SLC83

Non-reduced reflection factorizations of Coxeter elements

(joint work with S. Yahiatene)

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SLC83 A group W with presentation

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where

- $m_{ii} = 1$
- $m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for $i \neq j$
  ($m_{ij} = \infty$ indicates that there is no relation between s_i and s_j)

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finite Coxeter groups $\overset{1:1}{\longleftrightarrow}$ finite real reflection groups

$(A_n, B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m))$

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Associated to the generating sets R and S we have length functions

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A tuple $(r_1, \ldots, r_m) \in R^m$ is called a **reflection factorization** for $w \in W$ if $w = r_1 \cdots r_m$ and it is called a **reduced** reflection factorization if $m = \ell_R(w)$. We denote the set of all reduced reflection factorizations for w by $\mathrm{Red}_R(w)$.

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$$\sigma_i(r_1, \ldots, r_i, r_{i+1}, \ldots, r_m) = (r_1, \ldots, r_i r_{i+1} r_i, r_i, \ldots, r_m),$$
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Remark. Let $(r_1, \ldots, r_m) \sim (t_1, \ldots, t_m)$.

- $\langle r_1, \ldots, r_m \rangle = \langle t_1, \ldots, t_m \rangle$,
- $\{\{[r_1], \ldots, [r_m]\}\} = \{\{[t_1], \ldots, [t_m]\}\}$ (multiset of conj. classes).

Let $(W, \{s_1, \ldots, s_n\})$ be a Coxeter system. Then the element

$$c = s_{\pi(1)} \cdots s_{\pi(n)}$$

is called a **Coxeter element** for every $\pi \in \mathrm{Sym}(n)$.

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Lemma (Lewis–Reiner 2016). Let (W, S) be a finite Coxeter system with set of reflections R and let $w \in W$ with $\ell_R(w) = m$. If $w = t_1 \cdots t_{m+2k}$ with $t_i \in R$ and $k \in \mathbb{Z}_{\geq 0}$. Then

$$(t_1, \ldots, t_{m+2k}) \sim (\underbrace{r_1, \ldots, r_m}_{\in \mathrm{Red}_R(w)}, r_{i_1}, r_{i_1}, \ldots, r_{i_k}, r_{i_k})$$

for some $r_i \in R$.

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Theorem (Lewis–Reiner 2016). In a finite Coxeter group, two reflection factorizations of a Coxeter element lie in the same orbit under the Hurwitz action if and only if they share the same multiset of conjugacy classes.

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That is, if $c = r_1 \cdots r_m = t_1 \cdots t_m$, then

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The two main ingredients of the proof are the previous Lemma of Lewis and Reiner as well as the following result:

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Theorem (Deligne 1974, Igusa–Schiffler 2010). In an arbitrary Coxeter group, the Hurwitz action is transitive on the set of reduced reflection factorizations of a Coxeter element.

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Question: Does the Theorem of Lewis and Reiner hold for arbitrary Coxeter groups?

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For a Coxeter element $c \in W$ we have $\ell_S(c) = \ell_R(c)$. Therefore we are able to prove...

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Corollary. Let (W, S) be a Coxeter system such that all elements of S are conjugated (for example, if the Coxeter graph is connected and has a spanning tree with odd labels on all of its edges), then the Hurwitz action is transitive on equal length reflection factorizations of a Coxeter element.

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Thank you!

Outlook: Complex reflection groups

- G_4, G_5 (Z. Peterson)
- the Shephard groups $G(p, 1, n)$ (Lewis, Yahiatene)
- computational evidence for G_8 and G_{20}.