Simplectic Keys and Demazure atoms in type C: A frank discussion

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Overview

1. Weyl group of type C: tableau criterion for Bruhat order
2. Type C Crystals
3. Demazure crystal and Demazure atom crystal
4. Key tableaux and frank words in type C
5. Lusztig involution

Notation

- $n \in \mathbb{N}_{>0}$;
- $[n] := \{1 < \cdots < n\}$ and $[\pm n] := \{1 < \cdots < n < -n < \cdots < -1\}$;
- Vector: element of $\mathbb{Z}^n$;
- Word: tuple with entries in $[\pm n]$.

Weyl group of type C

Hyperoctahedral goup

Definition

The hyperoctahedral group is a Coxeter group with the following presentation:

$$B_n \cong \langle s_1, \ldots, s_{n-1}, s_n \,|\, s_i^2 = 1,\ 1 \leq i \leq n;\ (s_i s_{i+1})^3 = 1,\ 1 \leq i \leq n-2;$$
$$(s_{n-1} s_n)^4 = 1;$$
$$(s_i s_j)^2 = 1,\ 1 \leq i < j \leq n,\ |i-j| > 1 \rangle.$$

$\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle \leq B_n$
$|B_n| = 2^n n!$

Weyl group of type C

Bruhat order

Definition

Let $w \in B_n$ with reduced decomposition $\sigma_1 \dots \sigma_k$, where σ_i are generators of B_n, and u be two elements in B_n. Then

$$u \leq w \overset{\text{def}}{\Leftrightarrow} \exists 1 \leq i_1 < i_2 \dots < i_{k'} \leq k \text{ such that}$$
$$u = \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_{k'}}.$$

Example

$s_1 s_2 s_4 \leq s_1 s_2 s_1 s_4 s_3$

There is a biggest element of B_n, denoted as ω_0, known as the longest element.

Weyl group of type C

Actions on vectors

Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{Z}^n$.

- $vs_i = (v_1, \dots, v_{i+1}, v_i, \dots, v_n)$ if $i \in [n-1]$;
- $vs_n = (v_1, \dots, v_{n-1}, -v_n)$;
- $v\omega_0 = (-v_1, -v_2, \dots, -v_n)$.

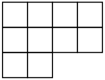
Define $x^v = x_1^{v_1} \dots x_n^{v_n}$.
s_i acts on $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by $(x^v)s_i = x^{vs_i}$.

Young diagram

Definition (Partition)

A vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ is a partition if $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Young diagram of shape $\lambda = (4, 4, 2, 0)$, $n = 4$:



Semi Standard Young Tableau

Definition (Semi Standard Young Tableaux)

A semi standard Young tableau (SSYT) of shape λ is a filling of the boxes of the Young diagram of shape λ with elements from a ordered alphabet such that they are non-decreasing in each row and strictly increasing in each column.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 3 & 3 & 3 & 4 \\ \hline 4 & 4 \\ \cline{1-2}\end{array}\ .$$

T is a SSYT, $\mathrm{sh}(T) = (4, 4, 2, 0)$, $wtT = (3, 0, 4, 3)$.

The (column) reading word of the tableau T, or word of T, is $wr(T) = 34\,13\,134\,134$.

The (column) insertion algorithm constructs a tableau from a given word. The insertion map is not injective and is the left inverse of the reading map.

Symplectic tableaux: Kashiwara-Nakashima tableaux

Admissible columns - Sheats, 1999

A column is a word whose letters are strictly increasing according to the alphabet $[\pm n] = \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$.

$$C_1 = \begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline \overline{5}\\\hline \overline{4}\\\hline\end{array} \quad \begin{array}{ccccc} \emptyset & 2 & \emptyset & 4 & 5\\ \emptyset & \emptyset & \emptyset & \overline{4} & \overline{5}\end{array} \qquad C_2 = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array} \quad \begin{array}{ccccc} \emptyset & 2 & 3 & 4 & \emptyset\\ \emptyset & \emptyset & \overline{3} & \overline{4} & \emptyset\end{array} \qquad C_3 = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline \overline{5}\\\hline \overline{1}\\\hline\end{array} \quad \begin{array}{ccccc} \emptyset & 2 & 3 & 4 & \emptyset\\ \overline{1} & \emptyset & \emptyset & \emptyset & \overline{5}\end{array}$$

A column is an admissible column if the diagram is such that there is a matching which sends each full slot to an empty slot to its left.

$$\ell C_1 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline \overline{5}\\\hline \overline{4}\\\hline\end{array} \quad \begin{array}{ccccc} 1 & 2 & 3 & \emptyset & \emptyset\\ \emptyset & \emptyset & \emptyset & \overline{4} & \overline{5}\end{array} \qquad rC_1 = \begin{array}{|c|}\hline 2\\\hline 4\\\hline 5\\\hline \overline{3}\\\hline \overline{1}\\\hline\end{array} \quad \begin{array}{ccccc} \emptyset & 2 & \emptyset & 4 & 5\\ \overline{1} & \emptyset & \overline{3} & \emptyset & \emptyset\end{array} \qquad \ell C_3 = C_3 = rC_3$$

Symplectic tableaux: Kashiwara-Nakashima tableaux

KN tableaux

Let T be a tableau with all columns admissible. $spl(T)$ is the tableau obtained after replacing each column C by the columns ℓC and rC.
T is a Kashiwara-Nakashima (KN) tableau if spl(T) is a SSYT.

Example

$$T_1 = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \bar{3} & \bar{3} \\ \hline \bar{1} & \\ \cline{1-1} \end{array}, \; spl(T_1) = \begin{array}{|c|c|c|c|}\hline 2 & 3 & 2 & 3 \\ \hline \bar{3} & \bar{2} & \bar{3} & \bar{2} \\ \hline \bar{1} & \bar{1} & & \\ \cline{1-2} \end{array} \text{ is not a KN tableau.}$$

$$T_2 = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \bar{2} \\ \hline \bar{2} & \\ \cline{1-1} \end{array}, \; spl(T_2) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 3 & 3 & \bar{2} & \bar{2} \\ \hline \bar{2} & \bar{1} & & \\ \cline{1-2} \end{array} \text{ is a KN tableau; } wt(T_2) = (0, -1, 2)$$

For KN tableaux there also exists an insertion algorithm that is the inverse of the reading map.

Symplectic Key Tableaux

Definition (Key tableau)

A key tableau is a KN tableau with nested columns and with no symmetric entries.

There is a bijection between symplectic key tableaux in the alphabet $[\pm n]$ and $\mathbb{Z}^n$.

Example

Consider $v = (-4, 0, 2, 4)$.

$$K(v) = \begin{array}{|c|c|c|c|} \hline 3 & 3 & 4 & 4 \\ \hline 4 & 4 & \overline{1} & \overline{1} \\ \hline \overline{1} & \overline{1} \\ \cline{1-2} \end{array} .$$

Tableau criterion for Bruhat order

Proposition

Let $v_1, v_2 \in \mathbb{Z}^n$ such that $v_1, v_2 \in \lambda B_n$.
Let $\sigma, \rho \in B_n$ minimal such that $\lambda\sigma = v_1$ and $\lambda\rho = v_2$. Then
$\sigma \leq \rho \Leftrightarrow K(v_1) \leq K(v_2)$

Example

$\lambda = (n, n-1, \ldots, 1)$, $\sigma = s_1 s_2 s_4 \leq \rho = s_1 s_2 s_1 s_4 s_3$.
$\lambda\sigma = (3, 2, 4, \overline{1})$ and $\lambda\rho = (2, 3, \overline{1}, 4)$.

$$K(\lambda\sigma) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 3 & \\ \cline{1-3} 3 & 3 & & \\ \cline{1-2} \overline{4} & & & \\ \cline{1-1} \end{array} \leq \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 4 & \\ \cline{1-3} 4 & 4 & & \\ \cline{1-2} \overline{3} & & & \\ \cline{1-1} \end{array} = K(\lambda\rho)$$

Knuth relations/plactic monoid in type A

Knuth equivalence in type A is the equivalence relation that identifies words in the alphabet $[n]$ with the same insertion tableau. It is the symmetric and transitive closure of the relations

K1: $\gamma\beta\alpha \sim \beta\gamma\alpha$, where $\gamma < \alpha \leq \beta$

K2: $\alpha\beta\gamma \sim \alpha\gamma\beta$, where $\gamma \leq \alpha < \beta$

Knuth-equivalent words have the same weight.

Knuth relations/plactic monoid in type C (Lecouvey, 2002)

Knuth equivalence in type C is the equivalence relation that identifies words in the alphabet $[\pm n]$ with the same insertion tableau. It is the symmetric and transitive closure of the relations

- K1: $\gamma\beta\alpha \sim \beta\gamma\alpha$, where $\gamma < \alpha \leq \beta$ and $(\beta, \gamma) \neq (\overline{x}, x)$;
- K2: $\alpha\beta\gamma \sim \alpha\gamma\beta$, where $\gamma \leq \alpha < \beta$ and $(\beta, \gamma) \neq (\overline{x}, x)$;
- K3: $\overline{y}y\beta \sim y+1\overline{y+1}\beta$, where $y < \beta < \overline{y}$;
- K4: $\beta\overline{y}y \sim \beta y+1\overline{y+1}$, where $y < \beta < \overline{y}$;
- K5: $w \sim w \setminus \{z, \overline{z}\}$, where w is non admissible column, but any of its factors form admissible columns, and z is minimal such that z and $\overline{z}$ appear in w and there are more than z letters in w with absolute value less or equal than z.

Knuth-equivalent words have the same weight.

Knuth relations/plactic monoid in type C (Lecouvey, 2002)

Knuth equivalence in type C is the equivalence relation that identifies words in the alphabet $[\pm n]$ with the same insertion tableau. It is the symmetric and transitive closure of the relations

- K1: $\gamma\beta\alpha \sim \beta\gamma\alpha$, where $\gamma < \alpha \leq \beta$ and $(\beta, \gamma) \neq (\overline{x}, x)$;
- K2: $\alpha\beta\gamma \sim \alpha\gamma\beta$, where $\gamma \leq \alpha < \beta$ and $(\beta, \gamma) \neq (\overline{x}, x)$;
- K3: $\overline{y}y\beta \sim y+1\overline{y+1}\beta$, where $y < \beta < \overline{y}$;
- K4: $\beta\overline{y}y \sim \beta y+1\overline{y+1}$, where $y < \beta < \overline{y}$;
- K5: $w \sim w \setminus \{z, \overline{z}\}$, where w is non admissible column, but any of its factors form admissible columns, and z is minimal such that z and $\overline{z}$ appear in w and there are more than z letters in w with absolute value less or equal than z.

Knuth-equivalent words have the same weight.

Example

$\underline{2\overline{2}2}\overline{1} \xrightarrow{K3} \overline{1}\underline{12\overline{1}} \xrightarrow{K5} \overline{1}2$

Kashiwara Crystal

Definition

Let Φ be a root system with simple roots $\{\alpha_i \mid i \in I\}$ in an Euclidian vector space V. A Kashiwara crystal of type Φ is a nonempty set $\mathfrak{B}$ together with maps:

$$e_i, f_i : \mathfrak{B} \to \mathfrak{B} \sqcup \{0\} \quad \varepsilon_i, \varphi_i : \mathfrak{B} \to \mathbb{Z} \sqcup \{-\infty\} \quad wt : \mathfrak{B} \to \Lambda \subseteq V$$

such that

1. if $a, b \in \mathfrak{B}$ then $e_i(a) = b \Leftrightarrow f_i(b) = a$. In this case, we also have $wt(b) = wt(a) + \alpha_i$, $\varepsilon_i(b) = \varepsilon_i(a) - 1$ and $\varphi_i(b) = \varphi_i(a) + 1$;
2. for all $a \in \mathfrak{B}$, we have $\varphi_i(a) = \langle wt(a), \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle} \rangle + \varepsilon_i(a)$.

$\varphi_i(a) = \max\{k \in \mathbb{Z} \geq 0 \mid f_i^k(a) \neq 0\}$ and
$\varepsilon_i(a) = \max\{k \in \mathbb{Z} \geq 0 \mid e_i^k(a) \neq 0\}$
Highest weight element: $u \in \mathfrak{B}$ such that $e_i(u) = 0$ for all $i \in I$.
Lowest weight element: $u \in \mathfrak{B}$ such that $f_i(u) = 0$ for all $i \in I$.
Cristal graph: $\mathfrak{B}$ is the vertex set and $b \xrightarrow{i} b'$ iff $b' = f_i(b)$, $i \in I$.

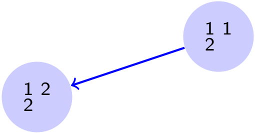
Kashiwara Crystal

Given $n \in \mathbb{N}$ and a partition λ with at most n parts, the KN tableaux of shape λ on the alphabet $[\pm n]$ form a connected crystal $\mathfrak{B}^\lambda$ with highest weight $K(\lambda)$ and lowest weight $K(\lambda\omega_0)$.

Proposition

Let $i \in [n]$ and $\varphi_i(v) \neq 0$. $v \overset{Knuth}{\simeq} w$ iff $f_i(v) \overset{Knuth}{\simeq} f_i(w)$.

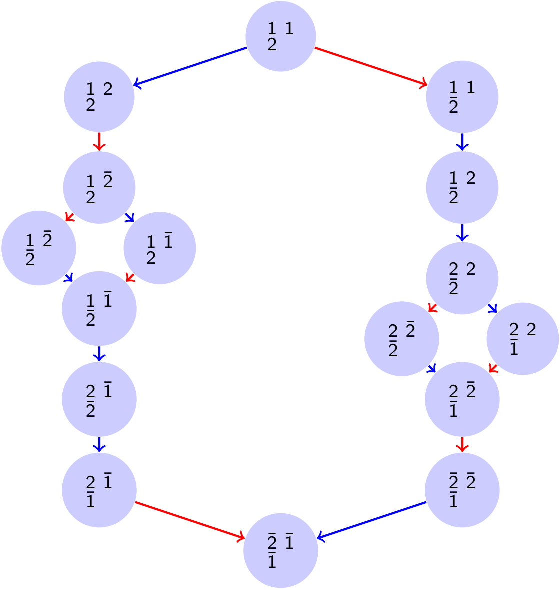
Example of crystal: $\mathfrak{B}^{(2,1)}$ (Type A)



$\rightarrow f_1$

$$s_{(2,1)}(\boldsymbol{x}) = x_1^2 x_2^1 + x_1^1 x_2^2$$

Example of crystal: $\mathfrak{B}^{(2,1)}$ (Type C)



$\rightarrow f_1$

$\rightarrow f_2$

$$s_{(2,1)}(\boldsymbol{x}) = x_1^2 x_2^1 + x_1^1 x_2^2$$
$$sp_{(2,1)}(\boldsymbol{y}) = \sum_{T \in \mathfrak{B}^{(2,1)}} \boldsymbol{y}^{wt\ T}$$

Demazure crystal and Demazure atom crystal in type C

Given X a subset of the crystal $\mathfrak{B}^\lambda$, we define the operator $\mathfrak{D}_i$ on X, $i \in [n]$:

$$X\mathfrak{D}_i = \{x \in \mathfrak{B}^\lambda \mid e_i^k(x) \in X \text{ for some } k \geq 0\}.$$

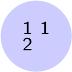
Let $v = \lambda\sigma$ where $\sigma \in B_n$ is minimal with reduced decomposition $s_{i_1} \dots s_{i_k}$. The Demazure Crystal $\mathfrak{B}_v := \{K(\lambda)\}\mathfrak{D}_{i_1} \dots \mathfrak{D}_{i_k}$.
Since $e_i^0(x) = x$, we have that if $\rho \leq \sigma$ then $\mathfrak{B}_{\lambda\rho} \subseteq \mathfrak{B}_{\lambda\sigma}$.
The Demazure Atom Crystal is $\hat{\mathfrak{B}}_v = \hat{\mathfrak{B}}_{\lambda\sigma} := \mathfrak{B}_{\lambda\sigma} \setminus \bigcup_{\rho<\sigma} \mathfrak{B}_{\lambda\rho}$.

Definition

Type C Key polinomial: $K_{\lambda\sigma}(x) := \sum_{T \in \mathfrak{B}_{\lambda\sigma}} x^{\mathrm{wt}\, T}$

Type C Demazure atom: $\hat{K}_{\lambda\sigma}(x) := \sum_{T \in \hat{\mathfrak{B}}_{\lambda\sigma}} x^{\mathrm{wt}\, T}$

Demazure crystal - Example $\mathfrak{B}_\lambda = \hat{\mathfrak{B}}_\lambda$

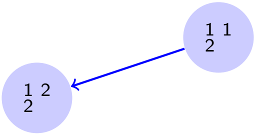


$\color{blue}{\rightarrow}$ f_1

$\color{red}{\rightarrow}$ f_2

$$K_{(2,1)}(x) = \hat{K}_{(2,1)}(x) = x_1^2 x_2$$

Demazure crystal - Example $\mathfrak{B}_{\lambda s_1}$

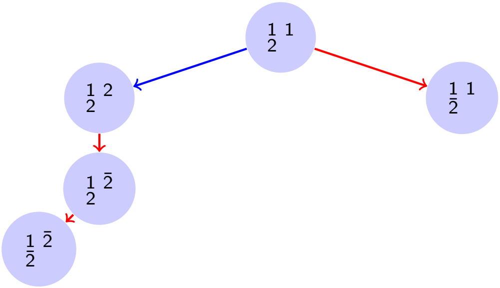


$\rightarrow f_1$

$\rightarrow f_2$

$$K_{(1,2)}(x) = x_1^2 x_2 + x_2^2 x_1$$

Demazure crystal - Example $\mathfrak{B}_{\lambda s_1 s_2}$

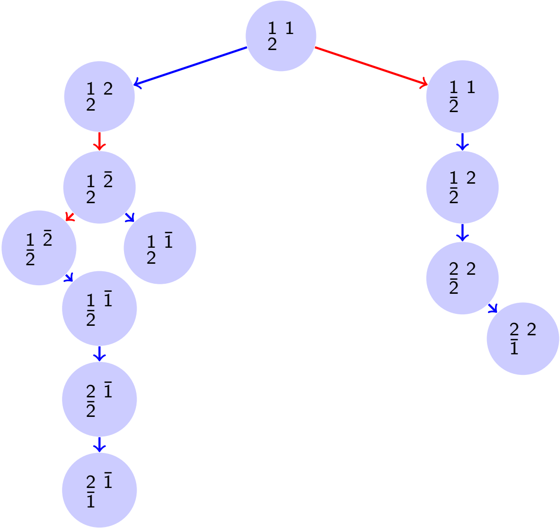


$\rightarrow f_1$

$\rightarrow f_2$

$$K_{(1,\bar{2})}(x) = \sum_{T \in \mathfrak{B}_{(1,\bar{2})}} x^{wt\ T}$$

Demazure crystal - Example $\mathfrak{B}_{\lambda s_1 s_2 s_1}$



$\rightarrow f_1$

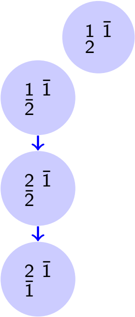
$\rightarrow f_2$

$$K_{(\bar{2},1)}(x) = \sum_{T \in \mathfrak{B}_{(\bar{2},1)}} x^{wt\ T}$$

Demazure atom crystal - Example $\hat{\mathfrak{B}}_{\lambda s_1 s_2 s_1}$

$\rightarrow f_1$

$\rightarrow f_2$



$$\hat{K}_{(\overline{2},1)}(x) = \sum_{T \in \hat{\mathfrak{B}}_{(\overline{2},1)}} x^{wt\ T}$$

Frank words in type C

Lascoux-Schützenberger, 1988, in type A

The column decomposition of a word w is the tuple of maximal subwords of w that are columns.

Example

$T = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} & \\ \cline{1-1} \end{array}$, $w = \mathrm{wr}(T) = 22\bar{2}$. Columns of w: 2, $2\bar{2}$

Proposition

Consider a KN tableau T and $w = wr(T)$ and consider the sequence of column lengths.
For every permutation of those lengths there is a word w' Knuth-equivalent to w with that sequence of column lengths.

Such a word w' is said to be a frank word.

Frank word - Examples

Example

$$T = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \; w = 2\,2\bar{2} \text{ has column lengths } (1,2).$$

$22\bar{2} \xrightarrow{K4} 2\bar{1}1 = w'$ has column lengths $(2,1)$, so it is a frank word.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad w = 1123 \text{ has column lengths } (1,3).$$

$\underline{112}3 \xrightarrow{K2} 1213$ has column lengths $(2,2)$, hence it is not a frank word.

$1\underline{213} \xrightarrow{K2} 1231$ has column lengths $(3,1)$, hence it is a frank word.

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:
1) Start with the Young diagram with same shape as T.
2) For every column length of T find a frank word whose first column has that length.
3) Split each of those columns.
4)Fill the Young diagram with the right column of each split.

Example

$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \\ \cline{1-1}\end{array}$, $\mathrm{wr}(T) = 2\,1\,\bar{2}$

$K_+(T) = \begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}$

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:
1) Start with the Young diagram with same shape as T.
2) For every column length of T find a frank word whose first column has that length.
3) Split each of those columns.
4)Fill the Young diagram with the right column of each split.

Example

$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, $\mathrm{wr}(T) = 2\,1\,\overline{2} \overset{Knuth}{\simeq} 2\,\overline{2}\,1$

$K_+(T) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:

1) Start with the Young diagram with same shape as T.

2) For every column length of T find a frank word whose first column has that length.

3) Split each of those columns.

4)Fill the Young diagram with the right column of each split.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \ \mathrm{wr}(T) = 2\,1\,\overline{2} \overset{Knuth}{\simeq} 2\,\overline{2}\,1$$

$$K_+(T) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad C_1 = \begin{array}{|c|}\hline 2 \\ \hline\end{array}$$

$$C_2 = \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$$

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:
1) Start with the Young diagram with same shape as T.
2) For every column length of T find a frank word whose first column has that length.
3) Split each of those columns.
4)Fill the Young diagram with the right column of each split.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\ ,\ \mathrm{wr}(T) = 2\,1\,\overline{2} \overset{Knuth}{\simeq} 2\overline{2}\,1$$

$$K_+(T) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad C_1 = \begin{array}{|c|}\hline 2 \\ \hline\end{array}\ \begin{smallmatrix} \emptyset & 2 \\ \emptyset & \emptyset \end{smallmatrix} \text{ hence } rC_1 = C_1$$

$$C_2 = \begin{array}{|c|}\hline 2 \\ \hline \overline{2} \\ \hline\end{array}$$

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:

1) Start with the Young diagram with same shape as T.

2) For every column length of T find a frank word whose first column has that length.

3) Split each of those columns.

4)Fill the Young diagram with the right column of each split.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \ \mathrm{wr}(T) = 2\,1\,\bar{2} \overset{Knuth}{\simeq} 2\,\bar{2}\,1$$

$$K_+(T) = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$C_1 = \begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{matrix} \emptyset & 2 \\ \emptyset & \emptyset \end{matrix} \ \text{hence } rC_1 = C_1$$

$$C_2 = \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline\end{array} \quad \begin{matrix} \emptyset & 2 \\ \emptyset & \bar{2} \end{matrix} \rightarrow \begin{matrix} \emptyset & 2 \\ \color{red}{\bar{1}} & \emptyset \end{matrix} \ \text{hence } rC_2 = \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array}$$

Right Keys

Consider a KN tableau T. To construct $K_+(T)$:
1) Start with the Young diagram with same shape as T.
2) For every column length of T find a frank word whose first column has that length.
3) Split each of those columns.
4)Fill the Young diagram with the right column of each split.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \ \mathrm{wr}(T) = 2\,1\,\bar{2} \overset{Knuth}{\simeq} 2\bar{2}\,1$$

$$K_+(T) = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \bar{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$C_1 = \begin{array}{|c|}\hline 2 \\ \hline \end{array} \quad \begin{matrix} \emptyset & 2 \\ \emptyset & \emptyset \end{matrix} \ \text{hence } rC_1 = C_1$$

$$C_2 = \begin{array}{|c|}\hline 2 \\ \hline \bar{2} \\ \hline \end{array} \quad \begin{matrix} \emptyset & 2 \\ \emptyset & \bar{2} \end{matrix} \rightarrow \begin{matrix} \emptyset & 2 \\ \bar{1} & \emptyset \end{matrix} \ \text{hence } rC_2 = \begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline \end{array}$$

Left Keys

Consider a KN tableau T. To construct $K_-(T)$:
1) Start with the Young diagram with same shape as T.
2) For every column length of T find a frank word whose last column has that length.
3) Split each of those columns.
4)Fill the Young diagram with the left column of each split.

Example

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{2} & \\ \cline{1-1}\end{array}, \ \mathrm{wr}(T) = 2\,1\bar{2} \overset{Knuth}{\simeq} 2\bar{2}\,1 \qquad K_-(T) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \bar{2} & \\ \cline{1-1}\end{array}$$

Main Theorem

Consider a vector $v \in \mathbb{N}_0^n$.
Define $\mathfrak{U}(v) = \{T \in SSYT(\lambda, n) \mid K_+(T) = K(v)\}$.

Theorem (Lascoux-Schützenberger, 1988)

Consider $v \in \mathbb{N}_0^n$, λ a partition and $\sigma \in \mathfrak{S}_n$ minimal such that $v = \lambda\sigma$. Then

$$\begin{aligned}\hat{\mathfrak{B}}_{\lambda\sigma} &= \{T \mid K_+(T) = K(\lambda\sigma)\} = \mathfrak{U}(\lambda\sigma) \\ \mathfrak{B}_{\lambda\sigma} &= \{T \in \mathfrak{U}(\lambda\rho) \mid \rho \leq \sigma\} \\ &= \{T \mid K_+(T) \leq K(\lambda\sigma)\}\end{aligned}$$

Main Theorem

Consider a vector $v \in \mathbb{Z}^n$.
Define $\mathfrak{U}(v) = \{T \in \mathcal{KN}(\lambda, n) \mid K_+(T) = K(v)\}$.

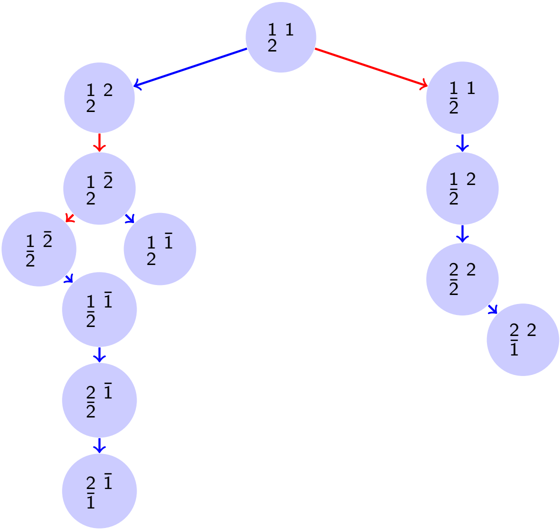
Theorem (JS, 2019)

Consider $v \in \mathbb{Z}^n$, λ a partition and $\sigma \in B_n$ minimal such that $v = \lambda\sigma$. Then

$$\hat{\mathfrak{B}}_{\lambda\sigma} = \{T \mid K_+(T) = K(\lambda\sigma)\} = \mathfrak{U}(\lambda\sigma)$$
$$\mathfrak{B}_{\lambda\sigma} = \{T \in \mathfrak{U}(\lambda\rho) \mid \rho \leq \sigma\}$$
$$= \{T \mid K_+(T) \leq K(\lambda\sigma)\}$$

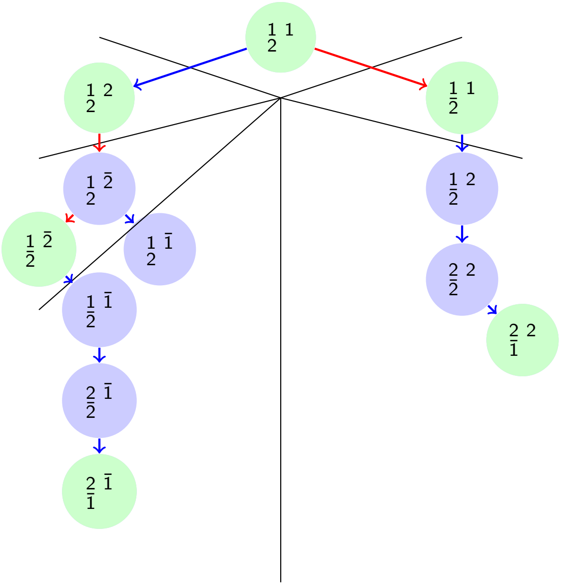
Demazure crystal - Atom Decomposition

Example $B_{\lambda s_1 s_2 s_1}$



Demazure crystal - Atom Decomposition

Example $B_{\lambda s_1 s_2 s_1}$



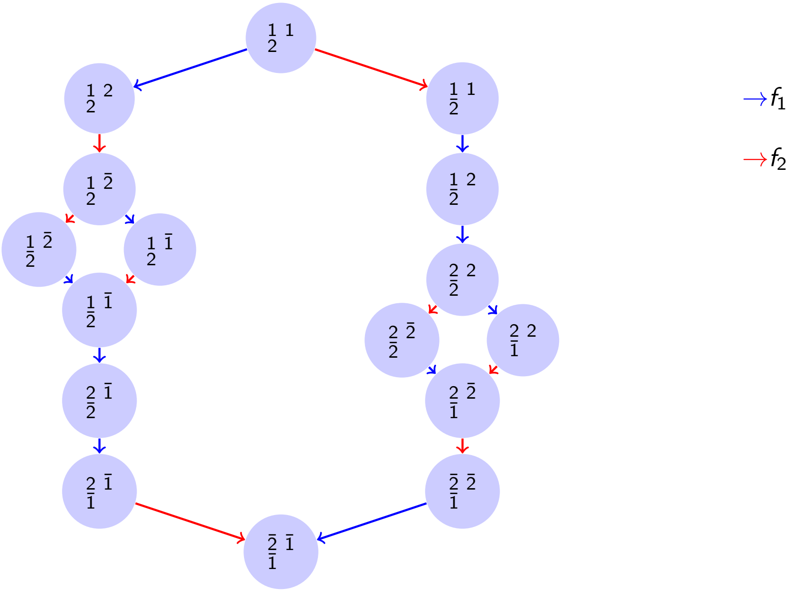
Lusztig/Schützenberger involution

Let $\mathfrak{B}^{\lambda}$ be the crystal of KN tableaux of shape λ.
$L : \mathfrak{B} \rightarrow \mathfrak{B}$ is the Lusztig involution if the following holds:

1. $wt(L(T)) = wt(x)\omega_0$
2. $e_i(L(T)) = L(f_i(T))$
3. $f_i(L(T)) = L(e_i(T))$

In type C, $v\omega_0 = -v$, $i \in [n]$.

Example of crystal: $\mathfrak{B}^{(2,1)}$ (Type C)



Lusztig involution and Keys

Proposition

$L(K_+(T)) = K_-(L(T))$

Example

Consider $T = \begin{array}{|c|c|}\hline 2 & \overline{1} \\ \hline \overline{2} & \\ \cline{1-1} \end{array}$.

Then $L(K_+(T)) = L\left(\begin{array}{|c|c|}\hline 2 & \overline{1} \\ \hline \overline{1} & \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \overline{2} & \\ \cline{1-1} \end{array}$.

$K_-(L(T)) = K_-\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \overline{2} & \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \overline{2} & \\ \cline{1-1} \end{array}$.

Thank you!