

Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights

Guillaume Chapuy, Theo Douvropoulos

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The number of reduced reflection factorizations of c

Theorem (Hurwitz, 1892)

There are n^{n-2} (minimal length) factorizations $t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n$ where the t_i 's are transpositions.

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Theorem (Deligne-Arnol'd-Bessis)

For a well-generated, complex reflection group W and a Coxeter element c , there are $\frac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1 \cdots t_n = c$ where $h = |c|$.

Arbitrary length reflection factorizations of c

If \mathcal{R} denotes the set of reflections of W , we write

$$\text{Fact}_{W,c}(N) := \#\{(t_1, \dots, t_N) \in \mathcal{R}^n \mid t_1 \cdots t_N = c\}.$$

Now, consider the exponential generating function:

$$\text{FAC}_{S_n,c}(t) = \sum_{N \geq 0} \text{Fact}_{S_n,c}(N) \frac{t^N}{N!}.$$

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Theorem (Jackson, '88)

If $c = (12 \cdots n) \in S_n$, then

$$\text{FAC}_{S_n,c}(t) = \frac{e^{t \binom{n}{2}}}{n!} (1 - e^{-tn})^{n-1}.$$

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Notice that

$$\left[\frac{t^{n-1}}{(n-1)!} \right] \text{FAC}_{S_n,c}(t) = \frac{1}{n!} \cdot (n)^{n-1} \cdot (n-1)! = n^{n-2}.$$

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The Burman-Zvonkine weighted enumeration for S_n

We consider $\binom{n}{2}$ parameters $\boldsymbol{\omega} := (\omega_{ij})_{i < j}$ that form a weight system $\mathbf{w}((ij)) = \omega_{ij}$ for the transpositions $(ij) \in S_n$. If \mathcal{C} denotes the class of the long cycles, we define:

$$FAC_{S_n}(t, \boldsymbol{\omega}) := \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} \sum_{\substack{(\tau_1, \tau_2, \dots, \tau_\ell, c) \in \mathcal{R}^\ell \times \mathcal{C} \\ \tau_1 \cdots \tau_\ell = c}} \mathbf{w}(\tau_1) \cdot \mathbf{w}(\tau_2) \cdots \mathbf{w}(\tau_\ell).$$

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Theorem (Burman and Zvonkine, 2010)

The exponential generating function above factors as

$$FAC_{S_n}(t, \boldsymbol{\omega}) = \frac{e^{t\mathbf{w}(\mathcal{R})}}{n} \cdot \prod_{i=1}^{n-1} (1 - e^{-t\lambda_i(\boldsymbol{\omega})}),$$

where $\mathbf{w}(\mathcal{R}) = \sum_{i < j} (\omega_{ij})$ and $\{\lambda_i(\boldsymbol{\omega})\}$ are the non-zero eigenvalues of the (weighted) Laplacian $L(K_n)$.

Enumeration for arbitrary W with parabolic weights

Consider a (maximal) tower of parabolic subgroups

$$T := (\{\mathbf{1}\} = W_0 \leq W_1 \leq W_2 \leq \cdots \leq W_n = W),$$

and a weight system \mathbf{w}_T on reflections $\tau \in \mathcal{R}$ with parameters $\omega := (\omega_i)$ given by

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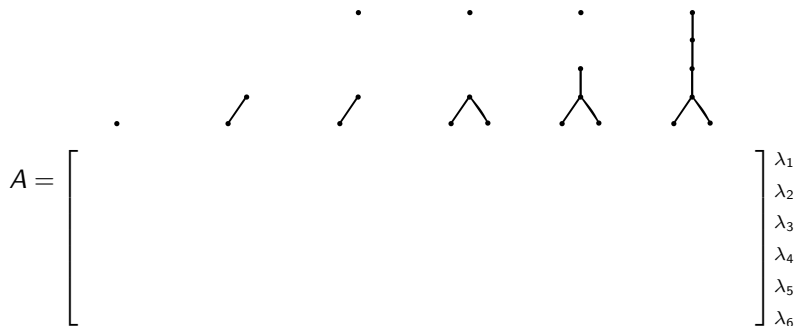
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where $\{\lambda_i^T(\boldsymbol{\omega})\}$ are the eigenvalues on the reflection representation V of the W -Laplacian $L_T(W) := \sum_{\tau \in \mathcal{R}} \mathbf{w}_T(\tau) \cdot (\mathbf{1} - \tau) \in \mathbb{C}[\boldsymbol{\omega}][W]$

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$$A = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{bmatrix}$$

The diagram shows a sequence of six graphs corresponding to the rows of matrix A. Each graph has a central node at the bottom. The first graph has one node. The second has two nodes connected by a diagonal edge. The third has two nodes connected by a diagonal edge, with a single node above the central node. The fourth has two nodes connected by a diagonal edge, with two nodes above the central node. The fifth has two nodes connected by a diagonal edge, with three nodes above the central node. The sixth has two nodes connected by a diagonal edge, with four nodes above the central node.

But, what do the eigenvalues look like?

$$A = \begin{bmatrix} 2 & 1 & & & & \\ 0 & 3 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix}$$

Diagrams illustrating the eigenvalues λ_1 through λ_6 of the matrix A :

- λ_1 : A single vertex.
- λ_2 : A vertex with one edge.
- λ_3 : A vertex with two edges.
- λ_4 : A vertex with three edges.
- λ_5 : A vertex with four edges.
- λ_6 : A vertex with five edges.

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Diagrams illustrating graph structures corresponding to the eigenvalues λ_1 through λ_6 :

- λ_1 : A single vertex (represented by a dot).
- λ_2 : A single edge (represented by a line segment between two vertices).
- λ_3 : A path of length 2 (represented by two edges sharing a central vertex).
- λ_4 : A star graph with 3 edges (represented by three edges meeting at a central vertex).
- λ_5 : A path of length 3 (represented by three edges in a chain).
- λ_6 : A star graph with 4 edges (represented by four edges meeting at a central vertex).

But, what do the eigenvalues look like?

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 & & \\ 0 & 3 & 0 & 1 & & \\ 0 & 0 & 2 & 0 & & \\ 0 & 0 & 0 & 4 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

The matrix A is associated with six graph structures shown above it, corresponding to the columns of the matrix. The structures are:

- Column 1: A single vertex (degree 2).
- Column 2: A path of two vertices (degree 3).
- Column 3: A path of two vertices (degree 2).
- Column 4: A path of three vertices (degree 4).
- Column 5: A star graph with three vertices (degree 3).
- Column 6: A star graph with four vertices (degree 4).

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$$A = \begin{bmatrix}
 \cdot & & & & & & \\
 \cdot & & & & & & \\
 & \diagup & & & & & \\
 & & \diagup & & & & \\
 & & & \wedge & & & \\
 & & & & \vee & & \\
 & & & & & \text{Y-shape} & \\
 & & & & & & \text{I-shape}
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$$A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \diagup & \diagup & \wedge & \vee & \vee \\ 2 & 1 & 0 & 1 & 2 & 4 \\ 0 & 3 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

$$\{\lambda_i(\omega)\} = \left\{ \begin{array}{l} 2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 2\omega_3 + 8\omega_6, \quad 4\omega_4 + 2\omega_5 + 4\omega_6, \\ 6\omega_5 + 4\omega_6, \quad 10\omega_6 \end{array} \right\}$$

Representation theoretic interpretation

The filtration by T defines natural analogs of the Jucys-Murphy elements:

$$\mathbb{C}[W] \ni J_i := \sum_{\tau \in \mathcal{R} \text{ and } \tau \in W_i \setminus W_{i-1}} \tau,$$

and we write $\mathbb{C}[\mathbf{J}] := \mathbb{C}[J_1, \dots, J_n]$ for the (commutative) algebra they generate.

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Definition

We say that two (virtual) characters ψ_1 and ψ_2 are *tower equivalent*, and write $\psi_1 \equiv \psi_2$, if they agree on the subalgebra $\mathbb{C}[\mathbf{J}]$ of $\mathbb{C}[W]$ for *any* choice of T .

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For any irreducible character $\chi \in \widehat{W}$ Gordon-Griffeth define a *Coxeter number* c_χ

$$c_\chi := (1/\chi(1)) \cdot \chi\left(\sum_{\tau \in \mathcal{R}} (\mathbf{1} - \tau)\right).$$

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Theorem (Chapuy, D. -version appropriate for induction)

Our enumeration theorem can be re-phrased via the tower equivalence relation as:

$$\sum_{\chi \in \widehat{W} \text{ s.th. } c_\chi = kh} \chi(c^{-1}) \cdot \chi \equiv (-1)^k \cdot \bigwedge^k (V_{\text{ref}}) \quad \text{and} \quad \sum_{c_\chi = m \text{ with } h \nmid m} \chi(c^{-1}) \cdot \chi \equiv 0.$$

- 1 A weighted version of the Frobenius Lemma

$$\text{FAC}_{W,c}(t) = \sum_{\chi \in \widetilde{W}} \dim(\chi) \cdot \chi(c^{-1}) \cdot \exp(\tilde{\chi}(\mathcal{R}) \cdot t)$$

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Ingredients of our proof

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- 2 A recursion to maximal parabolics to prove tower equivalence in $G(r, 1, n)$ and $G(r, r, n)$.
- 3 The eigenvalues of the W -Laplacian $L_T(W)$ in $\bigwedge^k(V_{\text{ref}})$ are the k -sums of its eigenvalues in V_{ref} (via Burman's Lie-like elements).

A remarkable W -Matrix forest theorem

Theorem (Chapuy, D. 2019)

If we write the characteristic polynomial for the W -Laplacian as

$$\chi(L_T(W), x) := \det(x \cdot \mathbf{1} - L_T(W)) = (-1)^n c_0^T + \cdots + (-1)^{n-k} c_k^T \cdot x^k + \cdots + c_n^T \cdot x^n,$$

then the coefficients c_k^T are given by

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Corollary

Let $\text{Hur}(W_X)$ denote the number of reduced reflection factorizations of a Coxeter element in W_X . Then,

$$\sum_{X \in \mathcal{L}} |W_X| \cdot \text{Hur}(W_X) \cdot \frac{t^{\text{codim}(X)}}{\text{codim}(X)!} = (ht + 1)^n.$$

Discussion, generalizations, advertisement

- 1 The algebra $\mathbb{C}[\mathbf{J}]$ is not worthy of the name *Gelfand-Tsetlin* algebra; often:

$$\mathbb{C}[\mathbf{J}] \subsetneq \langle Z(\mathbb{C}[W_1]), \dots, Z(\mathbb{C}[W_n]) \rangle \subsetneq \bigoplus_{\chi \in \widehat{W}} \text{Hom}(U_\chi, \mathbb{C}[W]).$$

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 - 1 Type B_n : All possible n^2 weights ω_{ij} .
 - 2 Type $G(r, 1, n)$: The reflections (ii^ξ) that fix the same hyperplane $x_i = 0$ get the same weight.

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$$\mathbb{C}[\mathbf{J}] \subsetneq \langle Z(\mathbb{C}[W_1]), \dots, Z(\mathbb{C}[W_n]) \rangle \subsetneq \bigoplus_{x \in \widehat{W}} \text{Hom}(U_x, \mathbb{C}[W]).$$

However, if it were, our theorem could never be true!

- 2 In the infinite families $G(r, 1, n)$ and $G(r, r, n)$ we get factoring formulas even if we use finer weight systems (but the W -Laplacian does *not* appear). Is there a combinatorial or representation theoretic significance for such maximal weighting systems?
 - 1 Type B_n : All possible n^2 weights ω_{ij} .
 - 2 Type $G(r, 1, n)$: The reflections (ij^ξ) that fix the same hyperplane $x_i = 0$ get the same weight.
- 3 Our elaboration on the Frobenius lemma allows experimentation in any group that has natural towers of subgroups. What about $GL_n(F_q)$?

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- 4

$$\chi(L_{\mathcal{A}}(\omega), t) = \sum_{X \in \mathcal{L}_{\mathcal{A}}} \text{q-det}(L_{\mathcal{A}_X}(\omega)) \cdot (-1)^{\text{codim}(X)} \cdot t^{\dim(X)}$$

Thank you!

