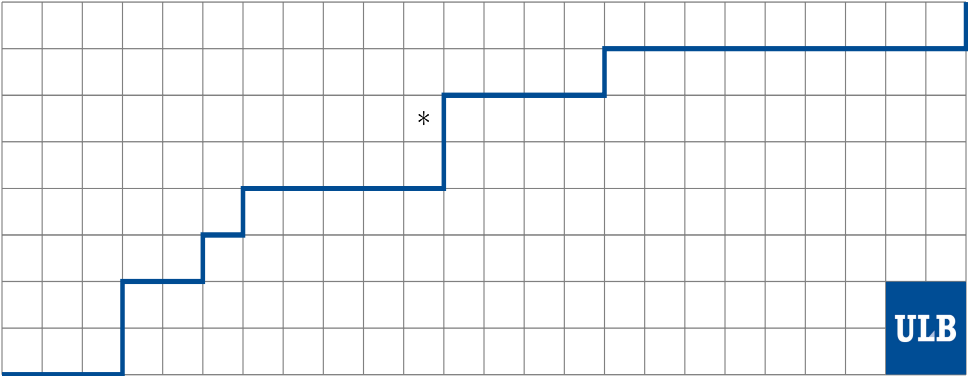
# The Compositional Delta Conjecture

Alessandro Iraci
joint work with Michele D'Adderio and Anna Vanden Wyngaerd

2 september, 2019



# Diagonal Harmonics

Let $R := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ on which $S_n$ acts *diagonally*:

$$\sigma f(x_1, \ldots, x_n, y_1, \ldots, y_n) := f(x_{\sigma_1}, \ldots, x_{\sigma_n}, y_{\sigma_1}, \ldots, y_{\sigma_n})$$

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We have

$$\mathcal{DH}_n = \bigoplus_{\substack{V \subseteq \mathcal{DH}_n \\ \text{irreducible}}} V$$

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$R$ is bi-graded, $I$ is homogeneous and invariant so $\mathcal{DH}_n$ is a bi-graded $S_n$-module.

We define its bi-graded Frobenius characteristic

$$\mathcal{F}_{q,t}(\mathcal{DH}_n) \coloneqq \sum_{\substack{V \subseteq \mathcal{DH}_n \\ \text{irreducible}}} q^{\deg_x(V)} t^{\deg_y(V)} s_{\lambda(V)} \in \Lambda_{\mathbb{Q}(q,t)}$$

# The nabla operator

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**Definition**

The **nabla operator** is the linear operator defined by

$$\begin{aligned}\nabla\colon \Lambda_{\mathbb{Q}(q,t)} &\to \Lambda_{\mathbb{Q}(q,t)}\\ H_\mu &\mapsto e_{|\mu|}[B_\mu]H_\mu\end{aligned}$$

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**Theorem (Haiman)**

The bi-graded Frobenius characteristic of $\mathcal{DH}_n$ is $\nabla e_n$.

# The Shuffle Theorem

**Theorem (Carlsson-Mellit)**

$$\nabla e_n = \sum_{D \in \mathsf{LD}(n)} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$

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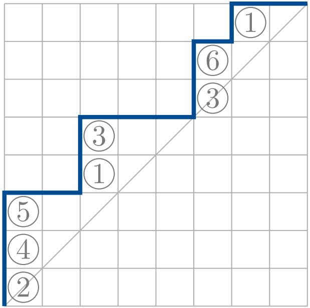


$\mathsf{LD}(n)$: labelled Dyck paths of size $n$

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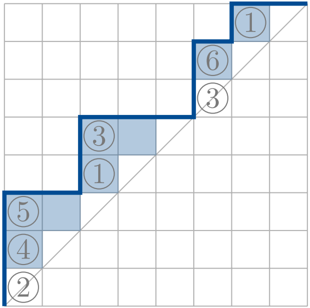


**Area:** number of whole squares between the path and $y = x$.

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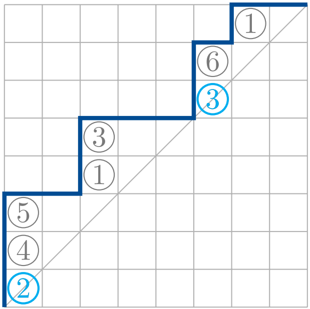


Dinv: count the number of pairs

- same diagonal,
  lower label < upper label
  (primary dinv)

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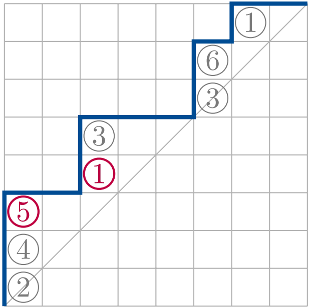


Dinv: count the number of pairs

- same diagonal, lower label $<$ upper label (primary dinv)
- lower step one diagonal above upper step lower label $>$ upper label (secondary dinv)

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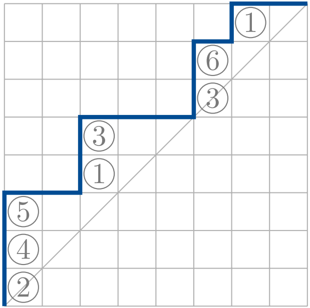



$$x^D := \prod_{i=1}^{n} x_{l_i(D)}$$

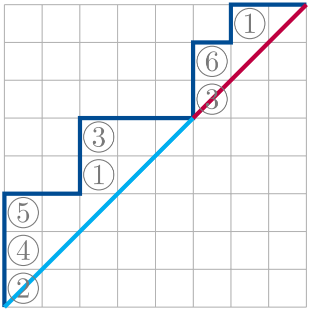
where $l_i(D)$ is the label of the $i$-th vertical step of $D$.

$$x_1^2 x_2 x_3^2 x_4 x_5 x_6$$

# The Compositional Shuffle Theorem

**Theorem (Carlsson-Mellit)**

$$\nabla C_\alpha = \sum_{D \in \mathsf{LD}(\alpha)} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$




We have

$$\sum_{\alpha \vDash n} C_\alpha = e_n$$

$\mathsf{LD}(\alpha)$: labelled Dyck paths with diagonal composition $\alpha$.

$$\alpha = (5, 3)$$

# Combinatorial recursion

**Theorem (Zabrocki)**

Set $\mathsf{D}_{q,t}(\alpha) = \sum_{D \in \mathsf{D}(\alpha)} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)}$. We have the recursion

$$\mathsf{D}_{q,t}(a,\alpha) = t^{a-1} \sum_{\beta \vDash a-1} q^{\ell(\alpha)} \mathsf{D}_{q,t}(\alpha,\beta)$$

for $a > 0$, with initial conditions $\mathsf{D}_{q,t}(\varnothing) = 1$.

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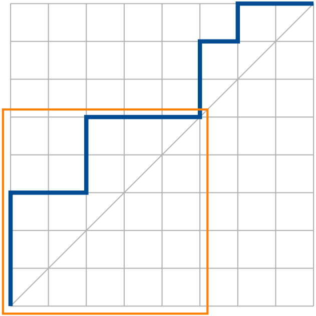

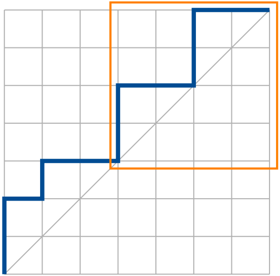

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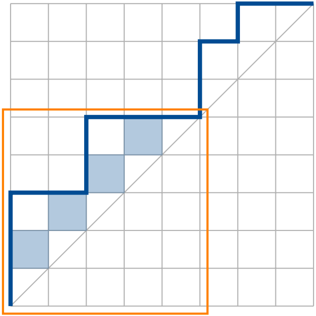

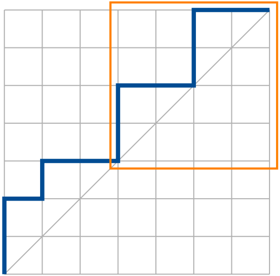

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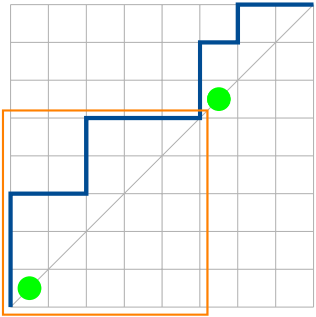

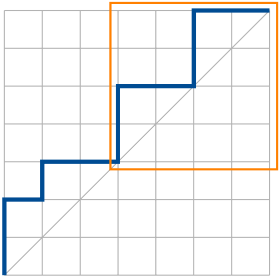

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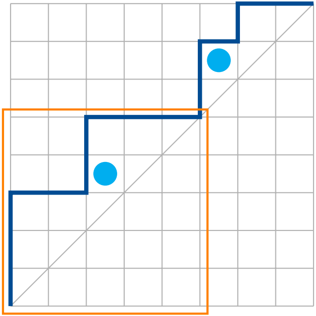

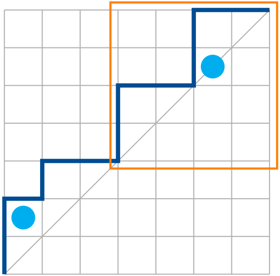

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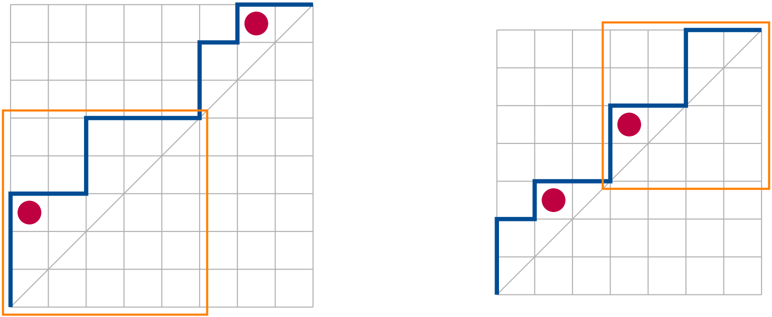

## Super-diagonal coinvariants and $\Delta_f$

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Consider again the diagonal action of $S_n$ on $R$ and let $I$ be the ideal generated by the constant free invariants of this action.

$$\mathcal{M}_n := R/I$$

$R$ is bi-graded, $I$ is homogeneous and invariant so $\mathcal{M}_n$ is a bi-graded $S_n$-module.

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For $f \in \Lambda_{\mathbb{Q}(q,t)}$ we define the Delta operators as

$$\Delta_f \colon \Lambda_{\mathbb{Q}(q,t)} \to \Lambda_{\mathbb{Q}(q,t)} : H_\mu \mapsto f[B_\mu] H_\mu$$
$$\Delta'_f \colon \Lambda_{\mathbb{Q}(q,t)} \to \Lambda_{\mathbb{Q}(q,t)} : H_\mu \mapsto f[B_\mu - 1] H_\mu$$

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**Conjecture (Zabrocki)**

The bi-graded Frobenius characteristic of the submodule of $\mathcal{M}_n$ in $\theta$-degree $k$ is $\Delta'_{e_{n-k-1}} e_n$.

# The Delta Conjecture

**Conjecture (Haglund-Remmel-Wilson)**

$$\Delta'_{e_{n-k-1}} e_n = \sum_{D \in \mathsf{LD}(n)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$

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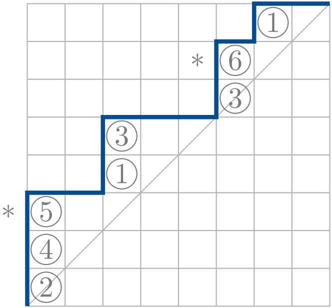


$\mathsf{LD}(n)^{*k}$: labelled Dyck paths of size $n$ with $k$ decorated rises

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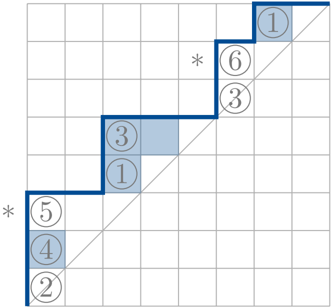


Area: number of whole squares between the path and $y = x$ and in rows not containing decorated rises.

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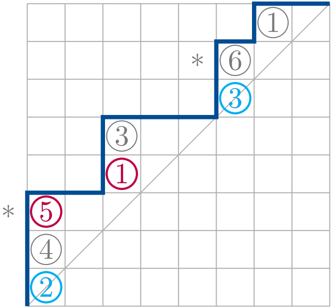


Dinv, $x^D$: same as for the undecorated case

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**Definition**

For $f \in \Lambda_{\mathbb{Q}(q,t)}$ we define the following operators on $\Lambda_{\mathbb{Q}(q,t)}$

$$\mathbf{\Pi} := \sum_{i \in \mathbb{N}} (-1)^i \Delta_{e_i} \qquad \Theta_f := \mathbf{\Pi} f \left[ \frac{X}{(1-q)(1-t)} \right] \mathbf{\Pi}^{-1}$$

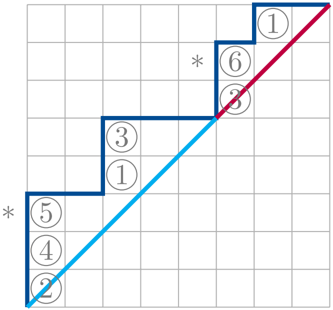
**Theorem (D'Adderio-Iraci-Vanden Wyngaerd)**

$$\Theta_{e_k} \nabla e_{n-k} = \Delta'_{e_{n-k-1}} e_n$$

# The Compositional Delta Conjecture

**Conjecture (D'Adderio-Iraci-Vanden Wyngaerd)**

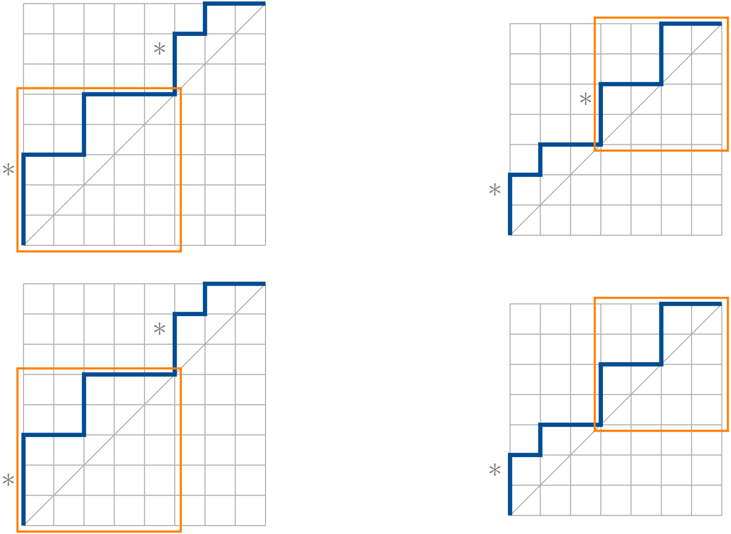
$$\Theta_{e_k} \nabla C_\alpha = \sum_{D \in \mathsf{LD}(\alpha)^{*k}} q^{\mathsf{dinv}(D)} t^{\mathsf{area}(D)} x^D$$




$\mathsf{LD}(\alpha)^{*k}$: labelled decorated Dyck paths with diagonal composition $\alpha$. Rows containing decorated rises do not count.

$$\alpha = (4, 2)$$

# Combinatorial recursion



# Combinatorial recursion

**Theorem (D'Adderio-Iraci-Vanden Wyngaerd)**

We have the combinatorial recursion, for $a, k > 0$,

$$\begin{aligned}\mathsf{D}_{q,t}(a,\alpha)^{*k} &= t^{a-1} \sum_{\beta \vDash a-1} q^{\ell(\alpha)} \mathsf{D}_{q,t}(\alpha,\beta)^{*k} \\ &+ t^{a-1} \sum_{\beta \vDash a} q^{\ell(\alpha)} \mathsf{D}_{q,t}(\alpha,\beta)^{*k-1}\end{aligned}$$

with initial conditions $\mathsf{D}_{q,t}(\varnothing)^{*k} = \delta_{k,0}$ and $\mathsf{D}_{q,t}(\alpha)^{*0} = \mathsf{D}_{q,t}(\alpha)$.

# Towards a Theta Conjecture

- Algebra side: diagonal coinvariants of

$$R \coloneqq \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n, \theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_n]$$

  where $\theta_i \theta_j = -\theta_j \theta_i$ and $\eta_i \eta_j = -\eta_j \eta_i$ in $\theta$-degree $k$ and $\eta$-degree $l$.
- Symmetric function side: $\Theta_{e_k} \Theta_{e_l} \nabla e_{n-k-l}$.
- Combinatorial side: labelled Dyck paths with decorated rises and valleys. Statistics?

Thank you for your attention!

| | | | |
|---|---|---|---|
| $t^3$ | | | |
| $t^2$ | | | |
| $t$ | $qt$ | $q^2t$ | |
| $1$ | $q$ | $q^2$ | $q^3$ |

For $f \in \Lambda_{\mathbb{Q}(q,t)}$ we set $f[B_\mu]$ to be $f$ evaluated in the content of this picture