Divided symmetrization and quasisymmetric functions

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Divided symmetrization

Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{Q}[x_1, \cdots, x_n]$, then its **divided symmetrization** $\big\langle f(x_1, \ldots, x_n) \big\rangle_n$ is defined by

$$\big\langle f(x_1, \ldots, x_n) \big\rangle_n := \sum_{w \in S_n} w \cdot \frac{f(x_1, \ldots, x_n)}{\prod_{1 \le i \le n-1} (x_i - x_{i+1})}.$$

Lemma: $\big\langle f \big\rangle_n$ is a polynomial, symmetric in $x_1, \cdots, x_n$.

Let f be homogeneous of degree $\deg(f) = d$.

- if $d < n - 1$, then $\big\langle f(x_1, \ldots, x_n) \big\rangle_n = 0$;
- if $d \ge n - 1$, then $\big\langle f(x_1, \ldots, x_n) \big\rangle_n$ has degree $d - n + 1$.

As a consequence, $\big\langle \cdot \big\rangle_n$ is a linear form on the space R_n of poynomials in $x_1, \cdots, x_n$ of degree $n - 1$.

Divided symmetrization

The space R_n has dimension $\binom{2n-2}{n-1}$, since it has a monomial basis given by $\mathbf{x}^{\mathbf{c}} := x_1^{c_1} \cdots x_n^{c_n}$ where $\sum_i c_i = n-1$.

Our goal is to study $\langle \cdot \rangle_n : R_n \to \mathbb{Q}$.

Motivation 1 Let $V_n(z_1, \dots, z_n)$ be the volume of the permutahedron $P_n(z_1, \dots, z_n)$, defined as the convex hull of $(z_1, \dots, z_n)$ and all points obtained by permuting the coordinates.

Proposition [Postnikov '06]

$$(n-1)! V_n(z_1, \dots, z_n) = \left\langle (z_1 x_1 + z_2 x_2 + \cdots + z_n x_n)^{n-1} \right\rangle_n$$

It is thus a polynomial in $z_1, \dots, z_n$ with coefficients given by the $\langle \mathbf{x}^{\mathbf{c}} \rangle_n$.

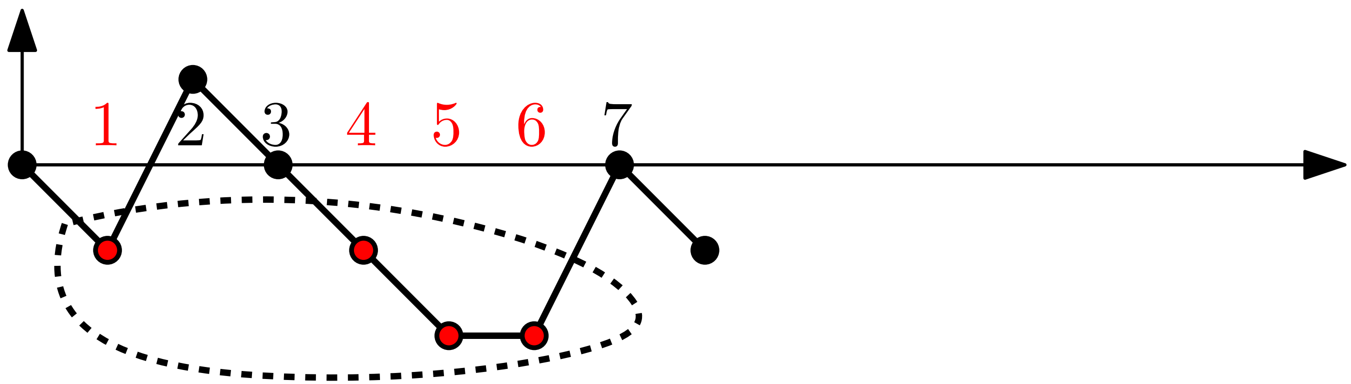
Value on monomials

The evaluation of $\left\langle \mathbf{x}^{\mathbf{c}} \right\rangle_n$ is also due to Postnikov. It was reformulated and reproved by Petrov (2016).

- Given $\mathbf{c} \in \mathbb{N}^n$ s.t. $\sum_i c_i = n-1$, one can attach to it a set $S_{\mathbf{c}} \subseteq [n-1]$.
- Let $\beta_{\mathbf{c}} := |w \in S_n$ such that for all i, $w_i > w_{i+1}$ iff $i \in S_c|$.

Proposition $\left\langle \mathbf{x}^{\mathbf{c}} \right\rangle_n = (-1)^{|S_{\mathbf{c}}|} \beta_{\mathbf{c}}$.

Definition of S_c: Build a path by attaching to c_i the step $(1, c_i - 1)$.



$\mathbf{c} = (0,3,0,0,0,1,3,0)$ with $S_{\mathbf{c}} = \{1,4,5,6\}$.

Quasisymmetric polynomials

If $\mathbf{a}$ is any vector in $\mathbb{N}^m$, let $\mathbf{a}^+$ be the composition obtained by deleting all 0's in $\mathbf{a}$, so $(0,3,0,0,1,2,0,0,0)^+ = (3,1,2)$.

A polynomial $P(x_1, \ldots, x_m)$ is **quasisymmetric** if the coefficients of $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in P are equal whenever $\mathbf{a}^+ = \mathbf{b}^+$.

This definition can be extended to series $f(x_1, x_2, \cdots)$ with bounded degree, called **quasisymmetric functions**.

Example $M_\alpha =$ the sum of all $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a}^+ = \alpha$ for a given composition α. Thus $M_{(3,1,2)} = \sum_{i<j<k} x_i^3 x_j x_k^2$.

Notation Let f be a quasisymmetric function ($f \in \mathrm{QSym}$).

• $f(x_1, \cdots, x_m) :=$ the quasisymmetric polynomial obtained by setting $x_i = 0$ for $i > m$

• $f(1^j) :=$ the value of $f(x_1, \cdots, x_j)$ at $(1, \ldots, 1)$.

If $f \in \mathrm{QSym}^{(k)}$, then $f(1^j)$ is a polynomial in j of degree $\leq k$

Action on quasisymmetric polynomials

Theorem [N.-Tewari '19] For any $f \in \mathrm{QSym}^{(n-1)}$,

$$\sum_{j\geq 0} f(1^j)t^j = \frac{\sum_{m=0}^{n-1} \left\langle f(x_1,\ldots,x_m)\right\rangle_n t^m}{(1-t)^n}.$$

Remark This gives expression for $\left\langle f(x_1,\cdots,x_m)\right\rangle_n$ as linear combinations of the values $f(1^j)$ for $j \leq m$.

Proof sketch: By linearity, enough to do this for the basis M_α. Here one can evaluate the l.h.s. and we get the following identity, equivalent to the theorem:
$\left\langle M_\alpha(x_1,\ldots,x_m)\right\rangle_n = (-1)^{m-\ell(\alpha)}\binom{n-1-\ell(\alpha)}{m-\ell(\alpha)}$
To prove it, one shows first that the l.h.s. only depends on $\ell(\alpha)$, and concludes by evaluating at a special α. □

Action on quasisymmetric polynomials

The preceding theorem is especially nice to apply in the case of F_α, the **fundamental quasisymmetric functions**.

Corollary If $|\alpha| = n-1$, one has $\left\langle F_\alpha(x_1,\ldots,x_m)\right\rangle_n = 0$ if $\ell(\alpha) < m$, and $= 1$ if $m = \ell(\alpha)$.

$$f = \sum_\alpha c_\alpha F_\alpha \in Qsym^{(n-1)} \Rightarrow \left\langle f(x_1,\ldots,x_m\right\rangle_n = \sum_{\ell(\alpha)=m} c_\alpha$$

This can be applied to several (quasi)symmetric functions for which combinatorial expansions are known.

Motivation 2 Our study of $\left\langle\cdot\right\rangle_n$ came from the investigation of the "cohomology class of the Peterson variety". Its coefficients in the Schubert basis are given precisely by $\left\langle\mathfrak{S}_w\right\rangle_n$, where $\mathfrak{S}_w$ is a **Schubert polynomial**. If w is a Grassmannian permutation, then $\mathfrak{S}_w$ is a Schur polynomial $s_\lambda(x_1,\ldots,x_m)$ and we can apply the result above.

Quotienting by quasisymmetric polynomials

If $f \in R_n$ has a homogeneous, *symmetric* factor of positive degree, then $\langle f \rangle_n = 0$.

$\Rightarrow$ By linearity, $\langle \cdot \rangle_n$ vanishes on $R_n \cap I_n$ where $I_n \subset \mathbb{Q}[\mathbf{x}_n]$ is the ideal generated by homogeneous symmetric polynomials in $x_1, \ldots, x_n$ of positive degree.

Theorem [N.-Tewari '19] The form $\langle \cdot \rangle_n$ vanishes on $R_n \cap J_n$ where the ideal $J_n \subset \mathbb{Q}[\mathbf{x}_n]$ is generated by homogeneous **quasisymmetric** polynomials in $x_1, \ldots, x_n$ of positive degree.

Now by the work of Aval-Bergeron-Bergeron (2004)

$$R_n = (R_n \cap J_n) \oplus \mathrm{Vect}\left(\mathbf{x}^{\mathbf{c}} \mid \sum_{k=1}^{i} c_k \geq i \text{ for } i \leq n-1 \right) \quad (*)$$

Corollary Write $f \in R_n$ as $f = g + h$ according to (*). Then $\langle f \rangle_n = h(1, \ldots, 1)$.