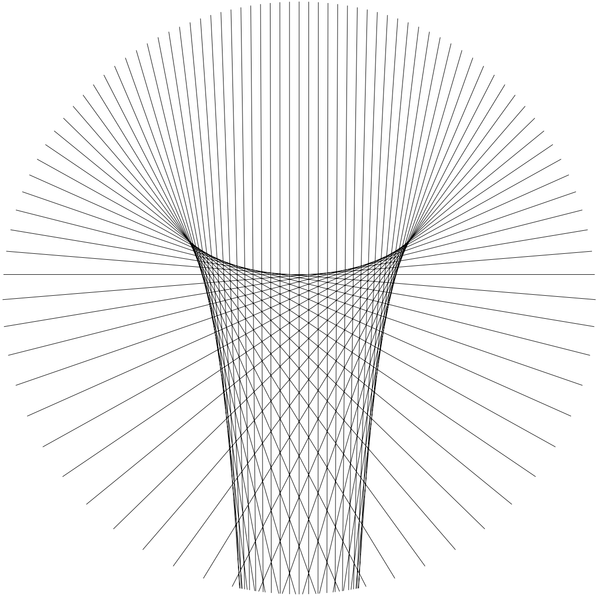


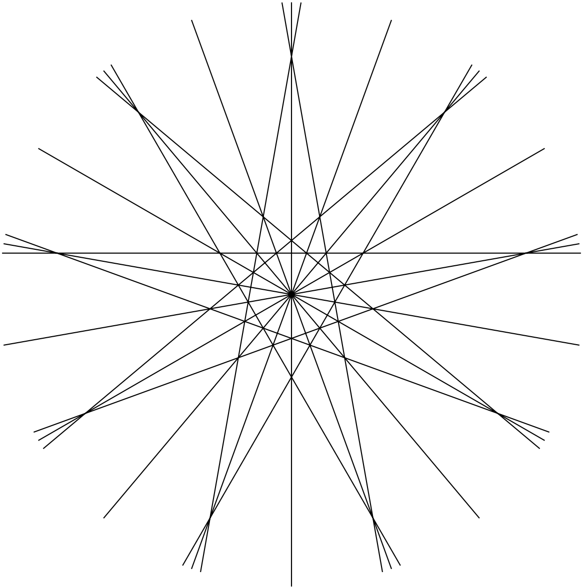


A free arrangement



A torsion subgroup of an elliptic curve





Simplicial arrangements

Let $\mathcal{A} := \{H_1, \ldots, H_n\}$ be a finite set of hyperplanes in $V = \mathbb{R}^r$.

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Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (**chambers**) of $V \setminus \bigcup_{H \in \mathcal{A}} H$.

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Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (**chambers**) of $V \setminus \bigcup_{H \in \mathcal{A}} H$.

Definition (Melchior, 1941)

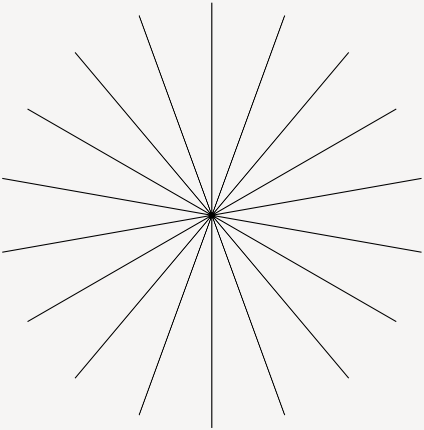
If every chamber K is an **open simplicial cone**, i.e. there exist $\beta_1, \ldots, \beta_r \in V$ such that

$$K = \left\{ \sum_{i=1}^{r} a_i \beta_i \mid a_i > 0 \quad \text{for all} \quad i = 1, \ldots, r \right\},$$

then $\mathcal{A}$ is called a **simplicial arrangement**.

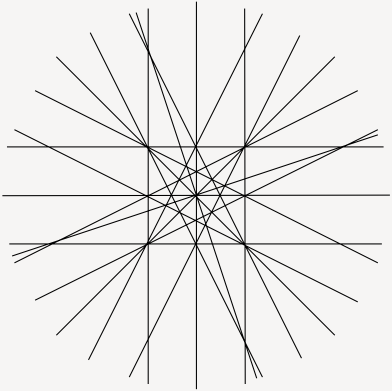
Simplicial arrangements

Example



Simplicial arrangements

Example



Source: Grünbaum, A catalogue of simplicial arrangements in the real projective plane.

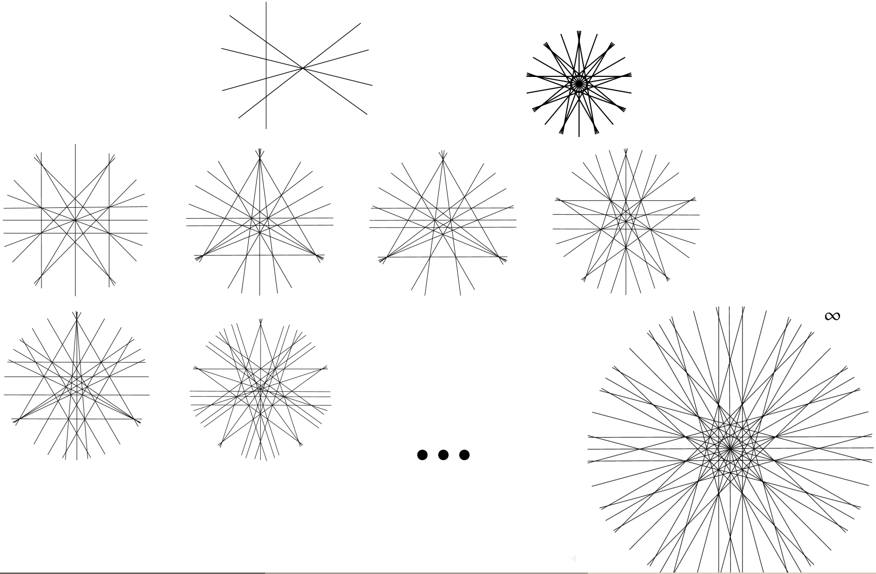
Simplicial arrangements

Theorem (Deligne, 1972)

The complement of a complexified finite simplicial arrangement is $K(\pi, 1)$.

Grünbaum's catalogue for the real projective plane

(Grünbaum, 1972–2009)

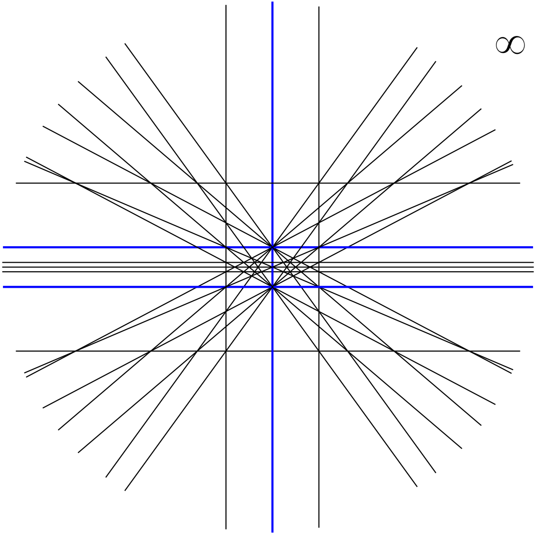


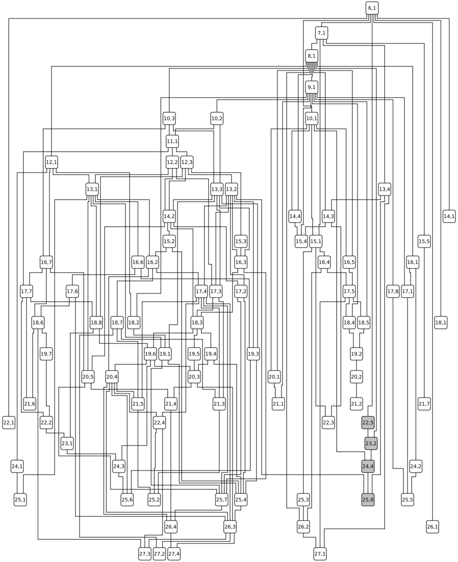
Theorem (C., 2012)

We have a complete list of simplicial arrangements in the real projective plane with at most 27 lines.

"New" simplicial arrangements (22,23,24,25 lines)

(C., 2012)





H. S. M. Coxeter:
"[...] the diagrams which profess to portray these known polygrams are strangely unintelligible."

Reducibility

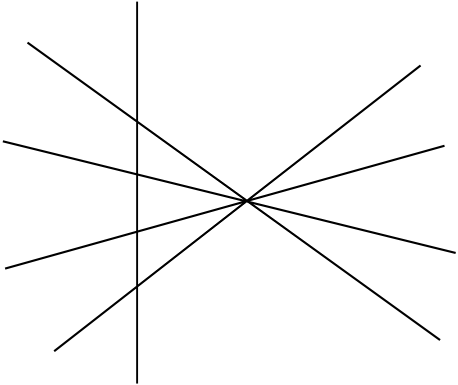
Definition

The **product** $(\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ of two arrangements $(\mathcal{A}_1, V_1)$, $(\mathcal{A}_2, V_2)$ is defined by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

If an arrangement $(\mathcal{A}, V)$ can be written as a non-trivial product $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$, then $\mathcal{A}$ is called **reducible**, otherwise **irreducible**.
The **rank** of an arrangement $(\mathcal{A}, V)$ is $\operatorname{rank} \mathcal{A} := \dim(V) - \dim(\bigcap_{H \in \mathcal{A}} H)$.

Reducibility – Near pencil



Reflections

Definition

Let K be a field, $r \in \mathbb{N}$, $V := K^r$, and H a hyperplane in V.

A **reflection** on V at H is a $\sigma \in \mathrm{GL}(V)$, $\sigma \neq \mathrm{id}$ of finite order which fixes H.

Notice that the eigenvalues of σ are 1 and ζ for some root of unity $\zeta \in K$.

In this lecture we always have $\zeta = -1$.

Example

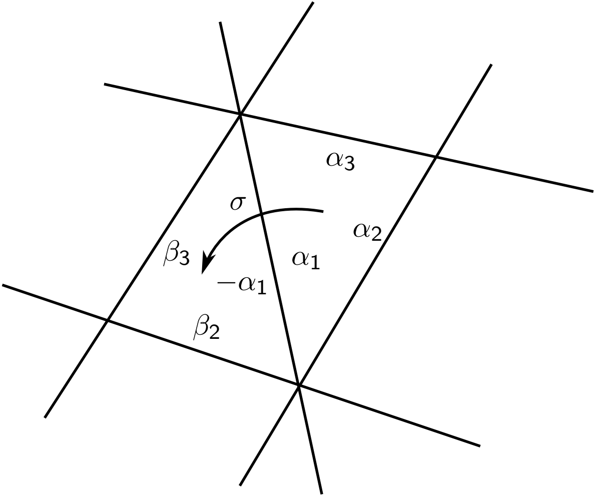
Let W be a real reflection group acting on $V = \mathbb{R}^r$, i.e. a finite group generated by reflections on V.

Let $\mathcal{R} \subseteq V^*$ be the set of roots of W.

Then $\mathcal{A} = \{\ker \alpha \mid \alpha \in \mathcal{R}\}$ is a simplicial arrangement.

The reflection arrangement is the most symmetric type of simplicial arrangement, one cannot "distinguish" the chambers, they all look the same.

Simplicial arrangements and reflections



Lemma

Let $\mathcal{A}$ be a simplicial arrangement and K a chamber, i.e. there is a basis $B^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\}$ of V such that $K = \langle B^\vee \rangle_{>0}$. Let $\tilde{K}$ be the chamber with

$$\overline{K} \cap \overline{\tilde{K}} = \langle \alpha_2^\vee, \dots, \alpha_r^\vee \rangle_{\geqslant 0}.$$

Then there is a unique $\beta^\vee \in V$ with

$$\tilde{K} = \langle \tilde{B}^\vee \rangle_{>0}, \quad \tilde{B}^\vee = \{\beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\}, \quad \textit{and} \quad |B \cap -\tilde{B}| = 1,$$

where $B := (B^\vee)^*$ and $\tilde{B} := (\tilde{B}^\vee)^*$ denote the dual bases.

Proof.

Choose $\beta^\vee \in V$ such that $\tilde{K} = \langle \beta^\vee, \alpha_2^\vee, \ldots, \alpha_r^\vee \rangle_{>0}$. Let $\mu_1, \ldots, \mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i \alpha_i^\vee$ (notice $\mu_1 \neq 0$).

Proof.

Choose $\beta^\vee \in V$ such that $\tilde{K} = \langle \beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee \rangle_{>0}$. Let $\mu_1, \dots, \mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i \alpha_i^\vee$ (notice $\mu_1 \neq 0$).

Let $\tilde{B} = \{\beta_1, \dots, \beta_r\}$ be the dual basis of $\{\beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\}$, and $B = \{\alpha_1, \dots, \alpha_r\}$ be dual to $B^\vee$.

Proof.

Choose $\beta^\vee \in V$ such that $\tilde{K} = \langle \beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee \rangle_{>0}$. Let $\mu_1, \dots, \mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i \alpha_i^\vee$ (notice $\mu_1 \neq 0$).

Let $\tilde{B} = \{\beta_1, \dots, \beta_r\}$ be the dual basis of $\{\beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\}$, and $B = \{\alpha_1, \dots, \alpha_r\}$ be dual to $B^\vee$.

Then $\beta_1 = \frac{1}{\mu_1}\alpha_1$ and $\beta_j = -\frac{\mu_j}{\mu_1}\alpha_1 + \alpha_j$ for $j > 1$.

Proof.

Choose $\beta^\vee \in V$ such that $\tilde{K} = \langle \beta^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee \rangle_{>0}$. Let $\mu_1, \dots, \mu_r \in \mathbb{R}$ be such that $\beta^\vee = \sum_{i=1}^r \mu_i \alpha_i^\vee$ (notice $\mu_1 \neq 0$).

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Then $\beta_1 = \frac{1}{\mu_1}\alpha_1$ and $\beta_j = -\frac{\mu_j}{\mu_1}\alpha_1 + \alpha_j$ for $j > 1$.

To obtain $|B \cap -\tilde{B}| = 1$ we need $-\alpha_1 = \beta_1 \in \tilde{B}$ and hence $\mu_1 = -1$, $\beta_1 = -\alpha_1$ and $\beta_j = \mu_j \alpha_1 + \alpha_j$ for $j > 1$.

Thus a $\beta^\vee$ as desired exists and is unique. □

Corollary

Using the notation of the proof of the Lemma, the map

$$\sigma : V^* \to V^*, \quad \alpha_i \mapsto \beta_i$$

is a reflection. With respect to $B = (B^\vee)^*$, it becomes the matrix

$$\begin{pmatrix} -1 & \mu_2 & \dots & \mu_r \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

Simplicial arrangements and reflections

Example

Let $R = \{(1,0),(0,1),(1,2)\} \in (\mathbb{R}^2)^*$, $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$.

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Then $K = \langle B^\vee \rangle_{>0}$ is a chamber if $B^\vee = \{\alpha_1^\vee = (1,0), \alpha_2^\vee = (0,1)\}$, $K' = \langle \tilde{B}^\vee \rangle_{>0}$ with $\tilde{B}^\vee = \{\tilde{\beta}^\vee = (-2,1), \alpha_2^\vee = (0,1)\}$ is an adjacent chamber.

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To obtain $\mu_1 = -1$, we need to choose $\beta^\vee = (-1, \frac{1}{2})$, hence $\mu_2 = \frac{1}{2}$. The unique reflection σ is

$$\begin{pmatrix} -1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

with respect to $B = (B^\vee)^*$.

$\mathcal{A}$ a simplicial arrangement, $K = \langle B^\vee \rangle_{>0}$, $B^\vee = \{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ a chamber, and $B = \{\alpha_1, \ldots, \alpha_r\}$ be dual to $B^\vee$.

Corollary: for K, B there are unique reflections $\sigma_1, \ldots, \sigma_r$, represented by

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ \mu_{i,1} & \cdots & -1 & \cdots & \mu_{i,r} \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix},$$

for certain $\mu_{i,j} \in \mathbb{R}$, $i \neq j$ with respect to B.

Reflections and Cartan matrices

Definition

The matrix $C^{K,B} = (c_{i,j})_{1 \leqslant i,j \leqslant r}$ with

$$c_{i,j} := \begin{cases} -\mu_{i,j} & \text{if } i \neq j \\ 2 & \text{if } i = j \end{cases}$$

is called the **Cartan matrix** of (K, B) in $\mathcal{A}$. Note that

$$\sigma_i(\alpha_j) = \alpha_j - c_{i,j}\alpha_i$$

for all $1 \leqslant i, j \leqslant r$.

We sometimes write $\sigma_i^{K,B}$ to emphasize that σ_i depends on K and B.

Example

1. Let $\mathcal{A}$ be as in the last example. Then the Cartan matrix of (K, B) is

$$C^{K,B} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -2 & 2 \end{pmatrix}.$$

Example

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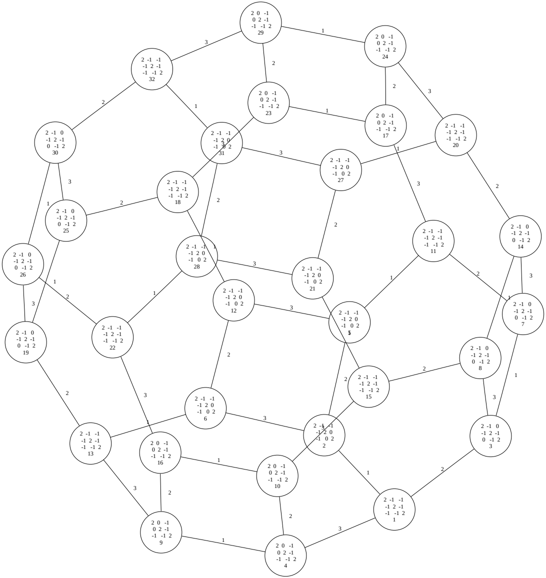
$$C^{K,B} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -2 & 2 \end{pmatrix}.$$

2. If W is a Weyl group with root system $\mathcal{R}$, then all Cartan matrices of (K, B) when B is a set of simple roots for the chamber K are equal and coincide with the classical Cartan matrix of W.

A Cartan graph



Definition

Let $\mathcal{A}$ be a simplicial arrangement in $V = \mathbb{R}^r$. We construct a category $\mathcal{C}(\mathcal{A})$ with

- objects: $\mathrm{Obj}(\mathcal{C}(\mathcal{A})) = \{B = (\alpha_1, \ldots, \alpha_r) \in (V^*)^r \mid \langle B^* \rangle_{>0} \in \mathcal{K}(\mathcal{A})\}$ (where the bases B are ordered).

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- morphisms: for each $B = (\alpha_1, \dots, \alpha_r) \in \mathrm{Obj}(\mathcal{C}(\mathcal{A}))$ and $i = 1, \dots, r$ there is a morphism $\sigma_i^{K,B} \in \mathrm{Mor}(B, (\sigma_i^{K,B}(\alpha_1), \dots, \sigma_i^{K,B}(\alpha_r)))$. All other morphisms are compositions of the generators $\sigma_i^{K,B}$.

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A **reflection groupoid** $\mathcal{W}(\mathcal{A})$ of $\mathcal{A}$ is a connected component of $\mathcal{C}(\mathcal{A})$.

A **Weyl groupoid** is a reflection groupoid for which all Cartan matrices are integral.

Using the so-called gate property, one can prove the existence of a type function for the chamber complex of a simplicial arrangement. In other words:

Proposition

Let $\mathcal{A}$ be a simplicial arrangement, $\mathcal{W}(\mathcal{A})$ a reflection groupoid, and $B_1 = (\alpha_1, \dots, \alpha_r)$, $B_2 = (\beta_1, \dots, \beta_r)$ two objects with $\langle B_1^* \rangle_{>0} = \langle B_2^* \rangle_{>0}$.

Then there exist $\lambda_1, \dots, \lambda_r$ such that $\alpha_i = \lambda_i \beta_i$ for all $i = 1, \dots, r$.

In particular, for a fixed reflection groupoid we obtain a unique labelling of the walls of each chamber with the labels $1, \dots, r$.

Definition

Let $\mathcal{A}$ be a simplicial arrangement, $\mathcal{W}(\mathcal{A})$ a reflection groupoid, and $K = \langle B^* \rangle_{>0}$ a chamber for $B = (\alpha_1, \dots, \alpha_r) \in \mathrm{Obj}(\mathcal{W}(\mathcal{A}))$.
For $i \in \{1, \dots, r\}$, let $\rho_i(K)$ be the chamber adjacent to K with common wall $\ker \alpha_i$. We thus obtain well defined maps

$$\rho_i : \mathcal{K}(\mathcal{A}) \mapsto \mathcal{K}(\mathcal{A})$$

which satisfy $\rho_i^2 = \mathrm{id}$ by the proposition.

Crystallographic arrangements

Definition (C., 2011)

Let $\mathcal{A}$ be a simplicial arrangement in V and $\mathcal{R} \subseteq V^*$ a finite set such that $\mathcal{A} = \{\ker \alpha \mid \alpha \in \mathcal{R}\}$ and $\mathbb{R}\alpha \cap \mathcal{R} = \{\pm\alpha\}$ for all $\alpha \in \mathcal{R}$.

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We call $(\mathcal{A}, V, \mathcal{R})$ a **crystallographic arrangement** if for all chambers $K \in \mathcal{K}(\mathcal{A})$:

$$\mathcal{R} \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha, \tag{3}$$

where

$$B^K = \{\alpha \in \mathcal{R} \mid \forall x \in K \ : \ \alpha(x) \geqslant 0, \ \langle \ker \alpha \cap \overline{K} \rangle = \ker \alpha\}$$

corresponds to the set of walls of K.

Definition

Two crystallographic arrangements $(\mathcal{A}, V, \mathcal{R})$, $(\mathcal{A}', V, \mathcal{R}')$ in V are called **equivalent** if there exists $\psi \in \mathrm{Aut}(V^*)$ with $\psi(\mathcal{R}) = \mathcal{R}'$. We then write $(\mathcal{A}, V, \mathcal{R}) \cong (\mathcal{A}', V, \mathcal{R}')$.

If $\mathcal{A}$ is an arrangement in V for which a set $\mathcal{R} \subseteq V^*$ exists such that $(\mathcal{A}, V, \mathcal{R})$ is crystallographic, then we say that $\mathcal{A}$ is **crystallographic**.

Example

1. Let $\mathcal{R}$ be the set of roots of the root system of a crystallographic reflection group (i.e. a Weyl group). Then $(\{\ker\alpha \mid \alpha\in\mathcal{R}\}, V, \mathcal{R})$ is a crystallographic arrangement.

Example

1. Let $\mathcal{R}$ be the set of roots of the root system of a crystallographic reflection group (i.e. a Weyl group). Then $(\{\ker\alpha \mid \alpha \in \mathcal{R}\}, V, \mathcal{R})$ is a crystallographic arrangement.
2. If $R_+ := \{(1,0),(3,1),(2,1),(5,3),(3,2),(1,1),(0,1)\}$, then $(\{\alpha^\perp \mid \alpha \in R_+\}, \mathbb{R}^2, R_+ \cup -R_+)$ is a crystallographic arrangement.

Crystallographic arrangements

$$R_+ := \{(1,0),(3,1),(2,1),(5,3),(3,2),(1,1),(0,1)\}$$

0	1	2	5	3	4	1	0						
	0	1	3	2	3	1	1	0					
		0	1	1	2	1	2	1	0				
			0	**1**	**3**	**2**	**5**	**3**	**1**	**0**			
				0	**1**	**1**	**3**	**2**	**1**	**1**	0		
					0	1	4	3	2	3	1	0	
						0	1	1	1	2	1	1	0

Crystallographic arrangements

Definition

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement and K a chamber. Fixing an ordering for B^K, we obtain a unique reflection groupoid $\mathcal{W}(\mathcal{A})$ and thus unique orderings for all $B^{K'}$, $K' \in \mathcal{K}(\mathcal{A})$ (type function). Hence we obtain a unique coordinate map

$$\Upsilon^K : V \to \mathbb{R}^r \quad \text{with respect to} \quad B^K.$$

The elements of the standard basis $\{\alpha_1, \ldots, \alpha_r\} = \Upsilon^K(B^K)$ are called **simple roots**.

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The elements of the standard basis $\{\alpha_1, \ldots, \alpha_r\} = \Upsilon^K(B^K)$ are called **simple roots**. The set

$$R^K := \{\Upsilon^K(\alpha) \mid \alpha \in \mathcal{R}\} \subseteq \mathbb{N}_0^r \cup -\mathbb{N}_0^r$$

is called the set of **roots** of $\mathcal{A}$ at K. The roots in $R_+^K := R^K \cap \mathbb{N}_0^r$ are called **positive**.

Let $1 \leqslant i,j \leqslant r$. Then it is easy to see that

$$c_{i,j}^K = \begin{cases} -\max\{k \in \mathbb{N}_{\geqslant 0} \mid k\alpha_i + \alpha_j \in R^K\} & i \neq j \\ 2 & i = j \end{cases},$$

where $C^K := (c_{i,j}^K)_{i,j}$ is the Cartan matrix of (K, B^K).

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Recall that for every $i = 1, \ldots, r$, we have a reflection $\sigma^K_i : \mathbb{Z}^r \to \mathbb{Z}^r$ defined by $\sigma^K_i(\alpha_j) = \alpha_j - c^K_{i,j}\alpha_i$ for all $1 \leqslant j \leqslant r$.

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Remark that if $\tilde{K}$ is the chamber adjacent to K with

$$\langle \overline{K} \cap \overline{\tilde{K}} \rangle = \ker \alpha \quad \text{for} \quad \alpha \in R \quad \text{with} \quad \Upsilon^K(\alpha) = \Upsilon^{\tilde{K}}(\alpha) = \alpha_i,$$

then the lemma implies $\sigma_i^K = \Upsilon^{\tilde{K}} \circ (\Upsilon^K)^{-1}$ and thus $\sigma_i^K(R^K) = R^{\tilde{K}}$.

Crystallographic arrangements

To avoid confusion, we use different fonts for the “global” set $\mathcal{R}$ and the “local” representations R^K.

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These local representations “are” the objects of the Weyl groupoid. Notice that in the crystallographic case we have

$$\mathrm{Mor}(B^K, B^{\tilde{K}}) = \{w^{K,\tilde{K}} := \Upsilon^{\tilde{K}} \circ (\Upsilon^K)^{-1}\}$$

for chambers K and $\tilde{K}$.

Volumes

Definition

Let $m, r \in \mathbb{N}$.

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By the Smith normal form there is a unique left $\mathrm{GL}(\mathbb{Z}^r)$-invariant right $\mathrm{GL}(\mathbb{Z}^m)$-invariant function $\mathrm{Vol}_m : (\mathbb{Z}^r)^m \to \mathbb{Z}$ such that

$$\mathrm{Vol}_m(a_1\alpha_1, \ldots, a_m\alpha_m) = |a_1 \cdots a_m| \quad \text{for all } a_1, \ldots, a_m \in \mathbb{Z}, \tag{4}$$

where $|\cdot|$ denotes absolute value, i.e. $\mathrm{Vol}_m(\beta_1, \ldots, \beta_m)$ is the product of the elementary divisors of the matrix with columns $\beta_1, \ldots, \beta_m$.

Volumes

If $m = 1$ and $\beta \in \mathbb{Z}^r \setminus \{0\}$, then $\mathrm{Vol}_1(\beta)$ is the greatest common divisor of the coordinates of β.

If $m = r$ and $\beta_1, \ldots, \beta_r \in \mathbb{Z}^r$, then $\mathrm{Vol}_r(\beta_1, \ldots, \beta_r)$ is the absolute value of the determinant of the matrix with columns $\beta_1, \ldots, \beta_r$.

Volumes

We obtain a "**volume**" for tuples of roots:

Definition

Let $(\mathcal{A}, V, \mathcal{R})$ be an irreducible crystallographic arrangement of rank r. By the crystallographic property (3), for chambers K, K', the bases B^K and $B^{K'}$ differ by a map in $\mathrm{GL}(\mathbb{Z}^r)$. Thus for $\beta_1, \ldots, \beta_m \in \mathcal{R}$,

$$\mathrm{Vol}_m(\Upsilon^K(\beta_1), \ldots, \Upsilon^K(\beta_m)) = \mathrm{Vol}_m(\Upsilon^{K'}(\beta_1), \ldots, \Upsilon^{K'}(\beta_m)).$$

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Hence we have a well-defined map

$$\mathrm{Vol}_m : \mathcal{R}^m \to \mathbb{Z}, \quad (\beta_1, \ldots, \beta_m) \mapsto \mathrm{Vol}_m(\Upsilon^K(\beta_1), \ldots, \Upsilon^K(\beta_m))$$

which does not depend on the choice of K.

Definition

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement and K a chamber. For a subspace $X \leqslant \mathbb{R}^r$, we call $S_{K,X} := X \cap R^K$ a **localization** of the crystallographic arrangement at K and X.

Localizations

Definition

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement and K a chamber. For a subspace $X \leqslant \mathbb{R}^r$, we call $S_{K,X} := X \cap R^K$ a **localization** of the crystallographic arrangement at K and X.

Notice that

$$S_{K,X} = S_{K,X_+} \dot\cup - S_{K,X_+} \quad \text{for} \quad S_{K,X_+} := X \cap R^K_+.$$

Localizations

Localizations in crystallographic arrangements define crystallographic arrangements.

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Localizations in crystallographic arrangements define crystallographic arrangements.

Lemma

Let $(\mathcal{A}, V, \mathcal{R})$ *be a crystallographic arrangement,* K *a chamber, and* $X \leqslant \mathbb{R}^r$*. Then there is a subset* $\Delta \subseteq X \cap R_+^K$ *which is a set of simple roots for the localization* $S_{K,X} = X \cap R^K$*, i.e.*

$$S_{K,X_+} \subseteq \sum_{\alpha \in \Delta} \mathbb{N}_0 \alpha.$$

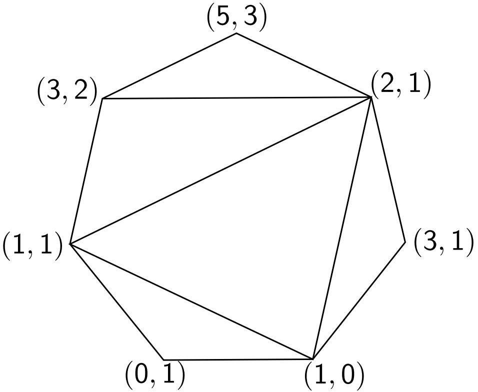
Definition

Define $\mathcal{F}$**-sequences** as finite sequences of length $\geqslant 2$ with entries in $\mathbb{N}_0^2$ given by the following recursion.

1. $((0,1),(1,0))$ is an $\mathcal{F}$-sequence.
2. If $(v_1,\dots,v_n)$ is an $\mathcal{F}$-sequence, then $(v_1,\dots,v_i,v_i+v_{i+1},v_{i+1},\dots,v_n)$ are $\mathcal{F}$-sequences for $i=1,\dots,n-1$.
3. Every $\mathcal{F}$-sequence is obtained recursively by (1) and (2).

Rank two

$$R_+ := \{(1,0),(3,1),(2,1),(5,3),(3,2),(1,1),(0,1)\}$$



Rank two

Theorem

Let $(\mathcal{A}, V)$ *be an arrangement of rank two and* $\mathcal{R} \subseteq V^*$ *such that* $\mathcal{A} = \{\ker \alpha \mid \alpha \in \mathcal{R}\}$ *and* $\mathbb{R}\alpha \cap \mathcal{R} = \{\pm\alpha\}$ *for all* $\alpha \in \mathcal{R}$*.*

Then $(\mathcal{A}, V, \mathcal{R})$ *is a crystallographic arrangement if and only if there exists a chamber* K *such that* R_+^K *is an* $\mathcal{F}$*-sequence.*

In this case, R_+^K *is an* $\mathcal{F}$*-sequence for all chambers* K*.*

Rank two

Remark

A crystallographic arrangement $\mathcal{A}$ of rank two and a chamber K define a sequence of negative Cartan entries

$$(c_1, \ldots, c_n) := (-c_{1,2}^{K}, -c_{2,1}^{\rho_1(K)}, -c_{1,2}^{\rho_2(\rho_1(K))}, \ldots)$$

$n = |\mathcal{A}|$, which is the **quiddity cycle** of a Conway-Coxeter frieze pattern.

Rank two

Corollary

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement of rank two and K a chamber.

1. *Any $\alpha \in R_+^K$ is either simple or the sum of two positive roots in R_+^K.*
2. *If α, β are simple roots and $k\alpha + \beta \in R_+^K$, then $\ell\alpha + \beta \in R_+^K$ for all $\ell = 0, \dots, k$.*

Arbitrary rank

The first claim of the corollary may be extended to arbitrary rank, we omit the proof because it involves the length function of a Weyl groupoid:

Theorem

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement, K a chamber, and $\alpha \in R_+^K$ a positive root. Then either α is simple, or it is the sum of two positive roots in R_+^K.

The second part of the corollary extends to arbitrary rank as well (we will see this later).

Localizations in rank three

Now assume that $r = 3$, i.e. $V = \mathbb{R}^3$.

Lemma

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement of rank three and K a chamber. Then $(\mathcal{A}, V)$ is reducible if $|R_+^K \cap \langle \alpha_1, \alpha_2 \rangle| = |R_+^K \cap \langle \alpha_1, \alpha_3 \rangle| = 2$.

Proof.

Since $\sigma_1^K(\alpha_2) = \alpha_2$, $\sigma_1^K(\alpha_3) = \alpha_3$, the chamber $\rho_1(K)$ is also adjacent to the localization $\langle \alpha_2, \alpha_3 \rangle$. But then any further $\beta \in R_+^K \setminus \{\alpha_1\}$ is in $\langle \alpha_2, \alpha_3 \rangle$, thus $\mathcal{A}$ is a so-called near pencil arrangement which is reducible. □

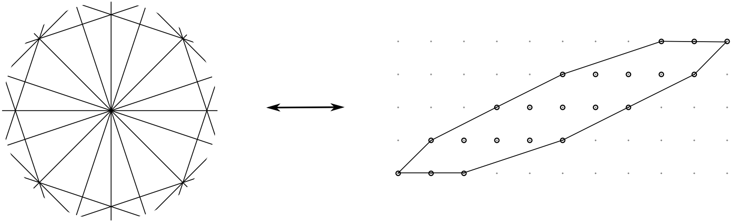


Figure: A localization and the roots on the boundary in the dual space.

Definition

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement, K_1 a chamber, $1 \leqslant i \neq j \leqslant r$, and $n := |\langle \alpha_i, \alpha_j \rangle \cap R_+^K|$. We denote the $2n$ chambers adjacent to the localization $\langle \alpha_i, \alpha_j \rangle$ by $K_1, \ldots, K_{2n}$: for $\ell > 1$, let

$$K_\ell := \begin{cases} \rho_i(K_{\ell-1}) & \text{if } \ell \text{ is even,} \\ \rho_j(K_{\ell-1}) & \text{if } \ell \text{ is odd.} \end{cases}$$

Notice that $K_{2n+1} = K_1$.

Localizations in rank three

Definition

This sequence of chambers yields two sequences of integers:

$$c_\ell := \begin{cases} -c_{i,j}^{K_\ell} & \text{if } \ell \text{ is odd,} \\ -c_{j,i}^{K_\ell} & \text{if } \ell \text{ is even,} \end{cases} \qquad d_\ell := \begin{cases} -c_{i,k}^{K_\ell} & \text{if } \ell \text{ is odd,} \\ -c_{j,k}^{K_\ell} & \text{if } \ell \text{ is even} \end{cases}$$

for $\ell = 1, \dots, 2n$ and the unique $k \notin \{i, j\}$ with $1 \leqslant k \leqslant r = 3$.

We call $(c_1, \dots, c_n)$ the **quiddity cycle**
and $(d_1, \dots, d_{2n})$ the **auxiliary cycle** of the localization $\langle \alpha_i, \alpha_j \rangle$.

Localizations in rank three

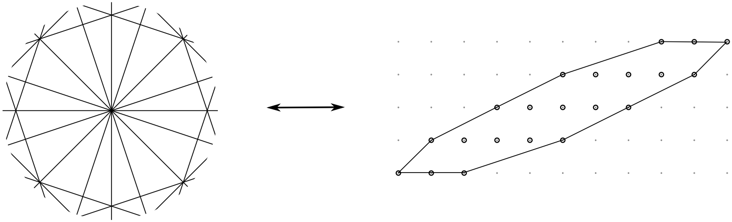


Figure: A localization and the roots on the boundary in the dual space.

Localizations in rank three

Proposition

Let $(\mathcal{A}, V, \mathcal{R})$ *be an irreducible crystallographic arrangement of rank three and* K *a chamber. Let* $\beta_1 = (0,1,0), \beta_2, \dots, \beta_{n-1}, \beta_n = (1,0,0)$ *be the roots in the localization* $\langle \alpha_1, \alpha_2 \rangle$ *ordered in such a way that* $(\beta_1, \dots, \beta_n)$ *"is" an* $\mathcal{F}$*-sequence. Let* $(d_1, \dots, d_{2n})$ *be the auxiliary cycle of the localization* $\langle \alpha_2, \alpha_1 \rangle$*.*

Localizations in rank three

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1.
$$\gamma_\ell := \alpha_3 + \sum_{k=1}^{\ell} d_k \beta_k, \quad \delta_\ell := \alpha_3 + \sum_{k=1}^{\ell} d_{2n+1-k} \beta_{n+1-k},$$

   $\ell = 0, \ldots, n$ are positive roots in R^K with third coordinate 1. These are the vertices of the convex set in the $(*, *, 1)$-plane.

Localizations in rank three

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2. *There are no consecutive d_ℓ's both equal to 0.*

Localizations in rank three

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Let $(\mathcal{A}, V, \mathcal{R})$ be an irreducible crystallographic arrangement of rank three and K a chamber. Let $\beta_1 = (0,1,0), \beta_2, \ldots, \beta_{n-1}, \beta_n = (1,0,0)$ be the roots in the localization $\langle \alpha_1, \alpha_2 \rangle$ ordered in such a way that $(\beta_1, \ldots, \beta_n)$ "is" an $\mathcal{F}$-sequence. Let $(d_1, \ldots, d_{2n})$ be the auxiliary cycle of the localization $\langle \alpha_2, \alpha_1 \rangle$. Then

1. $$\gamma_\ell := \alpha_3 + \sum_{k=1}^{\ell} d_k \beta_k, \quad \delta_\ell := \alpha_3 + \sum_{k=1}^{\ell} d_{2n+1-k} \beta_{n+1-k},$$

   $\ell = 0, \ldots, n$ are positive roots in R^K with third coordinate 1. These are the vertices of the convex set in the $(*,*,1)$-plane.
2. *There are no consecutive d_ℓ's both equal to 0.*
3. $|\{\gamma_\ell \mid \ell = 0, \ldots, n\}| \geqslant n/2$ *and* $\gamma_{\ell+1} - \gamma_\ell \in \mathbb{N}_0^3$.

The next lemma is a crucial tool. It extends the convexity which was observed in rank two to localizations and may be applied to pairs of roots in the $(*, *, 1)$-plane:

Lemma

Let $(\mathcal{A}, V, \mathcal{R})$ *be a crystallographic arrangement,* K *a chamber,* $k \in \mathbb{N}_{\geqslant 2}$*,* $\alpha \in R_+^K$, $\beta \in \mathbb{Z}^r$, $\dim\langle\alpha, \beta\rangle_{\mathbb{Q}} = 2$, $\alpha + k\beta \in R^K$, $\mathrm{Vol}_2(\alpha, \beta) = 1$, *and* $(-\mathbb{N}\alpha + \mathbb{Z}\beta) \cap \mathbb{N}_0^r = \varnothing$.

Then $\beta \in R^K$ *and* $\alpha + \ell\beta \in R^K$ *for all* $\ell = 0, \ldots, k$.
Moreover, there exists a chamber K' *and* $1 \leqslant i, j \leqslant r$ *such that* $-c_{i,j}^{K'} \geqslant k$.

Example

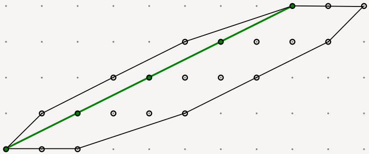


Figure: The lemma applied to the $(*, *, 1)$-plane.

With $\alpha = (0, 0, 1)$, $\beta = (2, 1, 0)$, and $k = 4$, the lemma implies the existence of the roots on the green line in the figure.

In fact, in this example the lemma implies that all lattice points in the convex set in the figure are roots.

Localizations in rank three

The next theorem is stronger than expected. If three roots have volume 1, then they are close to be the walls of a chamber:

Theorem

Let K be a chamber and $\alpha, \beta, \gamma \in R_+^K$. If $\mathrm{Vol}_3(\alpha, \beta, \gamma) = 1$ and none of $\alpha - \beta$, $\alpha - \gamma$, $\beta - \gamma$ are contained in R^K, then α, β, γ are the simple roots in R^K.

Localizations in rank three

Corollary

Let K be a chamber and $\gamma_1, \gamma_2, \alpha \in R^K$. Assume that γ_1, γ_2 are simple roots and that $\mathrm{Vol}_3(\gamma_1, \gamma_2, \alpha) = 1$. Then either α is a simple root or one of $\alpha - \gamma_1$, $\alpha - \gamma_2$ is contained in R^K.

Example

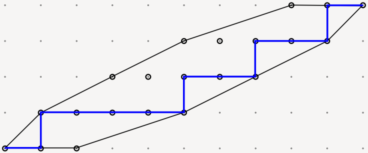


Figure: A path of roots in the $(*, *, 1)$-plane.

Repeatedly applying the corollary with $\gamma_1 = (1, 0, 0)$, $\gamma_2 = (0, 1, 0)$, and starting with $\alpha = (10, 4, 1)$ yields (for example) the blue path of roots displayed in the figure.

Remark

A short proof for the fact that all lattice points in the convex hull of the roots in the $(*, *, 1)$-plane are roots is still unknown.

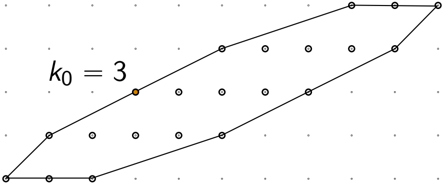
Localizations in rank three

Lemma

Let $(\mathcal{A}, V, \mathcal{R})$ *be an irreducible crystallographic arrangement of rank three and* K *a chamber. Then* $\alpha_1 + \alpha_2 + \alpha_3 \in R^K$.

Bounds

Bounds



Theorem

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement of rank three, K a chamber, and $|R_+^K \cap \langle \alpha_1, \alpha_2 \rangle| \geqslant 5$. Then

$$k_0 := \min\{k \in \mathbb{N}_0 \mid k\alpha_1 + 2\alpha_2 + \alpha_3 \in R^K\} \in \{0, \dots, 4\}$$

and $k_0 \leqslant 2$ if $c_{1,3}^K = 0$.

Proof.

Let $(c_1, \dots, c_n)$ be the quiddity cycle, $(d_1, \dots, d_{2n})$ the auxiliary cycle of $\langle \alpha_2, \alpha_1 \rangle$, and $\gamma_0, \dots, \gamma_n$ as before. Then

$$\gamma_0 = (0,0,1), \quad \gamma_1 = (0, d_1, 1), \quad \gamma_2 = (d_2, c_1 d_2 + d_1, 1),$$

$$\gamma_3 = (c_2 d_3 + d_2, c_1 c_2 d_3 + c_1 d_2 + d_1 - d_3, 1),$$

$$\gamma_4 = (c_2 c_3 d_4 + c_2 d_3 + d_2 - d_4, c_1 c_2 c_3 d_4 + c_1 c_2 d_3 + c_1 d_2 - c_1 d_4 - c_3 d_4 + d_1 - d_3, 1),$$

are positive roots. Moreover, $(1,1,1) \in R^K$.

Now we consider several cases:

Remark first that if $(0, c, 1) \in R^K$ for $c > 1$, then $(0, 2, 1) \in R^K$ by a lemma since $\gamma_0 = (0, 0, 1) \in R^K$. Similarly, if $(1, c, 1) \in R^K$ for $c > 1$, then $(1, 2, 1) \in R^K$ by a lemma since $(1, 1, 1) \in R^K$. Hence

$$(k, c, 1) \in R^K, \ k \leqslant 1, \ c > 1 \quad \Longrightarrow \quad k_0 \leqslant 1. \tag{5}$$

Now we consider all possible values for the cycles.
If $d_1 \geqslant 2$, then $k_0 \leqslant 1$ by (5) since $\gamma_1 \in R^K$. Hence assume $d_1 \leqslant 1$.
We first consider the case $c_1 > 1$.
If $d_1 = 0$, then $d_2 > 0$ (Prop.). Applying a lemma to $\gamma_0, (d_2, c_1 d_2, 1) = \gamma_2 \in R^K$ gives $(1, c_1, 1) \in R^K$, thus $k_0 \leqslant 1$ by (5).
If $d_1 = 1$, $d_2 > 0$, then $\gamma_2 = d_2(1, c_1, 0) + \gamma_1$, thus $(1, c_1 + 1, 1) \in R^K$ and $k_0 \leqslant 1$ by (5).
If $d_1 = 1$, $d_2 = 0$, then $d_3 > 0$, $\gamma_3 = d_3(c_2, c_1 c_2 - 1, 0) + \gamma_1$ thus $(c_2, c_1 c_2, 1) \in R^K$ which implies $(1, c_1, 1) \in R^K$ and $k_0 \leqslant 1$ by (5).
Now consider the case $c_1 = 1$. This implies $c_2 > 1$ since $|R^K_+ \cap \langle \alpha_1, \alpha_2 \rangle| \geqslant 5$.
If $d_1 = 1$, $d_2 > 0$, then $\gamma_2 = d_2(1, 1, 0) + \gamma_1$, thus $(1, 2, 1) \in R^K$ and $k_0 \leqslant 1$.
If $d_1 = 1$, $d_2 = 0$, then $d_3 > 0$, $\gamma_3 = d_3(c_2, c_2 - 1, 0) + \gamma_1$ thus $(c_2, c_2, 1) \in R^K$ which implies $(2, 2, 1) \in R^K$ and $k_0 \leqslant 2$.
The last remaining case is $d_1 = 0$, and thus $d_2 > 0$. Notice that $d_1 = 0$ also implies $(1, 0, 1) \in R^K$ since $\delta_1 = (d_{2n}, 0, 1) \in R^K$ and $d_{2n} > 0$. Recall also that we are still in the case $c_1 = 1$ and $c_2 > 1$.
If $d_2 \geqslant 2$, then $\gamma_2 = (d_2, d_2, 1) \in R^K$ and thus $(2, 2, 1) \in R^K$ and $k_0 \leqslant 2$. Hence we may assume $d_2 = 1$.
If $d_3 > 0$ then $\gamma_3 = (c_2 d_3 + 1, c_2 d_3 + 1 - d_3, 1) = d_3(c_2, c_2 - 1, 0) + (1, 1, 1)$, thus $(c_2 + 1, c_2, 1) \in R^K$. But $(c_2 + 1, c_2, 1) = c_2(1, 1, 0) + (1, 0, 1)$ which implies $(3, 2, 1) \in R^K$ and $k_0 \leqslant 3$.
Finally, assume that $d_3 = 0$, $d_4 > 0$. Then $\gamma_4 = d_4(c_2 c_3 - 1, c_2 c_3 - 1 - c_3, 0) + (1, 1, 1)$ implies $(c_2 c_3, c_2 c_3 - c_3, 1) = c_3(c_2, c_2 - 1, 0) + (0, 0, 1) \in R^K$.
If $c_2 > 2$, then $(c_2, c_2 - 1, 1) = (c_2 - 1)(1, 1, 0) + (1, 0, 1) \in R^K$ and thus $(3, 2, 1) \in R^K$ and $k_0 \leqslant 3$.
If $c_2 = 2$, then $(2c_3, c_3, 1) \in R^K$. If $c_3 > 1$ then this implies $(4, 2, 1) \in R^K$ and $k_0 \leqslant 4$. The case $c_3 = 1$ is excluded since it implies $|R^K_+ \cap \langle \alpha_1, \alpha_2 \rangle| = 4$: by a remark, the only quiddity cycles containing $(1, 2, 1)$ are $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$.
If $c^K_{1,3} = 0$ then $d_{2n} = c^K_{1,3} = 0$ implies $d_1 > 0$ by a Prop. All above cases with positive d_1 imply $k_0 \leqslant 2$.

Bounds

This allows to compute a global bound for Cartan entries in crystallographic arrangements of rank greater than two:

Theorem

Let $(\mathcal{A}, V, \mathcal{R})$ be a crystallographic arrangement of rank greater or equal to three.
Then all entries of the Cartan matrices are greater or equal to -7.

Proof.

Assume that K is a chamber with largest Cartan entry $-c^K_{1,2} \geqslant 8$, i.e. $|R^K_+ \cap \langle \alpha_1, \alpha_2 \rangle| \geqslant 5$.

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By the theorem there exists $k_0 \in \{0, 1, 2, 3, 4\}$ such that $\gamma := k_0\alpha_1 + 2\alpha_2 + \alpha_3 \in R^K_+$.
In the adjacent chamber $K' = \rho_1(K)$, we have

$$\gamma' := \sigma^K_1(\gamma) = (-k_0 - 2c^K_{1,2} - c^K_{1,3})\alpha_1 + 2\alpha_2 + \alpha_3 \in R^{K'}_+.$$

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Again by the theorem there exists $k'_0 \in \{0, 1, 2, 3, 4\}$ such that $\alpha := k'_0\alpha_1 + 2\alpha_2 + \alpha_3 \in R^{K'}_+$.

Bounds

Now applying a lemma to α and $\gamma' = \alpha + (-k_0 - 2c^K_{1,2} - c^K_{1,3} - k'_0)\alpha_1$ yields a chamber K'' with $1 \leqslant i, j \leqslant 3$ and

$$-c^{K''}_{i,j} \geqslant -k_0 - 2c^K_{1,2} - c^K_{1,3} - k'_0.$$

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We have

$$k_0 \leqslant \begin{cases} 2 & \text{if } -c^K_{1,3} = 0, \\ 4 & \text{if } -c^K_{1,3} > 0, \end{cases}$$

thus

$$-c^{K''}_{i,j} \geqslant \begin{cases} -c^K_{1,2} + 2 > -c^K_{1,2} & \text{if } -c^K_{1,3} = 0, \\ -c^K_{1,2} - c^K_{1,3} > -c^K_{1,2} & \text{if } -c^K_{1,3} > 0. \end{cases}$$

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This is a contradiction to the assumption that $-c^K_{1,2}$ is the largest Cartan entry. □

Bounds

Remark

In fact, entries of the Cartan matrices in rank greater or equal to three are always greater or equal to -6.

Bounds

Notice that there are infinitely many non-equivalent crystallographic arrangements of rank two with Cartan entries greater or equal to -7. (quiddity cycles over $\mathbb{N}$ with entries $\leqslant 7$)

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However:

Theorem

Any localization of rank two of an irreducible crystallographic arrangement of rank three has at most 128 *positive roots.*

Proof.

Without loss of generality, assume that $|R_+^K \cap \langle \alpha_1, \alpha_2 \rangle| > 128$ for some chamber K. Then by a previous proposition there are more than 64 roots of the form $k\alpha_1 + \ell\alpha_2 + \alpha_3$,

Bounds

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$$a \equiv a' \pmod 8, \quad b \equiv b' \pmod 8,$$

and by the same proposition we may assume $a \geqslant a'$ and $b \geqslant b'$.

Bounds

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$$a \equiv a' \pmod 8, \quad b \equiv b' \pmod 8,$$

and by the same proposition we may assume $a \geqslant a'$ and $b \geqslant b'$. But then

$$(a, b, 1) = (a', b', 1) + k((a - a')/k, (b - b')/k, 0)$$

for some $k \geqslant 8$ and coprime $(a - a')/k, (b - b')/k \in \mathbb{Z}$.
By the "green lemma", this implies the existence of a Cartan entry less or equal to -8, contradicting the theorem. □

Bounds

Corollary

There is a finite set $\mathcal{I}$ of equivalence classes of crystallographic arrangements of rank two such that every localization of rank two of an irreducible crystallographic arrangement of rank three belongs to one of the classes in $\mathcal{I}$.

Proof.

By the theorem, a localization of rank two of a crystallographic arrangement of rank three has at most 128 positive roots.
Since a crystallographic arrangement $(\mathcal{A}, V, \mathcal{R})$ of rank two corresponds to a triangulation of a convex $|\mathcal{R}|/2$-gon by non-intersecting diagonals, there are only finitely many non-equivalent such arrangements with at most 128 positive roots. □

Bounds

Corollary

There exists a bound m, such that for any irreducible crystallographic arrangement of rank $r > 2$ and $\alpha, \beta \in \mathcal{R}$,

$$\mathrm{Vol}_2(\alpha, \beta) \leqslant m.$$

Remark

In fact, the sharp bound is $m = 6$.

Bounds

Proof.

Viewing α and β as elements of the localization $\langle\alpha,\beta\rangle$, we may choose a chamber K such that $\Upsilon^K(\alpha)=\alpha_i$, $\Upsilon^K(\beta)=a\alpha_i+b\alpha_j$ for suitable $a,b\in\mathbb{Z}$, without loss of generality $i=1$, $j=2$.

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This implies $\mathrm{Vol}_2(\alpha,\beta)=|b|\leqslant m$. □

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Theorem (C., Heckenberger (2015); C. (2019))

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Hence there is a global bound for the number of positive roots. But the number of equivalence classes of irreducible crystallographic arrangements with bounded number of roots is bounded. □

Enumeration and classification

Enumeration and classification

Theorem

Let K be a chamber of an irreducible crystallographic arrangement.

Let $\alpha \in R_+^K$. Then either α is simple, or it is the sum of two positive roots.

Function **Enumerate**(R)

1. If R defines a crystallographic arrangement, output R and continue.
2. $Y := \{\alpha + \beta \mid \alpha, \beta \in R,\ \alpha \neq \beta\} \backslash R$.
3. For all $\alpha \in Y$ with $\alpha > \max R$:
    1. Compute all localizations in $R \cup \{\alpha\}$.
    2. If all Cartan entries are $\geqslant -7$, all localizations are crystallographic [and ... and ...] then call **Enumerate**$(R \cup \{\alpha\})$.

The algorithm terminates and yields the result:

Theorem (C., Heckenberger (2012))

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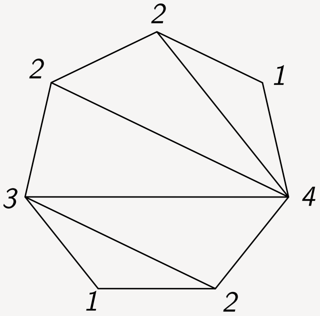
An analysis of Dynkin diagrams leads to a complete classification.

Classification

Theorem (C., Heckenberger, 2009/2010)

There are exactly three families of crystallographic arrangements:

1. *The family of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.*



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3. *Further 74 "sporadic" arrangements of rank r, $3 \leqslant r \leqslant 8$.*

Nichols algebras

Definition

Let V be a vector space,

$$c : V \otimes V \to V \otimes V$$

a linear isomorphism with

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Then c is a **braiding**, and (V, c) is a **braided vector space**.

Define a map $\rho : S_n \to \mathrm{End}(V^{\otimes n})$ by:

For a transposition $(i, i+1) \in S_n$ let

$$\rho((i, i+1)) := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where c acts in the copies i and $i+1$ of V.

If $\omega = \tau_1 \dots \tau_\ell$ is a reduced expression of $\omega \in S_n$, then

$$\rho(\omega) := \rho(\tau_1) \dots \rho(\tau_\ell).$$

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Definition

Let $\mathfrak{S}_n := \sum_{\omega \in S_n} \rho(\omega)$.

$$\mathfrak{B}(V) := \bigoplus_{n \geqslant 0} T^n(V)/\ker(\mathfrak{S}_n)$$

is called the **Nichols algebra** of (V, c).

Examples

- $c(x \otimes y) = y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = S(V)$ symmetric algebra

- $c(x \otimes y) = -y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = \Lambda(V)$ exterior algebra

Nichols algebras – Motivation

- Nichols (1978): construction of examples of Hopf algebras
- Woronowicz (1988): build a “quantum differential calculus”
- Lusztig (1993), Rosso (1994), Schauenburg (1996): abstract definition of quantized universal enveloping algebras
- Andruskiewitsch-Schneider (1998): essential tool in the classification of pointed Hopf algebras

Let (V, c) be a braided vector space.

- Is $\mathfrak{B}(V)$ finite dimensional?
- Compute the defining relations of $\mathfrak{B}(V)$.

Examples

Let $A = (a_{ij})_{1 \leqslant i,j \leqslant r}$ be a Cartan matrix of finite type and $d_1, \ldots, d_r \in \mathbb{N}_{>0}$ be such that $d_i a_{ij} = d_j a_{ji}$.

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Let V be a vector space over $\mathbf{k}$ with basis $x_1, \ldots, x_r$, and $q \in \mathbf{k}$, $c : V \otimes V \to V \otimes V$ given by $c(x_i \otimes x_j) = q^{d_i a_{ij}} x_j \otimes x_i$.

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Theorem (Lusztig)

If q is a root of unity of odd order N with $3 \nmid N$, then $\mathfrak{B}(V)$ is finite dimensional with basis [. . .].

$\mathfrak{B}(V)$ is the "positive part" of the Frobenius-Lusztig kernel of the Lie algebra associated to A.

Diagonal type

Definition

$\{x_1, \dots, x_r\}$ Basis of V,

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad q_{ij} \in \mathbb{C}.$$

Then c and $\mathfrak{B}(V)$ are called of **diagonal type**.

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Then c and $\mathfrak{B}(V)$ are called of **diagonal type**.

The numbers q_{ij}, $i, j = 1, \dots, r$ define a **bicharacter**

$$\chi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{C}, \quad ((a_1, \dots, a_r), (b_1, \dots, b_r)) \mapsto \prod_{i,j=1}^{r} q_{ij}^{a_i b_j}.$$

PBW basis for diagonal type

Let (V, c) be of diagonal type.

Theorem (Kharchenko, 1999)

There exists a totally ordered index set $(L, \leqslant)$ *and* $\mathbb{Z}^r$*-homogeneous elements* $X_\ell \in \mathfrak{B}(V)$, $\ell \in L$ *such that*

$$\{X_{\ell_1}^{m_1} \cdots X_{\ell_\nu}^{m_\nu} \quad | \quad \nu \geqslant 0, \quad \ell_1, \dots, \ell_\nu \in L, \quad \ell_1 > \dots > \ell_\nu, \\ 0 \leqslant m_i < h_{\ell_\nu} \ \forall i = 1, \dots, \nu\}$$

is a vector space basis of $\mathfrak{B}(V)$, *where*

$$h_\ell = \min\{m \in \mathbb{N} \mid 1 + q_\ell + \dots + q_\ell^{m-1} = 0\} \cup \{\infty\}$$

and $q_\ell = \chi(\deg X_\ell, \deg X_\ell)$, $\ell \in L$.

Finite dimensional Nichols algebras of diagonal type

Theorem (Heckenberger, 2006)

Let $\mathfrak{B}$ be a finite dimensional Nichols algebra of diagonal type.

Let R_+ be the set of degrees of the PBW generators of $\mathfrak{B}$.
Then $R_+ \cup -R_+$ is a root system of a finite Weyl groupoid.

Result (Angiono, 2013)

Explicit list of defining relations of a Nichols algebra of diagonal type with finite root system.

Yetter-Drinfeld modules

Definition

Let H be a Hopf algebra and V a module and a comodule over H. Then V is called a **Yetter-Drinfeld module** if

$$\delta_V(hv) = h_1 v_{-1} S(h_3) \otimes h_2 v_0 \quad \forall h \in H, v \in V.$$

A Yetter-Drinfeld module V is a braided vector space via

$$c : V \otimes V \to V \otimes V, \quad v \otimes w \mapsto v_{-1} w \otimes v_0.$$

Example

G a finite group, $H = \mathbb{C}G \Rightarrow$
Yetter-Drinfeld modules are representations of the quantum double $D(G)$.

Yetter-Drinfeld modules

Let V be a Yetter-Drinfeld module over $\mathbb{C}G$ where G is a finite group.

- G abelian $\Rightarrow \mathfrak{B}(V)$ of diagonal type.
- G non-abelian, V irreducible $\Rightarrow \mathfrak{B}(V)$ Nichols algebra of a rack.

The Weyl groupoid in diagonal type – rank two

Let $\mathbf{q} = (q_1, q, q_2)$ be a triple of numbers (in a commutative ring) and assume that

$$m_i := \min\{m \in \mathbb{N}_0 \mid 1 + q_i + q_i^2 + \ldots + q_i^m = 0 \text{ or } q_i^m q = 1\}$$

for $i = 1, 2$ are well defined integers.

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$$\begin{aligned}
\sigma_1(q_1, q, q_2) &= (q_1, q_1^{-2m_1} q^{-1}, q_1^{m_1^2} q^{m_1} q_2) \\
&= \begin{cases} (q_1, q_1^2 q^{-1}, q_1 q^{m_1} q_2) & \text{if } 1 + q_1 + q_1^2 + \ldots + q_1^{m_1} = 0 \\ (q_1, q, q_2) & \text{if } q_1^{m_1} q = 1 \end{cases}
\end{aligned}$$

and similarly

$$\sigma_2(q_1, q, q_2) = (q_1 q^{m_2} q_2^{m_2^2}, q_2^{-2m_2} q^{-1}, q_2).$$

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Thus σ_1, σ_2 produce new triples of numbers which possibly define new integers m_i, and notice that $\sigma_i(\sigma_i(q_1, q, q_2)) = (q_1, q, q_2)$.

The Weyl groupoid in diagonal type – rank two

Definition

Assuming that the new m_i are well defined again and again, the first triple $\mathbf{q}_0 := \mathbf{q} = (q_1, q, q_2)$ will produce an infinite sequence of the form

$$\dots \overset{\sigma_2}{\longleftrightarrow} \mathbf{q}_{-2} \overset{\sigma_1}{\longleftrightarrow} \mathbf{q}_{-1} \overset{\sigma_2}{\longleftrightarrow} \mathbf{q}_0 \overset{\sigma_1}{\longleftrightarrow} \mathbf{q}_1 \overset{\sigma_2}{\longleftrightarrow} \mathbf{q}_2 \overset{\sigma_1}{\longleftrightarrow} \dots$$

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where every σ_i has its own m_i, thus we obtain a sequence of integers

$$\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$$

which we call the **characteristic sequence** of $\mathbf{q} = (q_1, q, q_2)$, where the c_i correspond to the maps in the following way ($c_0 = m_1$, $c_{-1} = m_2$):

$$\ldots \xleftrightarrow{c_{-3}} \mathbf{q}_{-2} \xleftrightarrow{c_{-2}} \mathbf{q}_{-1} \xleftrightarrow{c_{-1}} \mathbf{q}_0 \xleftrightarrow{c_0} \mathbf{q}_1 \xleftrightarrow{c_1} \mathbf{q}_2 \xleftrightarrow{c_2} \ldots$$

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We say that a triple $\mathbf{q}$ is **broken** if the above procedure leads to a triple for which one of the m_i is not defined.

The Weyl groupoid in diagonal type – rank two

Example

Let $\zeta \in \mathbb{C}$ be a primitive 9-th root of unity and $\mathbf{q} = (\zeta^6, \zeta^8, \zeta^6)$. Then the above picture is

$$\dots \xleftrightarrow{5} (\zeta, \zeta^4, \zeta^6) \xleftrightarrow{2} (\zeta^6, \zeta^8, \zeta^6) \underset{\sigma_1}{\xleftrightarrow{2}} (\zeta^6, \zeta^4, \zeta) \xleftrightarrow{5} (\zeta^6, \zeta^4, \zeta) \xleftrightarrow{2} \dots$$

and the characteristic sequence is $(\dots, 2, 2, 5, 2, 2, 5, \dots)$, thus periodic with period $(2, 2, 5)$.

The Weyl groupoid in diagonal type – rank two

To determine the triple **q** from a given characteristic sequence, the knowledge of three consecutive entries c_i, c_{i+1}, c_{i+2} is (almost) sufficient.

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The Nichols algebra of diagonal type corresponding to a triple **q** *is finite dimensional if and only if the characteristic sequence of* **q** *is the quiddity cycle of a Conway-Coxeter frieze pattern.*

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Corollary

A local description ($\ell = 3$) of quiddity cycles leads to a complete classification of finite dimensional Nichols algebras of diagonal type in rank two.

Question

What about "infinite" Weyl groupoids?

Definition (C., Mühlherr, Weigel, 2014)

Let $\mathcal{A}$ be a set of linear hyperplanes in V and $\emptyset \neq T \subseteq V$ an open convex cone (called the **Tits cone**). We call $(\mathcal{A}, T)$ a **simplicial arrangement**, if

1. $H \cap T \neq \emptyset \quad \forall H \in \mathcal{A}$,
2. $\forall v \in T \ \exists \varepsilon > 0 \ : \ |\{H \in \mathcal{A} \mid H \cap U_\varepsilon(v) \neq \emptyset\}| < \infty$,
3. the connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are simplicial cones,
4. every wall is in $\mathcal{A}$.

The non-spherical case

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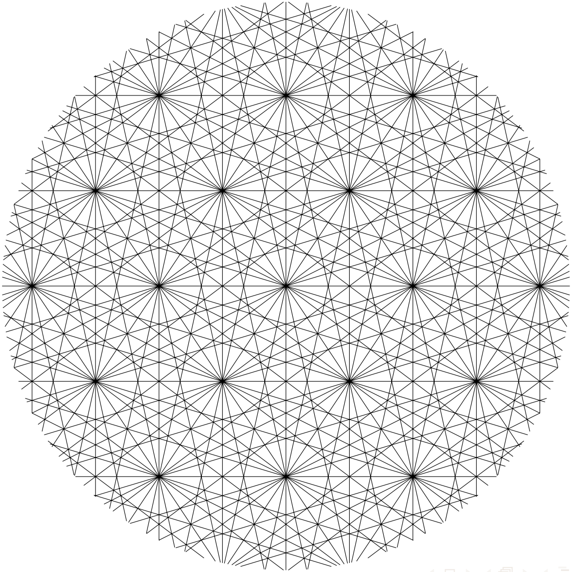
$(\mathcal{A}, T, R)$ is a **crystallographic arrangement**, if

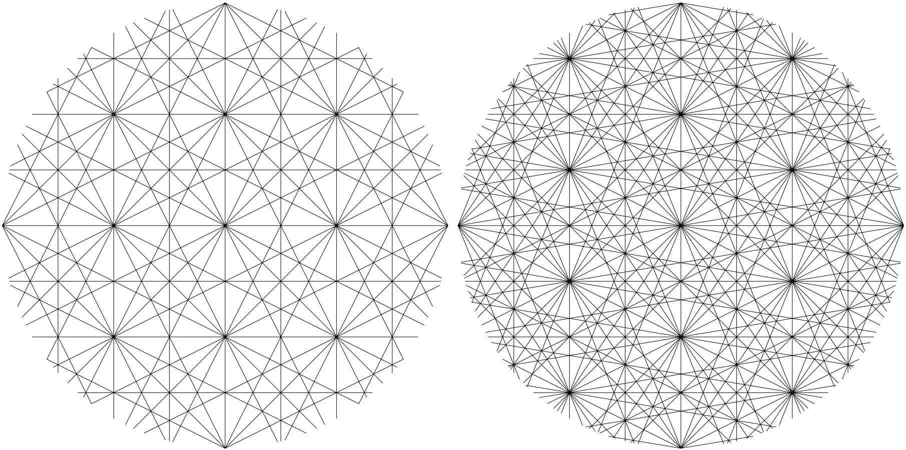
1. $(\mathcal{A}, T)$ is simplicial,
2. $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$,
3. for all $K \in \mathcal{K}(\mathcal{A})$:

$$R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha.$$

An affine simplicial arrangement



Affine crystallographic arrangements



Example

1. If $V = T$, then $\mathcal{A}$ is a finite simplicial arrangement.
2. If T is a half-space, then $(\mathcal{A}, T)$ is called **affine**.
3. An affine Weyl group defines an affine crystallographic arrangement.

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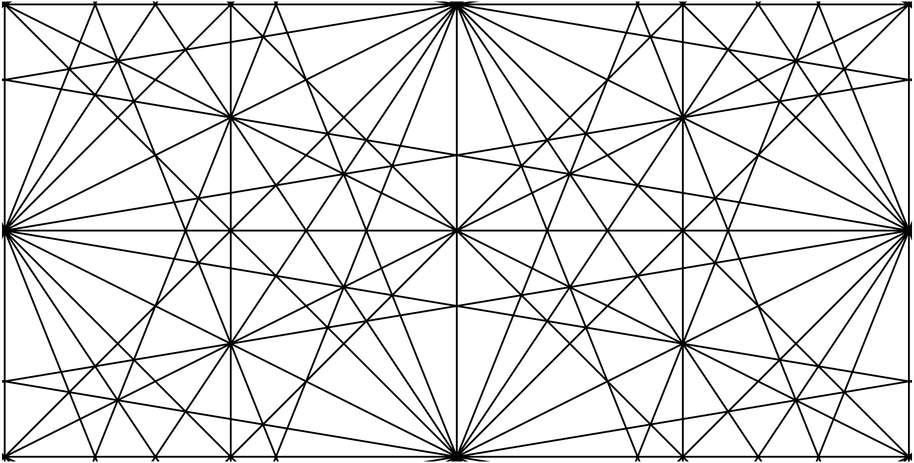
Theorem (C., Mühlherr, Weigel, 2014)

Correspondence: "Weyl groupoids" $\longleftrightarrow$ crystallographic arrangements.

Theorem (C., Mühlherr, 2013)

Characterization of Weyl groupoids of rank two with finitely many objects via periodic continued fractions.

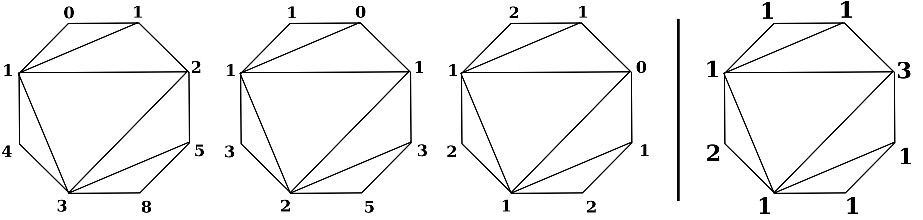
Affine crystallographic arrangements



Some numbers

Quiddity cycle: $c = (1, 2, 3, 2, 1, 4, 1, 4)$

$$m_i := \{j \in \{1, \ldots, n\} \mid c_{i,j} \geqslant c_{i,\ell} \text{ for all } \ell = 1, \ldots, n\}.$$



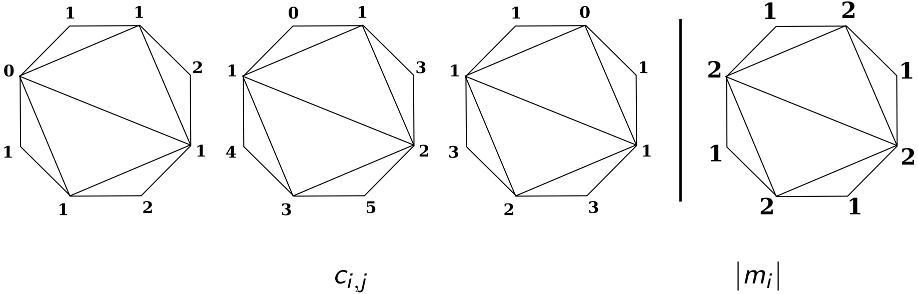
$c_{i,j}$ $|m_i|$

$\rightsquigarrow (|m_1|, |m_2|, \ldots) = (1, 1, 3, 1, 1, 1, 2, 1)$

More numbers

Quiddity cycle: $c = (1, 3, 1, 4, 1, 3, 1, 4)$

$$m_i := \{j \in \{1, \ldots, n\} \mid c_{i,j} \geqslant c_{i,\ell} \text{ for all } \ell = 1, \ldots, n\}.$$



$\rightsquigarrow (|m_1|, |m_2|, \ldots) = (1, 2, 1, 2, 1, 2, 1, 2)$

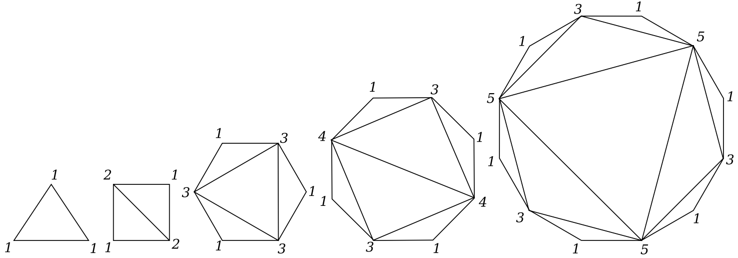
Dense sequences

Theorem (C., 2013)

Let c be a quiddity cycle such that for all i, $|m_i| > 1$ or $|m_{i+1}| > 1$. Then up to rotations, c is one of the following:

$$(1,1,1), \quad (1,2,1,2), \quad (1,3,1,3,1,3),$$

$$(1,3,1,4,1,3,1,4), \quad (1,3,1,5,1,3,1,5,1,3,1,5).$$



Affine simplicial arrangements

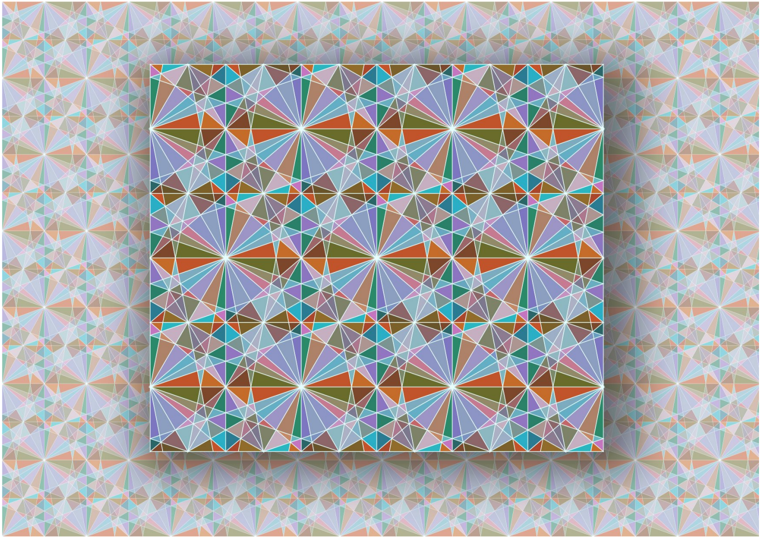
Theorem (C., 2013)

Let c be a quiddity cycle and $R \subseteq \mathbb{Z}^2$ its root system (at any object). If

$$\{(x,y,z)^\perp \mid (x,y) \in R,\ z \in \mathbb{Z}\}$$

is simplicial, then c is

$$(1,1,1),\ (1,2,1,2),\ (1,3,1,3,1,3),\ \textit{or}\ (1,3,1,5,1,3,1,5,1,3,1,5).$$



Thank you!