Today:

- Counting flags
- Quasi-symmetric functions

Obtain combinatorial identities by counting objects over <u>finite fields</u>

BASIC EXAMPLE

p prime number: consider residues mod p. They form field $\mathbb{F}_p$

EXTENSION: Take irreducible polynomial with coeffs in $\mathbb{F}_p$ (degree d), formally add its root to $\mathbb{F}_p$, obtain field $\mathbb{F}_q$

$q = p^d$.

$\mathbb{F}_q$ has exactly q elements.

MAIN IDEA: DONT CARE ABOUT ACTUAL IMPLEMENTATION OF $\mathbb{F}_q$.

COUNTING: $|\mathbb{F}_q| = q$

Projective space of dimension d ($\neq d$)

Definition

$$\mathbb{P}^d(\mathbb{F}_q) = \{(x_0, \dots, x_d) \in \mathbb{F}_q^{d+1} \mid (x_0, \dots, x_d) \neq (0, \dots, 0)\} / \sim$$

$$(x_0, \dots, x_d) \sim (\lambda x_0, \dots, \lambda x_d) \qquad \lambda \in \mathbb{F}_q,\ \lambda \neq 0$$

<u>Problem</u> compute $|\mathbb{P}^d(\mathbb{F}_q)|$

Start by looking at x_0

$x_0 \neq 0$: multiply by x_0^{-1}

$x_0 = 0$: proceed to x_1, and so on

any element of $\mathbb{P}^d(\mathbb{F}_q)$ has a unique representative of the form

$(0, \dots, 0, 1, a_1, \dots, a_k)$ $\quad k = 0, 1, \dots, d$

$a_i \in \mathbb{F}_q$

$$|\mathbb{P}^d(\mathbb{F}_q)| = \underbrace{|\cdot|}_{k=0} + \underbrace{|\mathbb{F}_q|}_{k=1} + \dots \underbrace{|\mathbb{F}_q^d|}_{k=d} =$$

$$= 1 + q + \dots + q^d = [d+1]_q$$

<u>Remark</u>

$$= \frac{q^{d+1} - 1}{q - 1} = \frac{|\mathbb{F}_q^{d+1} \setminus \{0\}|}{|\mathbb{F}_q \setminus \{0\}|}$$

$$\Rightarrow [d+1]_q = \frac{q^{d+1} - 1}{q - 1}$$

More complicated
Flag variety

Definition: Fix n, consider $\mathbb{F}_q^n$;
this is a vector space

$$Fl_n(\mathbb{F}_q) = \{ \{0\} = V_0 < V_1 < V_2 \dots V_n = \mathbb{F}_q^n : V_i \text{ is a linear subspace of } \mathbb{F}_q^n,\ \dim V_i = i \}$$

FLAG VARIETY

PROBLEM: COMPUTE $|Fl_n(\mathbb{F}_q)|$

GIVEN A FLAG $V_0 < \dots < V_n$
CHOOSE $x_1 \in V_1$,
COMPLETE TO A BASIS OF V_2, GET (x_1, x_2)
⋮
FINAL RESULT: $x_1, \dots, x_n \in \mathbb{F}_q^n$
SUCH THAT $x_1, \dots, x_k$ IS A BASIS OF V_k

OF CHOICES MADE:
$q-1$
$q(q-1)$
⋮
$q^{k-1}(q-1)$
⋮

COUNTING THESE:
$x_1 \in \mathbb{F}_q^n \setminus \{0\}$ $\quad q^n - 1$
x_2: ANYTHING IN $\mathbb{F}_q^n$, not in Span(x_1) $\quad q^n - q$
⋮
x_k: ANYTHING NOT IN SPAN$(x_1, \dots, x_{k-1})$ $\quad q^n - q^{k-1}$

OBTAIN: $(q^n-1)(q^n-q)\dots(q^n-q^{n-1})$
$=$ # of tuples $(x_1, \dots, x_n)$

$$|Fl_n(\mathbb{F}_q)| = \frac{(q^n-1)\dots(q^n-q^{n-1})}{(q-1)\,q(q-1)\cdots q^{n-1}(q-1)}$$

$$= \frac{(q^n-1)\dots(q-1)}{(q-1)^n} = [1]_q \dots [n]_q = [n]_q!$$

HOW TO COUNT $|Fl_n(\mathbb{F}_q)|$ DIRECTLY?

GAUSSIAN ELIMINATION: $\quad V_k = \mathrm{SPAN}(x_1, \dots, x_k)$

$$x_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \dots x_n = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix} \to \text{matrix} \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & & & \\ & x_{n-1,2} & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}$$

IF WE ARE LUCKY:
FIRST STEP:
MAKE $x_{n1} = 1$
NEXT: KILL x_{n2}, MAKE $x_{n-1,2} = 1$
AND SO ON

WHAT IF $x_{n,1} = 0$?

MORE GENERALLY, $\exists\, \pi_1$ SO THAT WE CAN MAKE FIRST COLUMN

$$\begin{matrix} x_{11} & & \\ \vdots & & \\ x_{1\,\pi_1-1} & & \\ 1 & 0 & \dots & 0 \\ 0 & & \\ \vdots & & \\ 0 & & \end{matrix}$$

THEN KILL ROW π_1
AND SO ON

IN THE END: 1 ON POSITIONS $(\pi_1, 1), (\pi_2, 2) \dots (\pi_n, n)$ for $\pi \in S_n$
0 ON POSITIONS
$(i, 1)$ FOR $i > \pi_1$
(π_1, j) FOR $j \geq 2$
(i, j) $\quad i > \pi(j)$ $\qquad i < \pi(j)$
(π_i, j) $\quad j > i$ $\qquad \pi^{-1}(i) > j$
(i, j) $\quad \pi^{-1}(i) > j$

NON ZEROES: (i, j) : $i < j$ $\quad \pi(i) > \pi(j)$

FOR A GIVEN π: EXACTLY $q^{\mathrm{inv}(\pi)}$ possibilities

$$\Rightarrow \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q!$$

MORE ADVANCED EXAMPLE

FOR A SEQUENCE

$c \models n$

$c = (c_1, \dots, c_k)$ $\quad \sum c_i = n$

$c_i > 0$

FLAGS $\quad V_0 = V_1 \subset V_2 \dots V_k$

$V_0 = 0$, $\quad V_k = n$

$\dim V_i = c_1 + \dots + c_i$

RESULT

$$\begin{bmatrix} n \\ c_1 \dots c_k \end{bmatrix}_q = \frac{[n]_q!}{[c_1]_q \dots [c_k]_q}$$

q-multinomial

ANOTHER COUNTING:

LET $m = 1^{c_1} 2^{c_2} \dots k^{c_k}$ SEQUENCE OF LENGTH n

$$\sum_{\substack{\text{permutations} \\ \text{of } m \\ \pi}} q^{\#\{i,j \mid i<j \ \pi_i > \pi_j\}}$$

(REMARK: EQUIVALENTLY, SUM OVER THE S_n / S_c

$S_c = S_{c_1} \times \dots \times S_{c_k} \subset S_n$)

GENERAL RULE:

WHEN COUNTING FLAGS FOR $c = \lambda$
MULTIPLY BY m_λ

monomial symmetric function

$$m_\lambda(x_1, \dots, x_N) = \sum_{\substack{\text{permutations } \pi \text{ of} \\ (\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, \dots, 0)}} x_1^{\pi_1} \cdots x_N^{\pi_N}$$

$$H_n = \sum_{\lambda \vdash n} m_\lambda(x_1, \dots, x_N) \cdot \left| \begin{array}{c} \text{FLAGS with} \\ \dim V_i = \lambda_1 + \dots + \lambda_i \end{array} \right| \text{ OVER } \mathbb{F}_q$$

WE HAVE:

$$H_n = \sum_{\lambda \vdash n} m_\lambda \cdot \begin{bmatrix} n \\ \lambda_1 \dots \lambda_\ell \end{bmatrix}_q$$

$$H_n = \sum_{\lambda \vdash n} m_\lambda \cdot \sum_{\substack{\pi \text{ perm} \\ \text{of } 1^{\lambda_1} \dots \ell^{\lambda_\ell}}} q^{\operatorname{inv}(\pi)}$$

$\ell = \text{length}(\lambda)$

$$= \sum_{i_1 \dots i_n} x_{i_1} \cdots x_{i_n} \, q^{\operatorname{inv}(i_1, \dots, i_n)}$$

QUASI-SYMMETRIC FUNCTIONS

CONSIDER $\quad M_c = \sum_{i_1 < \dots < i_{\ell(c)}} x_{i_1}^{c_1} \dots x_{i_\ell}^{c_\ell}$

$c \models n$

MONOMIAL Q-S FUNCTION

DEF

A Q-S FUNCTION IS A LINEAR COMBINATION OF M_c

PROP Q-S FUNS FORM A RING

$$M_c M_{c'} = \sum_{\substack{i_1 < \dots < i_\ell \\ j_1 < \dots < j_{\ell'}}} x_{i_1}^{c_1} \dots x_{i_\ell}^{c_\ell} x_{j_1}^{c'_1} \dots x_{j_{\ell'}}^{c'_{\ell'}}$$

$$= \sum_{\text{SHUFFLES}} M_{c''}$$

EVERY SYMM. FUNCTION IS QUASI-SYMMETRIC

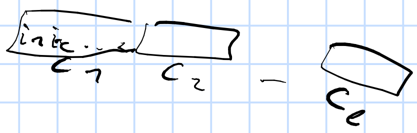
$$m_\lambda = \sum_{\substack{c = \text{permutation} \\ \text{of } \lambda}} M_c$$

FUNDAMENTAL QSFUN:

$$F_c = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \dots x_{i_n}$$

NUMBERS IN DIFFERENT BLOCKS ARE DISTINCT

BLOCKS:



$F_{(n)} = h_n$

$F_{1^n} = e_n$

THEOREM (GESSEL):

INVOLUTION ω SENDING h_n TO e_n CAN BE EXTENDED TO QUASI SYMMETRIC FUNCTIONS SO THAT

F_c GOES TO $F_{c'}$, where

PARTIAL SUMS OF c' ARE THE NON-PARTIAL SUMS OF c

DAY 2

Hopf algebras / Hall algebras

1) Hopf algebra structure on symm. functions

$F, G \in Sym$ ← symm. functions

product FG

coproduct: $\underset{Sym}{F} \to$ function in 2 sets of variables $x_1, \dots, x_N$; $y_1, \dots, y_N$

$$\Delta F = \underset{Sym \otimes Sym}{F(x_1, \dots, x_N, y_1, \dots, y_N)}$$

$$\Delta p_n = p_n \otimes 1 + 1 \otimes p_n$$

Plethystic notation

$X = \sum x_i$ $\quad$ $p_n(X) =$ "raise all variables to nth powers" $\sum x_i^n$

$$(p_{m_1} \dots p_{m_k})(X) = p_{m_1}(X) \dots p_{m_k}(X)$$

coproduct: $(\Delta F)(X, Y) = F(X+Y)$

$$F\left(\frac{X}{1-q}\right) \overset{?}{=} \quad \text{2 ways: 1) } p_n\left(\frac{X}{1-q}\right) = \frac{p_n(X)}{1-q^n}$$

2) $F(x_1, \dots, x_N, qx_1, \dots, qx_N, q^2 \quad)$

counit?

$$F(X) \xrightarrow{\Delta} F(X+Y) \xrightarrow[\text{Set } Y=0]{\varepsilon} F(X)$$

antipode

$$F(X) \to F(X+Y) \to F(X+Y+Z) \to F(X)$$

$Z = -Y$

$F(-X) = ?$

$$p_n(-X) = -p_n(X)$$

$$-x_1 - x_2 - \dots - x_N \to -x_1^n - x_2^n - \dots - x_N^n$$

$$h_n \to (-1)^n e_n$$

$$h_n(-X) = (-1)^n e_n(X) = (-1)^n \omega h_n$$

2) Quasi-symmetric functions:

$F, G \hookrightarrow F \cdot G$

$F(x_1, .., x_N)$ $\qquad (\Delta F)(x_1, .., x_N, y_1, .., y_N)$

$$= F(x_1, .., x_N, y_1, .., y_N)$$

example. $F = F_c \qquad c = (c_1, .., c_k)$

$$\Delta F_c = 1 \otimes F_c + F_{(1)} \otimes F_{(c_1 - 1, .., c_k)} + \dots$$

antipode:

$$F_c = (-1)^n F_{\overleftarrow{c'}}$$

c' as yesterday

$\overleftarrow{c'}$: backwards.

$$1 \otimes F_c + F_{(1)} \otimes F_{(c_1 - 1, .., c_k)} + \dots$$

apply antipode

need to check: get 0.

Obvious involution will cancel everything.

Prop $H_{(n)} = \sum_{i_1,\dots,i_n} q^{\mathrm{inv}(i_1,\dots,i_n)} x_{i_1}\cdots x_{i_n}$

$$H_{(n)} = \frac{h_n\left(\frac{X}{1-q}\right)}{h_n\left(\frac{1}{1-q}\right)}$$

Proof Idea: check that $H_{(n)}\left[(q-1)X\right]$ is proportional to $e_n(X)$

$H_{(n)}\left[(q-1)X\right]$:

$$H_{(n)}(X) \xrightarrow{\Delta} H_{(n)}(X+Y) \xrightarrow[\text{in } Y]{\text{antipode}} H_{(n)}(X-Y) \xrightarrow{\text{Scale } X \text{ by } q} H_{(n)}(qX-Y) \xrightarrow{Y=X} H_{(n)}(qX-X)$$

Reference: Haglund - Haiman - Loehr ... Macdonald polynomials

$(i_1,\dots,i_n) \to$ permutation π (standardization) such that

$i_k < i_{k'} \iff \pi_k < \pi_{k'}$ if $i_k \neq i_{k'}$

otherwise $\left.\begin{matrix} i_k = i_{k'} \\ k < k' \end{matrix}\right\} \Rightarrow \pi_k < \pi_{k'}$

2 1 2	1 2 2
↓	↓
2 1 3	1 2 3
2 2 1	
↓	
2 3 1	

$\mathrm{inv}(\pi) = \mathrm{inv}(i_1,\dots,i_n)$

$$H_{(n)} = \sum_{\pi\in S_n} q^{\mathrm{inv}} F_{\mathrm{des}(\pi^{-1})}$$

result:

$$H_{(n)}\left[(q-1)X\right] = \sum_{\substack{i_1,\dots,i_n \\ i_k = m \text{ or } \bar{m}}} x_{i_1}\cdots x_{i_n}\, q^{\#m \text{ without bar in } (i_1,\dots,i_n)} (-1)^{\#\bar{m} \text{ in } (i_1,\dots,i_n)} q^{\#\mathrm{inv}}$$

$x_{\bar{m}} = x_m$

to count inv: use order

$1 < \bar{1} < 2 < \bar{2} < \dots$

modification: $\bar{m}$ $\bar{m}$ count as inversion

trick: if some number repeats

$2\ \bar{2}\ \bar{2}\ 2 \dots 2\ \bar{2}$

flip the bar of the one before last

cancels terms with repetitions

$\Rightarrow$ $H_{(n)}\left((q-1)X\right)$ is proportional to $e_n(X)$

Another Hopf algebra

(Hall algebra)

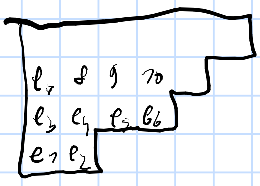
Game Fix a matrix N (over $\mathbb{F}_q$)

count flags preserved by N

$\rightsquigarrow H_N \in \mathrm{Sym}$.

Assume N is nilpotent

Let λ be a partition of n.

$\lambda =$ 

N_λ moves boxes down

$N_\lambda(e_7) = e_3$ $N_\lambda(e_6) = 0$

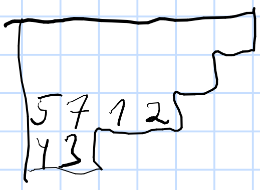
0 otherwise

how to count flags (let's say $\dim V_k = k$)

preserved by N_λ ?

visualizing flags.

permutation $\rightarrow$ filling of λ



$V_i = e_{\text{position of } i} + \sum_{c_{ij} \in \mathbb{F}_q} c_{ij}\, e_j$ j before position of i in the reading order $+ \pi_j > i$

$i = 2$ $e_8 + (?) \cdot e_7 + ? \cdot e_{2+} \; ? \cdot e_4$ (no e_5)

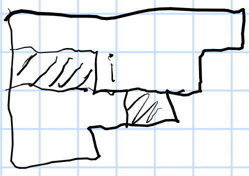
Ask $N_\lambda(V_i) < \mathrm{Span}(V_1, .., V_i)$

Property 1) $\boxed{\begin{smallmatrix} a \\ b \end{smallmatrix}} \Rightarrow V_b \leq V_a$

if 

what can coeff-s of V_i be ?

$\rightsquigarrow$ vector space whose dimension

$= \#$ 

of inversions in the shaded region between i and j

$j > i$ and j is in the shaded region

$\rightarrow$ Lascoux – Schützenberger formula

for the coeffs of Hall-Littlewood polynomials

Tomorrow: algebraic computation,

Macdonald polynomials

DAY 3

example

μ

$$N_\mu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

column-decreasing fillings:

2 3 1	3 2 1	3 1 2	
1	q	q	$\Sigma = 2q+1$
1 2 1	2 1 1		
1	q		

$$H_{2,1} = M_{(3)} + (q+1)\, m_{2,1} + (2q+1)\, m_{1^3}$$

Also algebraic way:

Hall algebra: formal sums $\sum [M_i] n_i \quad n_i \in Q$

M_i: nilpotent matrix / conjugation

1) Scalar product:

$$([M], [M']) = |\text{Isomorphisms from } M \text{ to } M'|$$

$$|N \in GL_n : NMN^{-1} = M'|$$

2) coproduct $\Delta[M] = \sum_{\substack{V \subset X \\ MV \subset V}} [M|_{X/V}] \otimes [M|_V]$

M lives on $X = \mathbb{F}_q^n$

3) product $[M_1], [M_2] \rightsquigarrow \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix}$ Σ over $*$

to be precise product is the dual of coproduct

$$E(u) = \sum_\lambda u^{|\lambda|} \frac{[N_\lambda]}{|Z(N_\lambda)|}$$

$$([M], E(u)) = u^{\text{dimension of } X}$$

(M on X)

$$\Delta E(u) = E(u) \otimes E(u)$$

$$I([M])(u_1, \dots, u_m) = ([M], E(u_1) \dots E(u_m))$$

$\in$ Sym

Hall algebra $\to$ Sym

$$(\Delta F, E(u_1) \dots E(u_k) \otimes E(v_1, \dots, v_m))$$

$$= (F, E(u_1) \dots E(u_k) \dots E(v_m))$$

$\Rightarrow$ commutes with coproduct

$$(F * G, E(u_1) \dots E(u_k))$$

$\Rightarrow$ commutes with product

$$[N_{1^n}] \to h_n$$

$$([N_{1^n}], [N_{1^n}]) = |\mathrm{Iso}(N_{1^n})|$$
$$= (q-1) q^{n-1}$$

$$(h_n[X], h_n[X(q-1)]) = (q-1) q^{n-1}$$

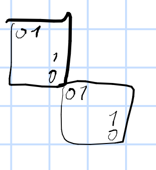
$$(h_i h_j, F) = (h_i \otimes h_j, \nabla F)$$

$$(\mathrm{Exp}[XY], F[X])_X = F[Y]$$

$$(\nabla \mathrm{Exp}[XY], F[X] \otimes G[X])$$
$$= \mathrm{Exp}[XY] \otimes \mathrm{Exp}[XY],$$
$$= F[Y] G[Y] =$$
$$(\mathrm{Exp}[XY], FG)$$

$$\mathrm{Exp}[XY] = \sum S_\lambda[X] S_\lambda[Y]$$

$$([N_\lambda], h_\mu) \neq 0 \Rightarrow$$



$$\lambda_1 \leq \ell(\mu)$$

$$\lambda \preceq \mu^t \iff \mu \prec \lambda^t$$

$$I([N_\lambda])[(q-1)X] \in \mathrm{span}(M_\mu \mid \mu \prec \lambda^t)$$

orthogonality $\Rightarrow I([N_\lambda])[-X] \in \mathrm{span}(m_\mu \mid \mu \prec \lambda)$

normalization $I([N_\lambda])(1) = 1.$

$\Downarrow$

$[N_\lambda]$ goes to $\tilde{H}_\lambda[X; q, 0]$

$\tilde{H}_\lambda$: modified Macdonald polynomial.

$$\tilde{H}_\lambda[X; q, 0] = \sum_\mu m_\mu \sum_{\substack{\text{fillings of shape } \lambda \\ \text{content } \mu \\ \text{column-nondecreasing}}} q^{\mathrm{inv}}$$

(specialization of Haglund Haiman Loehr formula)

COUNTING BUNDLES ON $\mathbb{P}^1$

BUNDLE ON $\mathbb{P}^1$ (OF RANK n)

=

TUPLE $(m_1, \ldots, m_n)$ $\quad m_i \in \mathbb{Z}_{\geq 0}$

only positive bundles

= THE SET OF n-TUPLES OF POLYNOMIALS $(p_1(z), \ldots, p_n(z))$ SATISFYING $\deg p_k \leq m_k$

NOTATION: $\mathcal{O}(m_1) \oplus \ldots \oplus \mathcal{O}(m_n)$

EXAMPLE $\quad m = (0, 0, \ldots, 0) \Rightarrow$ ALL POLYNOMIALS CONSTANT, GET $\mathbb{C}^n$

$m = (1, 1, \ldots, 1) \Rightarrow n$-TUPLES OF LINEAR POLYNOMIALS

EVALUATION AT z_0:

$z_0 \neq \infty \quad (p_1(z_0), \ldots, p_n(z_0))$

$z_0 = \infty$: TOP DEGREE TERMS

SYMMETRIES (=AUTOMORPHISMS), ENDOMORPHISMS
THESE ARE MATRICES WITH POLYNOMIAL ENTRIES PRESERVING THE BUNDLE $(\deg p_k \leq m_k)$

EXAMPLE

$m = (0, 0) \qquad Aut = GL_2(\mathbb{C}) \qquad End = Mat_2(\mathbb{C})$

(also for $m = (k, k)$, $k > 0$)

$m = (0, 1)$ $\quad Aut = \left\{ \begin{pmatrix} a & 0 \\ b + cz & d \end{pmatrix} \middle| \begin{array}{l} a, d \neq 0 \\ b, c \in \mathbb{C} \end{array} \right\}$

$End = \left\{ \begin{pmatrix} a & 0 \\ b + cz & d \end{pmatrix} \middle| \begin{array}{l} a, b, c, d \\ \text{arbitrary} \end{array} \right\}$

FORMULAS:

$$\sum_{m \in \mathbb{Z}^n_{\geq 0}/S_n} \frac{t^{\overbrace{\sum m_i}^{= \text{degree}}}}{\mathrm{Aut}(\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n))} \# \left\{ \begin{array}{l} \text{Nilpotent endomorphism} \\ \theta \text{ such that for almost} \\ \text{all } z_0 \quad \theta(z_0) \sim N_\lambda \end{array} \right\}$$

$$= \frac{1}{\prod_{\square \in \lambda} (q^{a+1} - t^{l})(q^{a} - t^{l+1})}$$

For instance,

$\lambda = (1,1)$: i.e. θ is nilpotent, but $\theta \neq 0$

$$\sum_{k=0}^{\infty} \frac{t^{2k}}{(q^2-1)(q^2-q)} \Big(\overbrace{q^2}^{\substack{\text{nilpotent} \\ 2\times 2 \text{ matrices}}} - \overbrace{1}^{\text{except } 0}\Big) + \sum_{m_1 < m_2} \frac{t^{m_1+m_2}}{(q-1)^2 q^{m_2-m_1+1}} \left(q^{m_2-m_1+1} - 1\right)$$

$$= \frac{1}{(q^2-q)(1-t^2)} + \frac{t}{(q-1)^2(1-t)(1-t^2)} - \frac{t/q^2}{(q-1)^2(1-t/q)(1-t^2)}$$

$$= \frac{1}{q(q-1)^2(q-t)(1-t)(1-t^2)} \Big(\underbrace{\underbrace{(q-1)(q-t)(1-t) + tq(q-t)}_{(q-t)(-1+q+t)} - t(1-t)}_{q^2 - qt + qt - q = q^2 - q} \Big)$$

$$= \frac{1}{(q-1)(q-t)(1-t)(1-t^2)}$$

$\lambda = 2$: $(\theta = 0)$

$$\sum_{k=0}^{\infty} \frac{t^{2k}}{(q^2-1)(q^2-q)} + \sum_{m_1 < m_2} \frac{t^{m_1+m_2}}{(q-1)^2 q^{m_2-m_1+1}}$$

$$= \frac{1}{(q^2-q)(q^2-1)(1-t^2)} + \frac{t/q^2}{(q-1)^2(1-t^2)(1-t/q)}$$

$$= \frac{1}{(q^2-q)(q^2-1)(1-t^2)(q-t)} \Big(\underbrace{q - t + t(q+1)}_{q(1+t)} \Big)$$

$$= \frac{1}{(q-1)(q^2-1)(1-t)(q-t)}$$

More generally: fix $z_1, \ldots, z_k \in \mathbb{P}^1(\mathbb{F}_q)$

$$\sum_{m \in \mathbb{Z}^n_{\geq 0}/S_n} \frac{t^{\deg}}{|\mathrm{Aut}|} \sum_{\substack{\theta \in \mathrm{End} \\ \text{nilpotent}}} \prod_{i=1}^{k} \tilde{H}_{\text{type } \theta(z_i)}[X_i; q, 0]$$

$$= \frac{\prod_{i=1}^{k} \tilde{H}_\lambda[X_i; q, t]}{\prod (q^{a+1} - t^{l})(q^{a} - t^{l+1})}$$

for instance, set $k=1$, obtain Macdonalds in terms of Hall-Littlewoods.

Proof is geometric

Open problem: find combinatorial proof to get HHL

Want to understand ∇

$$\nabla \tilde{H}_\lambda = q^{\cdot} t^{\cdot} \tilde{H}_\lambda = (-1)^n \tilde{H}_\lambda[-1; q, t] \, \tilde{H}_\lambda[X; q, t]$$

NEW PROOF OF THE SHUFFLE CONJECTURE

(joint with E. Carlsson) more general problem: $\nabla s_\mu = ?$

CAUCHY KERNEL $\quad M = (q-1)(t-1)$

$$\mathrm{Exp}\left[\frac{-XY}{M}\right] = \sum_\lambda \frac{\tilde{H}_\lambda[X;q,t]\,\tilde{H}_\lambda[Y;q,t]}{\prod_\square (q^{a+1}-t^{l})(q^{a}-t^{l+1})} = \sum_\lambda s_\lambda[X]\, s_\lambda\left[-\frac{Y}{M}\right]$$

WE WANT $\nabla(s_{1^n}^{n})$, WHERE $\quad = \sum_\lambda s_\lambda[X]\, s_{\lambda'}\left[\frac{Y}{M}\right](-1)^{|\lambda|}$

$$\nabla \tilde{H}_\lambda[X;q,t] = \tilde{H}_\lambda[-1;q,t]\cdot \tilde{H}_\lambda[X;q,t](-1)^{|\lambda|}$$

SO USE $\quad z_1 = 0,\ z_2 = 1,\ z_3 = \infty$

$\qquad X_1 = X,\ X_2 = Y,\ X_3 = -1$

OBTAIN:

$$\sum_{\lambda \vdash n} \left(\nabla s_\lambda[X]\right) s_{\lambda'}\left[\frac{Y}{M}\right] =$$

$$\underbrace{q^{\binom{n}{2}}}_{=(-1)^n \tilde{H}_{(n)}[-1;q,0]} \sum_{m \in \mathbb{Z}^n_{\geq 0}/S_n} \frac{t^{|m|}}{|\mathrm{Aut}|} \sum_{\substack{\theta \in \mathrm{End} \\ \theta(\infty)=0 \\ (\text{drop „nilpotent"})}} \tilde{H}_{\mathrm{type}\,\theta(0)}[X;q,0]\,\tilde{H}_{\mathrm{type}\,\theta(1)}[Y;q,0]$$

$$= \sum_{\substack{m \in \mathbb{Z}^n_{\geq 0} \\ a \in \mathbb{Z}^n_{\geq 1} \\ b \in \mathbb{Z}^n_{\geq 1}}/S_n} t^{|m|} \frac{\#\{\theta \in \mathrm{End} : \theta(\infty)=0,\ \theta(0), \theta(1) \text{ preserve the } a\text{-flag (resp. } b\text{-flag)}\}}{\#\{\theta \in \mathrm{Aut} : \theta(0), \theta(1) \text{ preserve the } a\text{-flag (resp. } b\text{-flag)}\}}\, q^{\binom{n}{2}} X_a Y_b$$

$$X_a = \prod_{i=1}^n X_{a_i}$$

$$Y_b = \prod_{i=1}^n Y_{b_i}$$

FINALLY:

$$\sum_{\lambda \vdash n} \left(\nabla s_\lambda[X]\right) s_{\lambda'}\left[\frac{Y}{\underset{(q-1)(t-1)}{M}}\right] = \sum_{m,a,b/S_n} \frac{t^{|m|} q^{\mathrm{dinv}(m,a,b)} X_a Y_b}{\prod[\text{multiplicity in } (m,a,b)]!\ (q-1)^n}$$

$$\mathrm{dinv}(m,a,b) = \#\left\{i,j : \begin{array}{l} m_i = m_j \ \&\ a_i < a_j \ \&\ b_i > b_j \quad \text{or} \\ m_i = m_j + 1 \ \&\ a_i > a_j \ \&\ b_i > b_j \end{array}\right\}$$

ORDER ACCORDING TO

$$m_1 \geq m_2 \geq \dots \geq m_n \qquad m_i = m_{i+1} \Rightarrow a_i \leq a_{i+1}$$

SAY i attacks j if $i<j$ and ($m_i = m_j$ or $m_i = m_j + 1$ & $a_i > a_j$)

REPLACE SUMMATION BY

$$\sum_{\substack{m,a \\ \text{ordered}}} \frac{t^{|m|}\, X_a}{\prod[\text{multiplicity in } (m,a)]!\ (q-1)^n} \sum_{b \in \mathbb{Z}^n_{\geq 1}} Y_b\, q^{\#\{i \text{ attacks } j :\ b_i > b_j\}}$$

$$\sum_{\substack{m,a\\ \text{ordered}}} \frac{t^{|m|}}{\prod[\text{multiplicity in }(m,a)]!}\,\frac{X_a}{(q-1)^n} \sum_{b\in\mathbb{Z}^n_{\geq 1}} Y_b\, q^{\#\{i \text{ attacks } j:\ b_i>b_j\}}$$

SUBSTITUTE $Y \leftarrow (q-1)Y$:

$$\sum_{\lambda\vdash n}\left(\nabla S_\lambda[X]\right) S_{\lambda'}\left[\frac{Y}{t-1}\right] = \sum_{\substack{m,a\\ \text{ordered}}} \frac{t^{|m|}}{\prod[\text{multiplicity in }(m,a)]!}\, X_a \sum_{\substack{b\in\mathbb{Z}^n_{\geq 1}\\ i \text{ attacks } j \Rightarrow\\ b_i\neq b_j}} Y_b\, q^{\#\{i \text{ attacks } j:\ b_i>b_j\}}$$

$$= \sum_{\substack{m,a\\ \text{ordered}}} t^{|m|} X_a \sum_{\substack{b\in\mathbb{Z}^n_{\geq 1}\\ i \text{ attacks } j \Rightarrow b_i\neq b_j\\ m_i=m_{i+1},\ a_i=a_{i+1}\Rightarrow b_i<b_{i+1}}} Y_b\, q^{\mathrm{dinv}(m,a,b)}$$

NOW ORDER BY b_i (PAIRS b_i, m_i DO NOT REPEAT)
$b_1\geq b_2\geq\dots\geq b_n$, $b_i=b_{i+1}\Rightarrow m_{i+1}>m_i$. NOTICE: dinv depends only on m, a.

$$\sum_{m,a} X_a t^{|m|} q^{\mathrm{dinv}(m,a)} \sum_{\substack{b_1\geq\dots\geq b_n\\ m_{i+1}\leq m_i \text{ or } m_{i+1}=m_i+1\ \&\ a_{i+1}>a_i \Rightarrow b_i>b_{i+1}}} Y_b$$

$m =$ area sequence of a Dyck path, a is a word P.F.

THIS IS THE PARKING FUNCTION CONDITION

THIS IS GESSEL'S QUASI-SYMMETRIC FUNCTION!

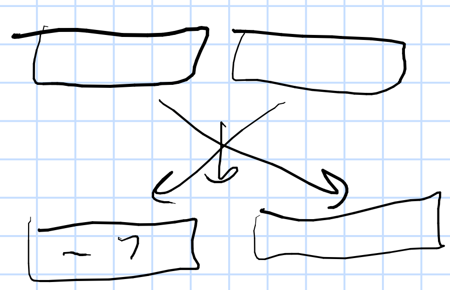
AFTER THE SUBSTITUTION $Y=1-t$
WE OBTAIN:

$$\nabla e_n = \sum_{m,a} X_a t^{|m|} q^{\mathrm{dinv}(m,a)} \sum_{k=0}^{n} (-t)^k\, \delta_{\substack{PF_i \text{ for } i<n-k-1\\ \text{NO } PF_i\ \ i\geq n-k}}$$

ROTATION (Zabrocki trick) $(m,a),k \mapsto (m',a'),k-1$

$m'_i = m_{i-1}+\delta_{i1} \quad a'_i = a_{i-1}$ (mod n)

IS A PERFECT MATCHING ON A SUBSET OF TRIPLES (m,a,k)



CANCELS TERMS THAT DO NOT SATISFY:

1) $k=0$
2) $m_1=0$

HENCE $\nabla e_n[X] = \sum_{\substack{m,a\\ PF,\ m_1=0}} X_a t^{|m|} q^{\mathrm{dinv}(m,a)}$ (SHUFFLE CONJECTURE)