An action of the cactus group on shifted tableau crystals

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Inês Rodrigues

Faculdade de Ciências - Universidade de Lisboa

Motivation: projective representations of $\mathfrak{S}_n$

- A **projective representation** of $\mathfrak{S}_n$ is an homomorphism $\rho : \mathfrak{S}_n \longrightarrow PGL(V) = GL(V)/\langle id \rangle$. This may be regarded as a *linear* representation of the **spin group** $\tilde{\mathfrak{S}}_n$ (double cover of $\mathfrak{S}_n$).
    - "Non-trivial" conjugacy classes $\longleftrightarrow$ **odd** partitions of n.
    - Irreducible representations "$\longleftrightarrow$" shifted diagrams of λ a **strict** partition of n.
    - Some of its characters ζ^λ are informed by **Schur Q-functions**:

$$Q_\lambda(\mathbf{x}) = \frac{1}{n!} \sum_{\substack{\mu \vdash n \\ \mu \text{ odd}}} 2^{\lceil \frac{\ell(\lambda)+\ell(\mu)}{2} \rceil} c_\mu \zeta^\lambda_\mu p_\mu(\mathbf{x})$$

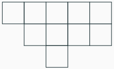
See also: [Stembridge '89, Hoffman, Humphreys '92, Matsumoto, Śniady '19]

Motivation: Schur P- and Q-functions

- Q-functions Q_λ were first introduced by [I. Schur, 1911] as Pfaffians of certain skew symmetric matrices indexed by **strict partitions**.
- They are special cases of Hall-Littlewood symmetric functions.
- A combinatorial definition was due to [Stembridge '89] in terms of shifted tableaux.
- Scaled to define Schur P-functions: $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$.
- Both Schur P- and Q-functions are symmetric and they constitute a basis for the subalgebra Ω of the symmetric functions generated by the **odd degree** power sums.

Shifted Tableaux

- A **strict partition** is a sequence of non-negative integers $\lambda = (\lambda_1 > \ldots > \lambda_k)$.They are represented by **shifted diagrams** (skew shapes defined as expected):



$\lambda = (5, 4, 1)$

- Primed alphabet $[n]' = \{1' < 1 < \ldots < n' < n\}$.
- A **(semistandard) shifted tableau** is a filling of a shifted shape λ/μ with letters of $[n]'$ such that:
  - Every row and every column is weakly increasing.
  - There is at most one i per column and one i' per row, for all i.
- **Canonical form**: the first i is unprimed.

$T =$

1	1	1	1	3′
	2	2	2	3′
		3		

$wt(T) = (4, 3, 3)$

$w(T) = 32223'11113'$

$\mathbf{x}^{wt(T)} = x_1^4 x_2^3 x_3^3$

Shifted Tableaux

- Back to Schur Q-functions:

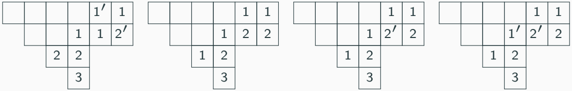
$$Q_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^{wt(T)}$$

where the sum is over all semistandard shifted tableaux of shape λ (not just in canonical form). Same definition for skew shapes λ/μ.

- Many well-known algorithms for Young tableaux have a shifted analogue:
  - Jeu-de-taquin [Worley '84, Sagan '87]
  - Insertion algorithm, RSK [idem]
  - Evacuation, reversal [Worley '84, Thomas, Yong '09, Choi, Nam, Oh '17]
  - Tableau switching [Choi, Nam, Oh '17]

Shifted LR rule

- A tableau of shape λ/μ and weight ν is said to be **Littlewood-Richardson-Stembridge** (LRS) if it rectifies to Y_ν (unique tableau of shape and weight ν). The number of such tableaux $f^{\lambda}_{\mu\nu}$ is called the **shifted Littlewood-Richardson coefficient**.
- For $\lambda = (6,5,2,1)$, $\mu = (4,2)$ and $\nu = (4,3,1)$, we have the following LRS tableaux:



hence $f^{\lambda}_{\mu\nu} = 4$.

Shifted LR rule

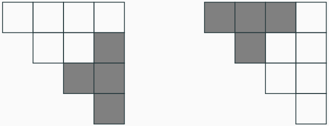
- Shifted LR coefficients are structure constants of the following linear expansions, concerning bases of Ω:

$$P_\mu P_\nu = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda \qquad\qquad Q_{\lambda/\mu} = \sum_\nu f_{\mu\nu}^\lambda Q_\nu$$

- They also appear in the context of orthogonal Grassmannian $OG(2n+1, n)$

$$\tau_\mu \tau_\nu = \sum_\lambda f_{\mu\nu}^\lambda \tau_\lambda$$

where τ_μ is a Schubert class in the cohomology ring of the orthogonal Grassmannian.



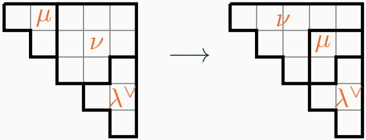
$\lambda = (4,2)$ $\qquad$ $\lambda^\vee = (3,1)$

(Shifted diagrams live in an ambient triangle)

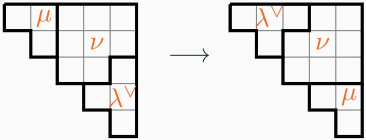
Shifted LR coefficients and its symmetries

Like the LR coefficients for the product of Schur functions, the shifted analogue exhibit symmetries under the action of $\mathfrak{S}_3$ on the triple (μ, ν, λ). Let $f_{\mu\nu\lambda} := f_{\mu\nu}^{\lambda^\vee}$

- $f_{\mu\nu\lambda} = f_{\nu\mu\lambda}$ (commutativity) $\longrightarrow$ P-functions product $P_\mu P_\nu = \sum_\nu f_{\mu\nu}^{\lambda} P_\lambda$ or **shifted tableau switching**.



- $f_{\mu\nu\lambda} = f_{\lambda\nu\mu}$ $\longrightarrow$ **shifted reversal** (together with a "reflection").



These two may be combined to obtain other symmetries.

Type A crystals

- A **Kashiwara crystal** of type A (for GL_n) is a non-empty set $\mathcal{B}$ together with partial maps $e_i, f_i : \mathcal{B} \longrightarrow \mathcal{B}$, lenght functions $\varepsilon_i, \phi_i : \mathcal{B} \longrightarrow \mathbb{Z}$, for $i \in I = [n-1]$, and weight function $wt : \mathcal{B} \longrightarrow \mathbb{Z}^n$ satisfying the axioms:

  (K1) For $x, y \in \mathcal{B}$, $e_i(x) = y$ iff $f_i(y) = x$. In that case,
  - $(\varepsilon_i(y), \phi_i(y)) = (\varepsilon_i(x) - 1, \phi_i(x) + 1)$
  - $wt(y) = wt(x) + \alpha_i$

  (K2) For $x \in \mathcal{B}$, $\phi_i(x) - \varepsilon_i(x) = \langle wt(x), \alpha_i \rangle$

  ($\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i \in I$, where $\{\mathbf{e}_i\}$ canonical base of $\mathbb{R}^n$)

- This may be regarded as a directed graph, with vertices in $\mathcal{B}$ and i-colored edges $y \xrightarrow{i} x$ iff $f_i(y) = x$, for $i \in I$.

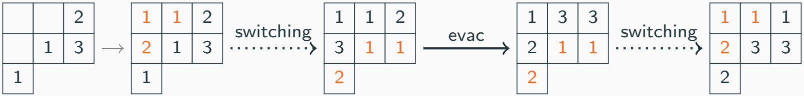
Type A crystals

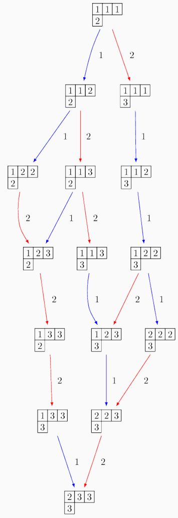
- **Semistandard Young tableaux** (SSYT) of a given shape, in the alphabet $[n]$, have a Kashiwara type A crystal structure, with *coplactic*[1] operators e_i and f_i. This crystal is isomorphic to the crystal basis of a $U_q(\mathfrak{gl}_n)$-module.
- The **Schützenberger involution** is defined on the type A crystal $\mathcal{B}$ of SSYT of shape λ on alphabet $[n]$ as the unique map $\xi : \mathcal{B} \longrightarrow \mathcal{B}$ such that, for $i \in I = [n-1]$:
  - $e_i \xi(x) = \xi f_{n-i}(x)$
  - $f_i \xi(x) = \xi e_{n-i}(x)$
  - $wt(\xi(x)) = \omega_{\{1,\ldots,n\}} \cdot wt(x)$

[1] i.e. they commute with the jeu de taquin.

Type A crystals

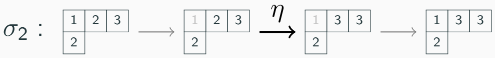
- The Schützenberger involution “flips” the crystal graph upside down (reverting the orientation of the arrows and its colors).
- For Young tableaux, it is realized by the **evacuation** (for normal shapes) or the **reversal** (the coplactic extension of the evacuation) involution.

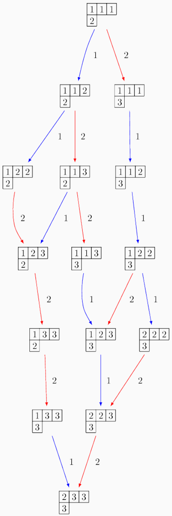




$\lambda = (3, 1), \ I = \{1, 2\}$

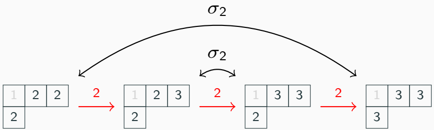
- In type A tableau crystals, there is an action of $\mathfrak{S}_n$, where the action of the simple transpositions s_i is realized by the **crystal reflection operators** σ_i, that corresponds to the *restriction* of the Schützenberger involution (or reversal) to the letters i and $i+1$.
- To obtain this restriction:
  - Temporarily forget about the letters different from i and $i+1$, obtaining a skew tableau.
  - Apply the Schützenberger involution to the obtained tableau.
  - Put the letters back again.





$\lambda = (3, 1),\ I = \{1, 2\}$

- These involutions take every string of color i to itself, “reflecting” it through the middle of the string:



The internal action of the cactus group on type A crystals

The n-**fruit cactus group** J_n is generated by $s_{p,q}$, for $1 \leq p < q \leq n$, subject to the following relations:

1. $s_{p,q}^2 = id$.
2. $s_{p,q}s_{k,l} = s_{k,l}s_{p,q}$ for $\{p, \dots, q\} \cap \{k, \dots, l\} = \emptyset$.
3. $s_{p,q}s_{k,l} = s_{p+q-l,p+q-k}s_{p,q}$ for $\{k, \dots, l\} \subseteq \{p, \dots, q\}$.

- For $n = 3$,

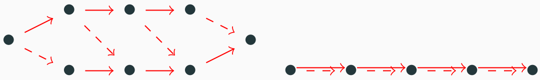
$$J_3 = \langle s_{1,2}, s_{1,3}, s_{2,3} | s_{1,2}^2 = s_{2,3}^2 = s_{2,3}^2 = 1, s_{1,3}s_{1,2} = s_{2,3}s_{1,3} \rangle.$$

- Surjection $J_n \twoheadrightarrow \mathfrak{S}_n$, $s_{p,q} \mapsto \omega_{\{p,\dots,q\}}$.
- Acts internally on type A tableau crystals through the restriction of the Schützenberger involution to letters $\{p < \dots < q\}$ [Halacheva '16].

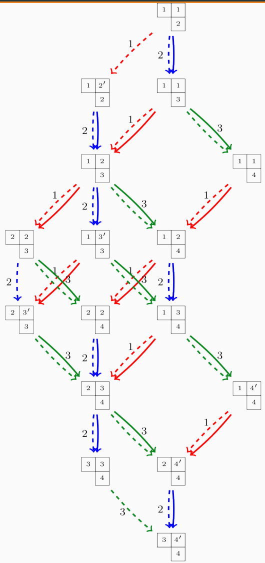
Shifted crystals

- [Gillespie, Levinson, Purbhoo '17] introduced a type A "crystal-like" structure for shifted tableaux. Let $\mathcal{B}(\lambda/\mu, n)$ be the set of semistandard shifted tableaux of shape λ/μ in the alphabet $[n]'$ and index set $I = [n-1]$ together with:
  - Primed and unprimed operators: E_i, E_i', F_i, F_i', defined by rules, for $i \in I$ (commute with jeu de taquin)
  - Lenght functions: $\varepsilon_i\,(\hat{\varepsilon}_i, \varepsilon_i')$ and $\phi_i\,(\hat{\phi}_i, \phi_i')$, for $i \in I$.
  - Weight function: $wt(T)$.
- This **shifted crystal** may be regarded as a directed graph, with vertices in $\mathcal{B}(\lambda/\mu, n)$ and i-colored edges, for $i \in I$:
  - $x \longrightarrow y$ iff $F_i(x) = y$ iff $E_i(y) = x$.
  - $x \dashrightarrow y$ iff $F_i'(x) = y$ iff $E_i'(y) = x$.

  Unlike type A tableau crystals, there are two possible arragements for i-colored strings:



Shifted crystals



$\lambda = (2, 1), \ I = \{1, 2, 3\}$

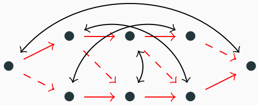
- Taking primed and unprimed operators *independently* yields Kashiwara type A crystals.
- $\mathcal{B}(\lambda, n)$ has a unique **highest weight** and **lowest weight** elements: Y_λ and its evacuation. Any shifted tableau of this shape and alphabet can be obtained from these.
- The character of $\mathcal{B}(\lambda/\mu, n)$ is the Schur Q-function $Q_{\lambda/\mu}(x_1, \ldots, x_n)$.
- $\mathcal{B}(\lambda/\mu, n) \simeq \bigsqcup_\nu \mathcal{B}(\nu, n)^{f^\lambda_{\mu\nu}}$.
- Taking characters of the connected components, it yields

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu$$

- The **Schützenberger involution** is defined in $\mathcal{B}(\lambda, n)$ as the unique map $\eta : \mathcal{B}(\lambda, n) \longrightarrow \mathcal{B}(\lambda, n)$ such that, for $1 \leq i \leq n-1$:
  - $E_i' \eta(T) = \eta F_{n-i}'(T), \qquad E_i \eta(T) = \eta F_{n-i}(T).$
  - $F_i' \eta(T) = \eta E_{n-i}'(T), \qquad F_i \eta(T) = \eta E_{n-i}(T).$
  - $wt(\eta(T)) = \omega_{\{1,\dots,n\}} \cdot wt(T).$
- It it realized by the **shifted evacuation** or **shifted reversal**.

Shifted crystals

- The **shifted reflection operators** σ_i may be defined using the crystal operators E_i', E_i, F_i', F_i.
  - It corresponds to the restriction of the Schützenberger involution to the letters $i', i, (i+1)', (i+1)$.
  - Acts as $s_i \in \mathfrak{S}_n$ on the *weight* of a tableau (in particular, it shows that Q-functions are symmetric functions).
  - Acts on strings by "double" reflection, through the vertical and horizontal middle axis (or rotation by π).



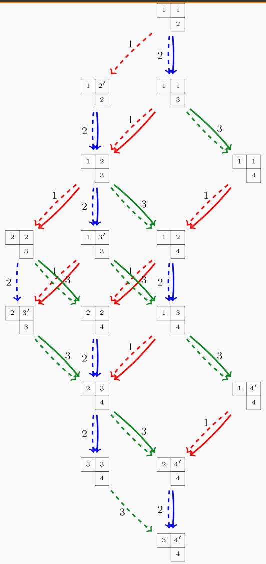
Shifted crystals

- We have $\sigma_i^2 = 1$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$, for $|i - j| \geq 2$.
- However, unlike the type A, the involutions σ_i do **not** realize an action of $\mathfrak{S}_n$ on $\mathcal{B}(\lambda, n)$, since the braid relations may not hold:

$$\sigma_1\sigma_2\sigma_1\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 3' \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array}\right) = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}$$

$$\sigma_2\sigma_1\sigma_2\left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 3' \\ & 2 & 2 & 3' & \\ & & 3 & & \end{array}\right) = \begin{array}{ccccc} 1 & 1 & 1 & 2' & 3' \\ & 2 & 3' & 3 & \\ & & 3 & & \end{array}$$

A cactus group action on $\mathcal{B}(\lambda, n)$



$\lambda = (2,1),\ I = \{1,2,3\}$

- The restriction of the Schützenberger involution to the letters $\{p, \ldots, q\}' \subseteq [n]'$, $\eta_{p,q}$, defines an action of the n-fruit cactus group in $\mathcal{B}(\lambda, n)$:

$$s_{p,q} \cdot T = \eta_{p,q}(T).$$

- Consider the *subgraph* $\mathcal{B}_{p,q}$, obtained from $\mathcal{B}(\lambda, n)$ considering only the vertices in which the letters $\{p, \ldots, q\}'$ appear and the edges colored in $\{p, \ldots, q-1\}$.
- Then $\eta_{p,q}$ acts on the connected components of $\mathcal{B}_{p,q}$ regarding its vertices as skew shifted tableaux on the alphabet $\{p, \ldots, q\}'$.

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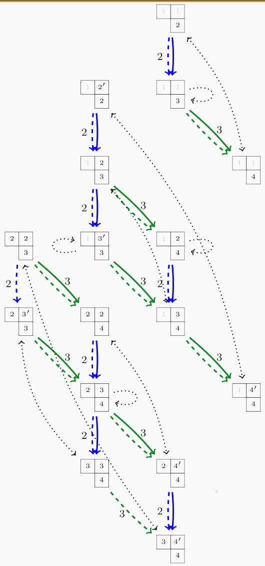

$\lambda = (2,1)$, $I = \{1,2,3\}$, action of $s_{2,4}$

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A cactus group action on $\mathcal{B}(\lambda, n)$ (sketch of proof)

- The relations

$$\eta_{p,q}^2 = id$$
$$\eta_{p,q}\eta_{k,l} = \eta_{k,l}\eta_{p,q} \text{ for } \{p, \ldots, q\} \cap \{k, \ldots, l\} = \emptyset$$

  are trivial.
- For the relation

$$s_{p,q}s_{k,l} = s_{p+q-l,p+q-k}s_{p,q} \text{ for } \{k, \ldots, l\} \subseteq \{p, \ldots, q\}$$

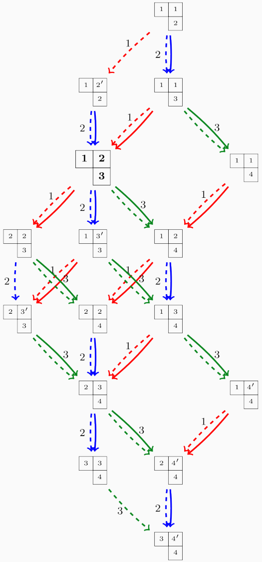
  if suffices to show that

$$\eta_{1,n}\eta_{p,q} = \eta_{1+n-q,1+n-p}\eta_{1,n}$$

- The subgraph $\mathcal{B}_{p,q}$ is an union of connected components, each one isomorphic to some $\mathcal{B}(\mu, q-p+1)$. Hence, each one has *unique* highest and lowest weights.
- $\eta = \eta_{1,n}$ takes each connected component $\mathcal{B}^0_{p,q}$ to another $\mathcal{B}^0_{1+n-q,1+n-p}$. Moreover, the highest weight of the former is sent to the lowest weight of the latter.

A cactus group action on $\mathcal{B}(\lambda, n)$ (sketch of proof)

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \in \mathcal{B}((2,1),4)$$



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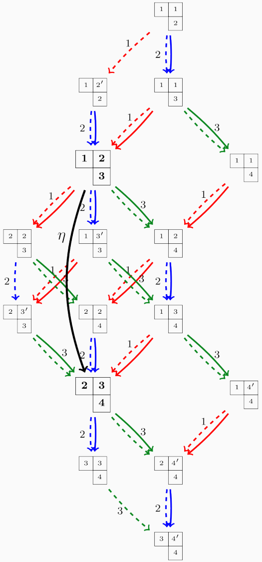
$$\eta_{1,4}(T) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 \\ \cline{2-2} \end{array}$$



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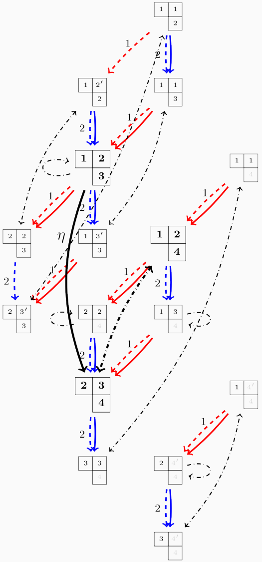

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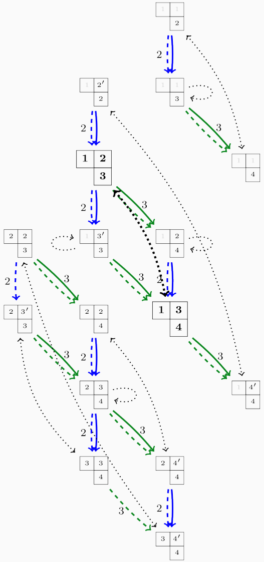

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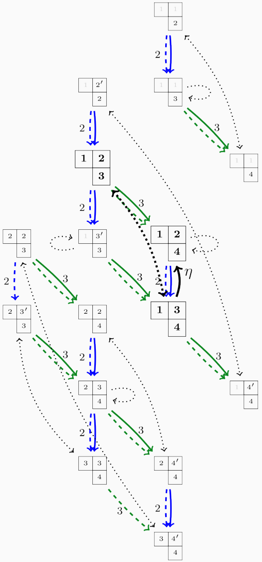

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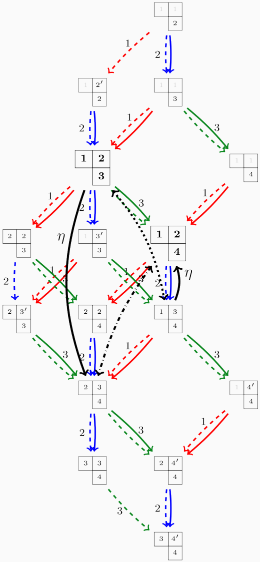
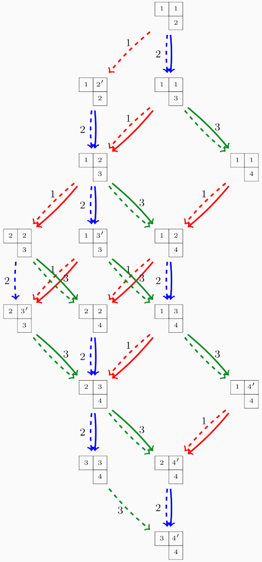


An application to the symmetries of shifted LR coefficients



- In particular, we have $s_{i,i+1} \cdot T = \sigma_i(T)$.
- The action of $s_{1,n}$ coincides with the Schützenberger involution in $\mathcal{B}(\lambda/\mu, n)$.
- For T a LRS tableau,

$$s_{1,n} \cdot T = \sigma_{i_1} \ldots \sigma_{i_k}(T)$$

  where $\omega_{\{1,\ldots,n\}} = s_{i_1} \ldots s_{i_k}$ is the longest permutation in $\mathfrak{S}_n$.
- It exhibits the symmetry $f_{\mu\nu\lambda} = f_{\lambda\nu\mu}$ (after "reflection").

Thank you!