Coxeter factorizations and the Matrix Tree theorem with generalized Jucys-Murphy weights

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The number of reduced reflection factorizations of c

Theorem (Hurwitz, 1892)

There are n^{n-2} (minimal length) factorizations $t_1 \cdots t_{n-1} = (12 \cdots n) \in S_n$ where the t_i's are transpositions.

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Theorem (Deligne-Arnol'd-Bessis)

For a well-generated, complex reflection group W and a Coxeter element c, there are $\frac{h^n n!}{|W|}$ (minimal length) reflection factorizations $t_1 \cdots t_n = c$ where $h = |c|$.

Arbitrary length reflection factorizations of c

If $\mathcal{R}$ denotes the set of reflections of W, we write

$$\mathrm{Fact}_{W,c}(N) := \#\{(t_1, \cdots, t_N) \in \mathcal{R}^n \mid t_1 \cdots t_N = c\}.$$

Now, consider the exponential generating function:

$$\mathrm{FAC}_{S_n,c}(t) = \sum_{N \geq 0} \mathrm{Fact}_{S_n,c}(N) \frac{t^N}{N!}.$$

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Theorem (Jackson, '88)

If $c = (12 \cdots n) \in S_n$, then

$$\mathrm{FAC}_{S_n,c}(t) = \frac{e^{t\binom{n}{2}}}{n!} \left(1 - e^{-tn}\right)^{n-1}.$$

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Notice that

$$\left[\frac{t^{n-1}}{(n-1)!}\right] \mathrm{FAC}_{S_n,c}(t) = \frac{1}{n!} \cdot (n)^{n-1} \cdot (n-1)! = n^{n-2}.$$

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The Burman-Zvonkine weighted enumeration for S_n

We consider $\binom{n}{2}$ parameters $\boldsymbol{\omega} := (\omega_{ij})_{i<j}$ that form a weight system $\mathbf{w}\big((ij)\big) = \omega_{ij}$ for the transpositions $(ij) \in S_n$. If $\mathcal{C}$ denotes the class of the long cycles, we define:

$$FAC_{S_n}(t, \boldsymbol{\omega}) := \sum_{\ell \geq 1} \frac{t^\ell}{\ell!} \sum_{\substack{(\tau_1, \tau_2, \cdots, \tau_\ell, c) \in \mathcal{R}^\ell \times \mathcal{C} \\ \tau_1 \cdots \tau_\ell = c}} \mathbf{w}(\tau_1) \cdot \mathbf{w}(\tau_2) \cdots \mathbf{w}(\tau_\ell).$$

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Theorem (Burman and Zvonkine, 2010)

The exponential generating function above factors as

$$FAC_{S_n}(t, \boldsymbol{\omega}) = \frac{e^{t\mathbf{w}(\mathcal{R})}}{n} \cdot \prod_{i=1}^{n-1} (1 - e^{-t\lambda_i(\boldsymbol{\omega})}),$$

where $\mathbf{w}(\mathcal{R}) = \sum_{i<j} (\omega_{ij})$ *and* $\{\lambda_i(\boldsymbol{\omega})\}$ *are the non-zero eigenvalues of the (weighted) Laplacian* $L(K_n)$.

Enumeration for arbitrary W with parabolic weights

Consider a (maximal) tower of parabolic subgroups

$$T := \big(\{\mathbf{1}\} = W_0 \leq W_1 \leq W_2 \leq \cdots \leq W_n = W\big),$$

and a weight system $\mathbf{w}_T$ on reflections $\tau \in \mathcal{R}$ with parameters $\boldsymbol{\omega} := (\omega_i)$ given by

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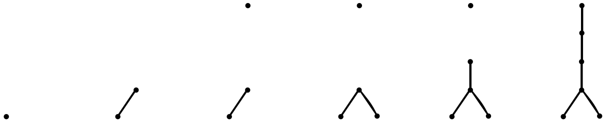
where $\{\lambda_i^T(\boldsymbol{\omega})\}$ *are the eigenvalues on the reflection representation* V *of the* W*-Laplacian* $L_T(W) := \sum_{\tau \in \mathcal{R}} \mathbf{w}_T(\tau) \cdot (\mathbf{1} - \tau) \in \mathbb{C}[\boldsymbol{\omega}][W]$

But, what do the eigenvalues look like?



$$A = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

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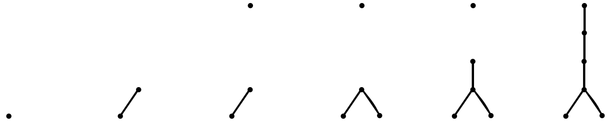

$$A = \begin{bmatrix} 2 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

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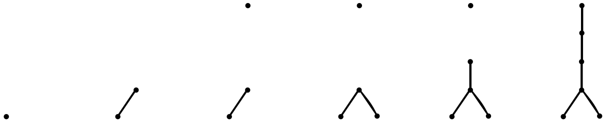

$$A = \begin{bmatrix} 2 & 1 & & & & \\ 0 & 3 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

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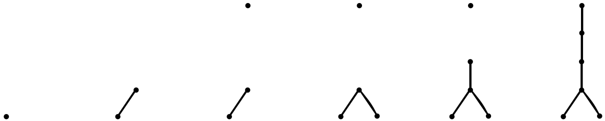

$$A = \left[\begin{array}{cccccc} 2 & 1 & 0 & & & \\ 0 & 3 & 0 & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{array}\right] \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{array}$$

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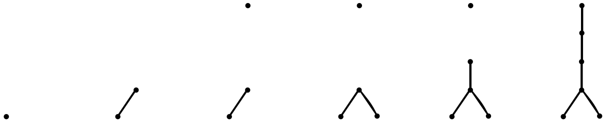

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 & & \\ 0 & 3 & 0 & 1 & & \\ 0 & 0 & 2 & 0 & & \\ 0 & 0 & 0 & 4 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

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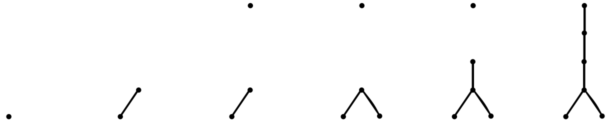

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 & 2 & \\ 0 & 3 & 0 & 1 & 2 & \\ 0 & 0 & 2 & 0 & 0 & \\ 0 & 0 & 0 & 4 & 2 & \\ 0 & 0 & 0 & 0 & 6 & \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \begin{matrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{matrix}$$

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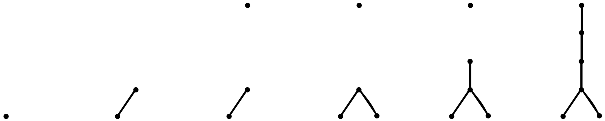

$$A = \left[\begin{array}{cccccc} 2 & 1 & 0 & 1 & 2 & 4 \\ 0 & 3 & 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 4 & 2 & 4 \\ 0 & 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{array}\right] \begin{array}{l} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \end{array}$$

$$\{\lambda_i(\boldsymbol{\omega})\} = \left\{\begin{array}{c} 2\omega_1 + \omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 3\omega_2 + \omega_4 + 2\omega_5 + 4\omega_6, \\ 2\omega_3 + 8\omega_6, \quad 4\omega_4 + 2\omega_5 + 4\omega_6, \\ 6\omega_5 + 4\omega_6, \quad 10\omega_6 \end{array}\right\}$$

Representation theoretic interpretation

The filtration by T defines natural analogs of the Jucys-Murphy elements:

$$\mathbb{C}[W] \ni J_i := \sum_{\tau \in \mathcal{R} \text{ and } \tau \in W_i \setminus W_{i-1}} \tau,$$

and we write $\mathbb{C}[\boldsymbol{J}] := \mathbb{C}[J_1, \cdots, J_n]$ for the (commutative) algebra they generate.

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Definition

We say that two (virtual) characters ψ_1 and ψ_2 are *tower equivalent*, and write $\psi_1 \equiv \psi_2$, if they agree on the subalgebra $\mathbb{C}[\boldsymbol{J}]$ of $\mathbb{C}[W]$ for *any* choice of T.

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For any irreducible character $\chi \in \widehat{W}$ Gordon-Griffeth define a *Coxeter number* c_χ

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Theorem (Chapuy, D. -version appropriate for induction)

Our enumeration theorem can be re-phrased via the tower equivalence relation as:

$$\sum_{\chi \in \widehat{W} \text{ s.th. } c_\chi = kh} \chi(c^{-1}) \cdot \chi \equiv (-1)^k \cdot \bigwedge^k (V_{ref}) \quad \textit{and} \quad \sum_{c_\chi = m \text{ with } h \nmid m} \chi(c^{-1}) \cdot \chi \equiv 0.$$

Ingredients of our proof

1. A weighted version of the Frobenius Lemma

$$\mathrm{FAC}_{W,c}(t) = \sum_{\chi \in \widetilde{W}} \dim(\chi) \cdot \chi(c^{-1}) \cdot \exp(\tilde{\chi}(\mathcal{R}) \cdot t)$$

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2. A recursion to maximal parabolics to prove tower equivalence in $G(r,1,n)$ and $G(r,r,n)$.
3. The eigenvalues of the W-Laplacian $L_T(W)$ in $\bigwedge^k(V_{ref})$ are the k-sums of its eigenvalues in V_{ref} (via Burman's Lie-like elements).

A remarkable W-Matrix forest theorem

Theorem (Chapuy, D. 2019)

If we write the characteristic polynomial for the W-Laplacian as

$$\chi\big(L_T(W),x\big) := \det\big(x\cdot\mathbf{1}-L_T(W)\big) = (-1)^n c_0^T+\cdots+(-1)^{n-k}c_k^T\cdot x^k+\cdots+c_n^T\cdot x^n,$$

then the coefficients c_k^T are given by

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Corollary

Let $\mathrm{Hur}(W_X)$ *denote the number of reduced reflection factorizations of a Coxeter element in* W_X. *Then,*

$$\sum_{X\in\mathcal{L}} |W_X| \cdot \mathrm{Hur}(W_X) \cdot \frac{t^{\mathrm{codim}(X)}}{\mathrm{codim}(X)!} = (ht+1)^n.$$

Discussion, generalizations, advertisement

1. The algebra $\mathbb{C}[\boldsymbol{J}]$ is not worthy of the name *Gelfand-Tsetlin* algebra; often:

$$\mathbb{C}[\boldsymbol{J}] \subsetneq \left\langle Z\big(\mathbb{C}[W_1]\big), \cdots, Z\big(\mathbb{C}[W_n]\big)\right\rangle \subsetneq \bigoplus_{\chi \in \widehat{W}} \mathrm{Hom}(U_\chi, \mathbb{C}[W]).$$

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2. In the infinite families $G(r,1,n)$ and $G(r,r,n)$ we get factoring formulas even if we use finer weight systems (but the W-Laplacian does *not* appear). Is there a combinatorial or representation theoretic significance for such maximal weighting systems?

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$$\mathbb{C}[\boldsymbol{J}] \subsetneq \langle Z(\mathbb{C}[W_1]), \cdots, Z(\mathbb{C}[W_n]) \rangle \subsetneq \bigoplus_{\chi \in \widehat{W}} \mathrm{Hom}(U_\chi, \mathbb{C}[W]).$$

   However, if it were, our theorem could never be true!

2. In the infinite families $G(r, 1, n)$ and $G(r, r, n)$ we get factoring formulas even if we use finer weight systems (but the W-Laplacian does *not* appear). Is there a combinatorial or representation theoretic significance for such maximal weighting systems?
   1. Type B_n: All possible n^2 weights ω_{ij}.
   2. Type $G(r, 1, n)$: The reflections (ii^{ξ}) that fix the same hyperplane $x_i = 0$ get the same weight.

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4. $$\chi(L_{\mathcal{A}}(\boldsymbol{\omega}), t) = \sum_{X\in\mathcal{L}_{\mathcal{A}}} \text{q-det}(L_{\mathcal{A}_X}(\boldsymbol{\omega})) \cdot (-1)^{\mathrm{codim}(X)} \cdot t^{\dim(X)}$$

Thank you!