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REFLECTING (ON) THE MODULO 9 KANADE–RUSSELL (CONJECTURAL) IDENTITIES

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To Doron Zeilberger, with Experimental Mathematics wishes, on his twentieth prime birthday

Abstract. We examine complexity and versatility of five modulo 9 Kanade–Russell identities through their finite (*aka* polynomial) versions and images under the $q \mapsto 1/q$ reflection.

1. Introduction

Every second paper about partition identities features the Rogers–Ramanujan identities as toy examples. Let us follow this tradition and consider one of those burdened with fame as the limiting case of the *finite* version

$$\sum_{n\ge0} q^{n^2}\begin{bmatrix}N\\n\end{bmatrix} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k+1)/2}\begin{bmatrix}2N\\N+2k\end{bmatrix} \tag{1}$$

due to Bressoud [8]. Here and in what follows we make use of standard q-hypergeometric notation:

$$\begin{bmatrix}N\\n\end{bmatrix} = \begin{bmatrix}N\\n\end{bmatrix}_q = \begin{cases}\dfrac{(q;q)_N}{(q;q)_n(q;q)_{N-n}}, & \text{for } n=0,1,\dots,N,\\ 0, & \text{otherwise,}\end{cases}$$

denotes a q-binomial coefficient,

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

is a q-shifted factorial (q-Pochhammer symbol), also meaningful for $n=\infty$, and

$$(a_1,a_2,\dots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n.$$

When $|q|<1$, the limit as $N\to\infty$ in (1) translates the equality of two polynomials into

$$\sum_{n\ge0}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k+1)/2} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{2}$$

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where Jacobi's triple product identity was applied to the sum on the right-hand side. On the other hand, we can *reflect* the identity (1) to obtain a different one by applying the involution $q \mapsto 1/q$. Since

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_{q^{-1}} = q^{-nm} \begin{bmatrix} n+m \\ m \end{bmatrix}_q,$$

we find that

$$q^{-N^2} \sum_{n\ge0} q^{nN} \begin{bmatrix} N \\ n \end{bmatrix} = q^{-N^2} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(3k-1)/2} \begin{bmatrix} 2N \\ N+2k \end{bmatrix}$$

(in the sum on the left-hand side we changed the summation index $n \mapsto N-n$). Apparently, this identity is not as interesting as the original (1): the polynomial obtained from the multiplication of either side by q^{N^2} tends to (boring) 1 as $N \to \infty$.

Let us take a look at a contemporary and more advanced (in particular, conjectural!) example that arises from the inspiring work [11] of Kanade and Russell. Denote by $s_N(n)$ the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n such that:

(a) the largest part of λ is at most N;
(b) λ has no parts of size 1;
(c) the difference of parts at distance 2 is at least 3;
(d) if consecutive parts differ by at most 1 then their sum is congruent to 2 (mod 3).

Then the generating function $S_N(q) = \sum_{n\ge0} s_N(n)q^n$ is (clearly) a polynomial for each $N \ge 0$, and the polynomials $S_N(q)$ satisfy the recursion

$$S_N(q) = S_{N-1}(q) + q^N \times \begin{cases} (1+q^{N-1})S_{N-3}(q) + q^{N-2}S_{N-4}(q), & \text{if } N \equiv 0 \pmod 3, \\ S_{N-2}(q) + q^N S_{N-3}(q), & \text{if } N \equiv 1 \pmod 3, \\ S_{N-2}(q), & \text{if } N \equiv 2 \pmod 3, \end{cases}$$

for $N = 1, 2, \dots$, with initial conditions $S_0(q) = 1$ and $S_N(q) = 0$ for $N < 0$. The generating function $S_\infty(q) = \lim_{N\to\infty} S_N(q)$ counts partitions with the condition on the largest part dropped, that is, satisfying hypotheses (b)–(d) above, and the conjecture from [11] predicts that

$$S_\infty(q) \overset{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3;q^9)_\infty} = \frac{1}{(q^2,q^3,q^5,q^8;q^9)_\infty}. \tag{3}$$

As we will see later, the series $S_\infty(q)$ can be written as a double-sum Rogers–Ramanujan type identity — this was found by Kurşungöz [14]. The conjecture in (3) has a surprisingly different (and supposedly more difficult) counterpart, which was observed by Warnaar [21]:

$$\begin{aligned}\lim_{M\to\infty} q^{M(3M+2)}S_{3M}(q^{-1}) &\stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty(q^3,q^9,q^{12},q^{21},q^{30},q^{36},q^{39};q^{45})_\infty},\\ \lim_{M\to\infty} q^{M(3M+5)}S_{3M+1}(q^{-1}) &\stackrel{?}{=} \frac{1}{(q^2;q^3)_\infty(q^3,q^{12},q^{18},q^{21},q^{27},q^{30},q^{39};q^{45})_\infty},\\ \lim_{M\to\infty} q^{(M+1)(3M+2)}S_{3M+2}(q^{-1}) &\stackrel{?}{=} \lim_{M\to\infty} q^{M(3M+2)}S_{3M}(q^{-1})\\ &\qquad + q^2\lim_{M\to\infty} q^{M(3M+5)}S_{3M+1}(q^{-1}). \qquad (4)\end{aligned}$$

One goal of this work is to analyze this $q\mapsto 1/q$ phenomenon from a more general perspective, in particular, to provide other modulo 45 product sides of all other (five in total) Kanade–Russell modulo 9 partition counting functions. On the way, we give explicit finite versions of the functions (for example, of $S_N(q)$ above) and explicit (Rogers–Ramanujan type) sums for the reflected identities. One pleasing outcome of this routine is a *proof* of (4) as well as other similar cases.

Doron Zeilberger's combined interests in combinatorics of generalized Rogers–Ramanujan identities [9], algorithmic aspects of q-identities and Experimental Mathematics have served as a fruitful motivation for many, junior and senior, to produce beautiful research pieces. One notable recent illustration is the discovery of novel identities for classical partitions by Doron's mathematical descendant, Matthew Russell, and the latter's then Rutgers graduate mate, Shashank Kanade [11, 12, 17]. We are happy to dedicate this note to Doron.

2. Capparelli's finite sums and their reflections

Before immersing into the topic of the Kanade–Russell identities, we first examine a similar but more familiar ground. We return to the theme of the Rogers–Ramanujan identity (2) but now from the perspective of another finite version,

$$\sum_{n\ge0} q^{n^2}\begin{bmatrix}N-n\\ n\end{bmatrix} = \sum_{k\in\mathbb{Z}}(-1)^k q^{k(5k+1)/2}\begin{bmatrix}N\\ \lfloor (N+5k+1)/2\rfloor\end{bmatrix},$$

found by Schur and explicitly stated by Andrews [2] (see also [18, Identity 3.18]). Substituting $1/q$ for q and multiplying the result by $q^{\lfloor (N/2)^2\rfloor}$, we obtain, depending on whether $N=2M$ or $N=2M+1$,

$$\begin{aligned}\sum_{n\ge0} q^{n^2}\begin{bmatrix}M+n\\ 2n\end{bmatrix} &= \sum_{k\in\mathbb{Z}}(-1)^k q^{\lfloor (5k+1)/2\rfloor^2-k(5k+1)/2}\begin{bmatrix}2M\\ M+\lfloor (5k+1)/2\rfloor\end{bmatrix},\\ \sum_{n\ge0} q^{n^2+n}\begin{bmatrix}M+n+1\\ 2n+1\end{bmatrix} &= \sum_{k\in\mathbb{Z}}(-1)^k q^{\lfloor 5k/2\rfloor^2+\lfloor 5k/2\rfloor-k(5k+1)/2}\begin{bmatrix}2M+1\\ M+\lfloor 5k/2\rfloor+1\end{bmatrix}.\end{aligned}$$

(We made the change $n\mapsto M-n$ in the sums on the left-hand sides.) These two (different!) identities are due to Andrews [3, 4] and listed in [18, Identities 3.79

and 3.94]; their limiting cases as $M\to\infty$,

$$\sum_{n\ge0}\frac{q^{n^2}}{(q;q)_{2n}}=\frac{1}{(q;q^2)_\infty(q^4,q^{16};q^{20})_\infty},$$
$$\sum_{n\ge0}\frac{q^{n^2+n}}{(q;q)_{2n+1}}=\frac{1}{(q,q^2,q^8,q^9;q^{10})_\infty(q^5,q^6,q^{14},q^{15};q^{20})_\infty},$$

appear in Rogers' famous "second memoir" [16].

Secondly, it is natural to draw parallels of the story below with some finite sums from [5, 6] underlying Capparelli's identities [10]. One of this, corresponding to the analytic counterpart [12, eq. (7.11)], [13, Theorem 8] of the first Capparelli identity

$$\sum_{m,n\ge0}\frac{q^{2(m^2+3mn+3n^2)}}{(q;q)_m(q^3;q^3)_n}=(-q^2,-q^4;q^6)_\infty(-q^3;q^3)_\infty, \tag{5}$$

is given in [6, Theorem 4.3]:

$$\sum_{m,n\ge0}q^{2(m^2+3mn+3n^2)}\begin{bmatrix}3N-3m-6n\\m\end{bmatrix}_q\begin{bmatrix}2N-2m-3n\\n\end{bmatrix}_{q^3}$$
$$=\sum_{l\ge0}q^{3(N-2l)(N-2l-1)/2}\begin{bmatrix}N\\2l\end{bmatrix}_{q^3}(-q^2,-q^4;q^6)_l.$$

By considering separately the cases $N=2M$ and $N=2M-1$ and performing the reflection $q\mapsto1/q$, we arrive at

$$q^{-6M^2}\sum_{a,b\ge0}q^{2(a^2-3ab+3b^2)}\begin{bmatrix}3b\\2a\end{bmatrix}_q\begin{bmatrix}M+a\\2b\end{bmatrix}_{q^3}=q^{-6M^2}\sum_{c\ge0}q^{3c}\begin{bmatrix}2M\\2c\end{bmatrix}_{q^3}(-q^2,-q^4;q^6)_{M-c},$$
$$q^{-6M(M+1)}\sum_{a,b\ge0}q^{2(a^2-3ab+3b^2)+2a-3b-1}\begin{bmatrix}3b\\2a+1\end{bmatrix}_q\begin{bmatrix}M+a\\2b\end{bmatrix}_{q^3}$$
$$=q^{-6M(M+1)}\sum_{c\ge0}q^{3c}\begin{bmatrix}2M\\2c+1\end{bmatrix}_{q^3}(-q^2,-q^4;q^6)_{M-c},$$

respectively, and we find that

$$\sum_{a,b\geq 0}\frac{q^{2(a^2-3ab+3b^2)}}{(q^3;q^3)_{2b}}\begin{bmatrix}3b\\2a\end{bmatrix}=(-q^2,-q^4;q^6)_\infty\sum_{c\geq 0}\frac{q^{3c}}{(q^3;q^3)_{2c}}$$
$$=\frac{1}{(q^2,q^{10};q^{12})_\infty(q^3,q^6,q^9,q^9,q^{12},q^{15},q^{15},q^{21},q^{27},q^{33},q^{33},q^{36},q^{39},q^{39},q^{42},q^{45};q^{48})_\infty},$$
$$\sum_{a,b\geq 0}\frac{q^{2(a^2-3ab+3b^2)+2a-3b-1}}{(q^3;q^3)_{2b}}\begin{bmatrix}3b\\2a+1\end{bmatrix}=(-q^2,-q^4;q^6)_\infty\sum_{c\geq 0}\frac{q^{3c}}{(q^3;q^3)_{2c+1}}$$
$$=\frac{1}{(q^2,q^{10};q^{12})_\infty}$$
$$\times\frac{1}{(q^3,q^3,q^9,q^{12},q^{15},q^{18},q^{21},q^{21},q^{27},q^{27},q^{30},q^{33},q^{36},q^{39},q^{45},q^{45};q^{48})_\infty},$$

where the right-hand sides were classically summed.

The same identity (5) admits a different finite version [5, Theorem 7.1]:

$$\sum_{m,n\geq 0}\frac{q^{2(m^2+3mn+3n^2)}(q^3;q^3)_N}{(q;q)_m(q^3;q^3)_n(q^3;q^3)_{N-m-2n}}=\sum_{l=-N}^{N}q^{l(3l+1)}\begin{bmatrix}2N\\N+l\end{bmatrix}_{q^3}.$$

Its $q\mapsto 1/q$ reflection after multiplication of both sides by $q^{N(3N+1)}$ reads

$$\sum_{k,n\geq 0}\frac{(-1)^n q^{n(3n+1)/2+k(3N+1)}(q^3;q^3)_N}{(q;q)_{N-k-2n}(q^3;q^3)_k(q^3;q^3)_n}=\sum_{l=-N}^{N}q^{N-l}\begin{bmatrix}2N\\N+l\end{bmatrix}_{q^3},$$

and the limit as $N\to\infty$ is uninspiringly equal to $1/(q;q^3)_\infty$.

3. Finite versions of the Kanade–Russell–Kurşungöz style double series

The five modulo 9 conjectures about partition generating functions were originally displayed in [11, 17] through difference equations; one of these four is already given in the introduction. Their double-sum Rogers–Ramanujan type versions read

$$\mathrm{KR}_1(q)=\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2}}{(q;q)_m(q^3;q^3)_n},\tag{6}$$

$$\mathrm{KR}_2(q)=\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2+m+3n}}{(q;q)_m(q^3;q^3)_n},\tag{7}$$

$$\mathrm{KR}_3(q)=\sum_{m,n\geq 0}\frac{q^{m^2+3mn+3n^2+2m+3n}}{(q;q)_m(q^3;q^3)_n},\tag{8}$$

$$\mathrm{KR}_4(q) = \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+m+2n}}{(q;q)_m(q^3;q^3)_n}, \tag{9}$$

$$\mathrm{KR}_5(q) = \sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+2m+4n}(1+q+q^{m+3n+2})}{(q;q)_m(q^3;q^3)_n}. \tag{10}$$

Here, entries (6)–(9) were found by Kurşungöz [14], and they correspond to the I_1–I_4 instances of Kanade and Russell [11], respectively. The expression (10) corresponds to the later found asymmetric mod 9 conjecture; its combinatorial version is represented in Russell's thesis [17], and the analytic sum is constructed by us using Kurşungöz's technique. The product sides,

$$\mathrm{KR}_1(q) \overset{?}{=} \frac{1}{(q,q^3,q^6,q^8;q^9)_\infty}, \tag{11}$$

$$\mathrm{KR}_2(q) \overset{?}{=} \frac{1}{(q^2,q^3,q^6,q^7;q^9)_\infty}, \tag{12}$$

$$\mathrm{KR}_3(q) \overset{?}{=} \frac{1}{(q^3,q^4,q^5,q^6;q^9)_\infty}, \tag{13}$$

$$\mathrm{KR}_4(q) \overset{?}{=} \frac{1}{(q^2,q^3,q^5,q^8;q^9)_\infty}, \tag{14}$$

$$\mathrm{KR}_5(q) \overset{?}{=} \frac{1}{(q,q^4,q^6,q^7;q^9)_\infty}, \tag{15}$$

are precisely the conjectures of Kanade and Russell [11, 17]. Notice that (14) is an equivalent form of identity (3). We call the products on the right-hand sides of (11)-(13) *symmetric* as, for any residue class i mod 9, the residue class $-i$ mod 9 is also present in the product. The products on the right-hand sides of (14) and (15) will then be deemed *asymmetric*.

Following the footsteps of the recent articles [5, 19, 20] of the first author, some in collaboration with Berkovich, and using the combinatorial interpretation of the sum sides from [11, 17], we can write finite generating functions as follows:

$$\mathrm{KR}_1(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2} \begin{bmatrix} N-m-3n+1 \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3}, \tag{16}$$

$$\mathrm{KR}_2(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2+m+3n} \begin{bmatrix} N-m-3n \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n \\ n \end{bmatrix}_{q^3}, \tag{17}$$

$$\mathrm{KR}_3(q,N) = \sum_{m,n\geq 0} q^{m^2+3mn+3n^2+2m+3n} \begin{bmatrix} N-m-3n-1 \\ m \end{bmatrix}_q^* \begin{bmatrix} \lfloor \frac{2}{3}N \rfloor - m - n \\ n \end{bmatrix}_{q^3}, \tag{18}$$

$$\mathrm{KR}_4(q,N) = \sum_{m,n\ge 0} q^{m^2+3mn+3n^2+m+2n} \begin{bmatrix} N-m-3n \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}(N-1)\rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3}, \tag{19}$$

$$\begin{aligned}\mathrm{KR}_5(q,N) = &\sum_{m,n\ge 0} q^{m^2+3mn+3n^2+2m+4n}(1+q) \\ &\quad\cdot \begin{bmatrix} N-m-3n-1 \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}(N-2)\rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3} \\ &+ \sum_{m,n\ge 0} q^{m^2+3mn+3n^2+3m+7n+2} \\ &\quad\cdot \begin{bmatrix} N-m-3n-2 \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor \frac{2}{3}(N-2)\rfloor - m - n + \delta_{3|(N-2)} \\ n \end{bmatrix}_{q^3},\end{aligned} \tag{20}$$

where the asterisk in (18) means that the q-binomial $\left[\begin{smallmatrix} N-m-3n-1 \\ m \end{smallmatrix}\right]_q$ is understood as 1 when it becomes $\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]_q$ (in other words, when $m = N-m-3n = 0$, which may only occur when $N \equiv 0 \pmod 3$); furthermore $\delta_{a|b}$ used in (20) stands for 1 if $a \mid b$ and for 0 otherwise. Then clearly the limits as $N \to \infty$ of (16)–(20) become (6)–(10); it is also routine to verify that the polynomial (finite) versions of the latter double sums indeed satisfy the recurrence equations given in [11, 17]. Since we find the technique of converting infinite sums into finite versions important, we select it for preservation in Appendix A below.

In all the expressions (6)–(10) the exponent of q keeps track of the size of the counted partitions. In fact, the technique developed in [5,14,19,20] allows us to write the generating functions with an additional statistics by multiplying the (m,n)-th term in the sum by x^{2n+m} in (16)–(19) and taking, for instance,

$$\begin{aligned}\mathrm{KR}_5(q,N;x) = &\sum_{m,n\ge 0} q^{m^2+3mn+3n^2+2m+4n}(1+xq) \begin{bmatrix} N-m-3n-1 \\ m \end{bmatrix}_q \\ &\quad\cdot \begin{bmatrix} \lfloor \frac{2}{3}(N-2)\rfloor - m - n + 1 \\ n \end{bmatrix}_{q^3} x^{m+2n} \\ &+ \sum_{m,n\ge 0} q^{m^2+3mn+3n^2+3m+7n+2} \begin{bmatrix} N-m-3n-2 \\ m \end{bmatrix}_q \\ &\quad\cdot \begin{bmatrix} \lfloor \frac{2}{3}(N-2)\rfloor - m - n + \delta_{3|(N-2)} \\ n \end{bmatrix}_{q^3} x^{m+2n+1}.\end{aligned}$$

Then the exponent of x in these general generated functions keeps track of the number of parts in the counted partitions. We leave this as a side remark but keep a hope that the more involved series can be used for approaching conjectures (11)–(15).

4. Modulo 45 reflections

In our language here, Warnaar's conjectures from the introduction can be stated as follows. *Define*

$$\begin{aligned}\reflectbox{KR}_4(q,3M)&=q^{M(3M+2)}\,\mathrm{KR}_4(1/q,3M),\\ \reflectbox{KR}_4(q,3M+1)&=q^{M(3M+5)}\,\mathrm{KR}_4(1/q,3M+1),\\ \reflectbox{KR}_4(q,3M+2)&=q^{(M+1)(3M+2)}\,\mathrm{KR}_4(1/q,3M+2).\end{aligned}$$

Then

$$\begin{aligned}\reflectbox{KR}_4(q,3\infty)&=\lim_{M\to\infty}\reflectbox{KR}_4(q,3M)\\ &\overset{?}{=}\frac{1}{(q^2;q^3)_\infty(q^3,q^9,q^{12},q^{21},q^{30},q^{36},q^{39};q^{45})_\infty},\\ \reflectbox{KR}_4(q,3\infty+1)&=\lim_{M\to\infty}\reflectbox{KR}_4(q,3M+1)\\ &\overset{?}{=}\frac{1}{(q^2;q^3)_\infty(q^3,q^{12},q^{18},q^{21},q^{27},q^{30},q^{39};q^{45})_\infty},\\ \reflectbox{KR}_4(q,3\infty+2)&=\lim_{M\to\infty}\reflectbox{KR}_4(q,3M+2)\\ &\overset{?}{=}\reflectbox{KR}_4(q,3\infty)+q^2\,\reflectbox{KR}_4(q,3\infty+1).\end{aligned}$$

To convert the limits into infinite sums we use the finite sum representation (19). We deduce that

$$\begin{aligned}&\reflectbox{KR}_4(q,3M)=q^{M(3M+2)}\,\mathrm{KR}_4(1/q,3M)\\ &\quad=q^{3M^2+2M}\sum_{m,n\ge0}q^{m^2+3mn+3n^2-(2n+m)(3M+1)}\begin{bmatrix}3M-3n-m\\ m\end{bmatrix}_q\begin{bmatrix}2M-n-m\\ n\end{bmatrix}_{q^3}.\end{aligned}$$

By a change of the summation to a summation over $a=3M-2m-3n$, $b=2M-m-2n$ (equivalently, $m=3b-2a$, $n=M+a-2b$), the sum turns into

$$\reflectbox{KR}_4(q,3M)=\sum_{a,b\ge0}q^{a^2-3ab+3b^2+b}\begin{bmatrix}3b-a\\ a\end{bmatrix}_q\begin{bmatrix}M+a-b\\ b\end{bmatrix}_{q^3},$$

leading to

$$\begin{aligned}\reflectbox{KR}_4(q,3\infty)&=\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix}\\ &\overset{?}{=}\frac{1}{(q^2;q^3)_\infty(q^3,q^9,q^{12},q^{21},q^{30},q^{36},q^{39};q^{45})_\infty}.\end{aligned}\tag{21}$$

Similarly, we have

$$\text{ꓘЯ}_4(q,3M+1) = \sum_{a,b\geq 0} q^{a^2-3ab+3b^2+b-2} \begin{bmatrix} 3b-a-1 \\ a \end{bmatrix}_q \begin{bmatrix} M+a-b+1 \\ b \end{bmatrix}_{q^3},$$
$$\text{ꓘЯ}_4(q,3M+2) = \sum_{a,b\geq 0} q^{a^2-3ab+3b^2+b} \begin{bmatrix} 3b-a+1 \\ a \end{bmatrix}_q \begin{bmatrix} M+a-b \\ b \end{bmatrix}_{q^3},$$

hence

$$\begin{aligned} \text{ꓘЯ}_4(q,3\infty+1) &= \sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b-2}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a-1 \\ a \end{bmatrix} \\ &\overset{?}{=} \frac{1}{(q^2;q^3)_\infty (q^3,q^{12},q^{18},q^{21},q^{27},q^{30},q^{39};q^{45})_\infty}, \qquad (22) \\ \text{ꓘЯ}_4(q,3\infty+2) &= \sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b}}{(q^3;q^3)_b} \begin{bmatrix} 3b-a+1 \\ a \end{bmatrix} \\ &= \text{ꓘЯ}_4(q,3\infty) + q^2\, \text{ꓘЯ}_4(q,3\infty+1). \qquad (23) \end{aligned}$$

Here the expression (23) follows from

$$\sum_{a,b\geq 0} \frac{q^{a^2-3ab+3b^2+b}}{(q^3;q^3)_b} \left(\begin{bmatrix} 3b-a-1 \\ a \end{bmatrix} + \begin{bmatrix} 3b-a \\ a \end{bmatrix} - \begin{bmatrix} 3b-a+1 \\ a \end{bmatrix} \right) = 0,$$

which is a consequence of the simple single-sum evaluation

$$\sum_{a=0}^{L+1} q^{a(a-L)} \left(\begin{bmatrix} L-a-1 \\ a \end{bmatrix} + \begin{bmatrix} L-a \\ a \end{bmatrix} - \begin{bmatrix} L-a+1 \\ a \end{bmatrix} \right) = 0, \quad \text{for } L=0,1,2,\dots. \qquad (24)$$

Quite remarkably, a similar structure is inherited by the other ‘asymmetric’ conjecture (15). *Define*

$$\begin{aligned} \text{ꓘЯ}_5(q,3M) &= q^{M(3M+1)}\, \mathrm{KR}_5(1/q,3M), \\ \text{ꓘЯ}_5(q,3M+1) &= q^{(M+1)(3M+1)}\, \mathrm{KR}_5(1/q,3M+1), \\ \text{ꓘЯ}_5(q,3M+2) &= q^{(M+2)(3M+1)}\, \mathrm{KR}_5(1/q,3M+2). \end{aligned}$$

Then

$$\lim_{M\to\infty} \text{ꓘЯ}_5(q,3M+1) \overset{?}{=} \frac{1}{(q;q^3)_\infty (q^6,q^9,q^{15},q^{24},q^{33},q^{36},q^{42};q^{45})_\infty},$$
$$\lim_{M\to\infty} \text{ꓘЯ}_5(q,3M+2) \overset{?}{=} \frac{1}{(q;q^3)_\infty (q^6,q^{15},q^{18},q^{24},q^{27},q^{33},q^{42};q^{45})_\infty},$$
$$\lim_{M\to\infty} \text{ꓘЯ}_5(q,3M) \overset{?}{=} \lim_{M\to\infty} \text{ꓘЯ}_5(q,3M+1) + q^2 \lim_{M\to\infty} \text{ꓘЯ}_5(q,3M+2).$$

Our analysis as before translates the expectations into

$$\begin{aligned}
\reflectbox{KR}_5(q,3\infty+1) &= \sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+2b}(1+q)}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix} \\
&\qquad+\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+5b+2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+1\\ a\end{bmatrix} \\
&\overset{?}{=}\frac{1}{(q;q^3)_\infty(q^6,q^9,q^{15},q^{24},q^{33},q^{36},q^{42};q^{45})_\infty}, \qquad (25) \\
\reflectbox{KR}_5(q,3\infty+2) &= \sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+2b-2}(1+q)}{(q^3;q^3)_b}\begin{bmatrix}3b-a-1\\ a\end{bmatrix} \\
&\qquad+\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+5b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix} \\
&\overset{?}{=}\frac{1}{(q;q^3)_\infty(q^6,q^{15},q^{18},q^{24},q^{27},q^{33},q^{42};q^{45})_\infty}, \qquad (26) \\
\reflectbox{KR}_5(q,3\infty) &= \sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+2b}(1+q)}{(q^3;q^3)_b}\begin{bmatrix}3b-a+1\\ a\end{bmatrix} \\
&\qquad+\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+5b+2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+2\\ a\end{bmatrix} \\
&= \reflectbox{KR}_5(q,3\infty+1)+q^2\,\reflectbox{KR}_5(q,3\infty+2). \qquad (27)
\end{aligned}$$

Again, the equality in (27) follows from (24).

In spite of the suggested simplicity of the reflections in the asymmetric cases, our investigation of the symmetric ones (11)–(13) brings to life somewhat more sophisticated expectations:

$$\begin{aligned}
&\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-1}}{(q^3;q^3)_b}\begin{bmatrix}3b-a-1\\ a\end{bmatrix}_q = \lim_{M\to\infty} q^{3M(M+1)}\,\mathrm{KR}_1(q^{-1},3M) = \reflectbox{KR}_1(q,3\infty) \\
&\qquad\overset{?}{=}\langle 2,8,11,20\rangle + q^3\langle 2,14,20,22\rangle - q^8\langle 17,19,20,22\rangle \\
&\qquad= \langle 1,8,13,20\rangle - q\langle 4,7,13,20\rangle + q^5\langle 7,16,17,20\rangle, \qquad (28) \\
&\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix}_q = \lim_{M\to\infty} q^{3(M+1)^2}\,\mathrm{KR}_1(q^{-1},3M+2) = \reflectbox{KR}_1(q,3\infty+2) \\
&\qquad\overset{?}{=}\langle 1,7,11,20\rangle + q^6\langle 11,13,14,20\rangle - q^6\langle 8,14,19,20\rangle \\
&\qquad= \langle 1,4,17,20\rangle - q^4\langle 2,16,19,20\rangle - q^5\langle 4,16,20,22\rangle, \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+3b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\a\end{bmatrix}_q &= \lim_{M\to\infty} q^{3M(M+1)}\,\mathrm{KR}_2(q^{-1},3M) = \text{ꓘꓤ}_2(q,3\infty)\\
&\overset{?}{=} \langle 2,5,14,22\rangle - q^2\langle 5,7,16,17\rangle - q^5\langle 5,17,19,22\rangle\\
&= q^{-3}\langle 1,5,8,13\rangle - q^{-3}\langle 2,5,8,11\rangle - q^{-2}\langle 4,5,7,13\rangle, \qquad (30)
\end{aligned}$$

$$\begin{aligned}
\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+3b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+1\\a\end{bmatrix}_q&\\
&= \lim_{M\to\infty} q^{3(M+1)^2-1}\,\mathrm{KR}_2(q^{-1},3M+2) = \text{ꓘꓤ}_2(q,3\infty+2)\\
&\overset{?}{=} \langle 2,5,16,19\rangle - q^2\langle 5,8,14,19\rangle + q^2\langle 5,11,13,14\rangle\\
&= q^{-4}\langle 1,4,5,17\rangle - q^{-4}\langle 1,5,7,11\rangle - q\langle 4,5,16,22\rangle, \qquad (31)
\end{aligned}$$

$$\begin{aligned}
1+\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+a}}{(q^3;q^3)_b}\begin{bmatrix}3b-a-2\\a\end{bmatrix}_q&\\
&= \lim_{M\to\infty} q^{3M(M+1)}\,\mathrm{KR}_3(q^{-1},3M+1) = \text{ꓘꓤ}_3(q,3\infty)\\
&\overset{?}{=} \langle 4,7,10,13\rangle - q^4\langle 7,10,16,17\rangle - q^7\langle 10,17,19,22\rangle\\
&= q^{-1}\langle 1,8,10,13\rangle - q^{-1}\langle 2,8,10,11\rangle - q^2\langle 2,10,14,22\rangle, \qquad (32)
\end{aligned}$$

$$\begin{aligned}
\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+a-2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a-1\\a\end{bmatrix}_q&\\
&= \lim_{M\to\infty} q^{3N(M+2)}\,\mathrm{KR}_3(q^{-1},3M+2) = \text{ꓘꓤ}_3(q,3\infty+2)\\
&\overset{?}{=} \langle 2,10,16,19\rangle + q\langle 4,10,16,22\rangle - q^2\langle 8,10,14,19\rangle\\
&= q^{-4}\langle 1,4,10,17\rangle - q^{-4}\langle 1,7,10,11\rangle - q^2\langle 10,11,13,14\rangle, \qquad (33)
\end{aligned}$$

where the summands are (modular) products

$$\langle c_1,c_2,c_3,c_4\rangle = \frac{(q^{45};q^{45})_\infty}{(q^3;q^3)_\infty\prod_{j=1}^4(q^{c_j},q^{45-c_j};q^{45})_\infty}. \qquad (34)$$

One can notice that the conjectural right-hand sides in all the above instances (28)–(33) are represented by two equal sums of three products; the equality of the two in each case is not difficult to verify, because they belong to a finite-dimensional space over $\mathbb{Q}$ — this is dictated by the modularity of these symmetric products (after multiplication of them by the appropriate powers $q^{N(c_1,c_2,c_3,c_4)}$). Nevertheless, it is quite remarkable that in each of the six examples sums of three products record the shortest representatives within the space and that there are exactly two such representations. Furthermore, identity (24) and its modifications also give us (provably!)

$$\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+1\\ a\end{bmatrix}_q=\lim_{M\to\infty}q^{3M(M+1)+1}\,\mathrm{KR}_1(q^{-1},3M+1)$$
$$=\reflectbox{KR}_1(q,3\infty+1)=q\,\reflectbox{KR}_1(q,3\infty)+\reflectbox{KR}_1(q,3\infty+2), \quad (35)$$

$$\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2-a+3b}}{(q^3;q^3)_b}\begin{bmatrix}3b-a+2\\ a\end{bmatrix}_q=\lim_{M\to\infty}q^{3M(M+1)}\,\mathrm{KR}_2(q^{-1},3M+1)$$
$$=\reflectbox{KR}_2(q,3\infty+1)=\reflectbox{KR}_2(q,3\infty)+\reflectbox{KR}_2(q,3\infty+2), \quad (36)$$

$$\sum_{a,b\ge0}\frac{q^{a^2-3ab+3b^2+a}}{(q^3;q^3)_b}\begin{bmatrix}3b-a\\ a\end{bmatrix}_q=\lim_{M\to\infty}q^{3M(M+1)}\,\mathrm{KR}_3(q^{-1},3M+1)$$
$$=\reflectbox{KR}_3(q,3\infty+1)=\reflectbox{KR}_2(q,3\infty)+q^2\,\reflectbox{KR}_2(q,3\infty+2). \quad (37)$$

One interesting outcome of these conjectures is the positivity of the q-expansions of the linear combinations of products that appear on the right-hand sides for $\reflectbox{KR}_i(q,3\infty+l)$ for $i=1,2,3$ and $l=0,1,2$. In fact, we have checked numerically that the combinations remain positive after each of them is multiplied by $(q^3;q^3)_\infty/(q^{45};q^{45})_\infty$, but for that we have no explanation.

The complexity of the reflections serves as a good reason for lack of simple single-sum evaluations for the original sums $\mathrm{KR}_i(q,N)$, where $i=1,2,3$.

One may also argue that the (combinatorially motivated!) finite versions (16)–(20) may be not best in approaching the Kanade–Russell identities (11)–(15) and one can possibly try more general sums like

$$F(N,M):=\sum_{m,n\ge0}q^{m^2+3mn+3n^2}\begin{bmatrix}N-m-3n\\ m\end{bmatrix}_q\begin{bmatrix}M-m-n\\ n\end{bmatrix}_{q^3}$$

for different specializations $M=M(N)\to\infty$ as $N\to\infty$ instead. Although using [1] it is easy to observe that the $F(N,M)$'s satisfy simple-looking recurrences such as

$$F(N,M)=F(N-1,M)+q^{N-1}F(N-2,M-1)\quad\text{for } 3\nmid N,$$

none of these variations seem to produce (linear combinations of) products for the limits of the sums reflected under $q\mapsto 1/q$. In other words, the combinatorics of the Kanade–Russell partition functions are a delicate sensor of arithmetic features.

5. Modular remarks

Without giving precise definitions of modular (and mock modular) functions, for which the reader may consult some standard sources (e.g., [15, 22]), we notice that it was the modularity of the product sides in (11)–(13) that led us to reasonable guesses for (28)–(37). The use of the servers at the Research Institute for Symbolic Computation (RISC) for the actual calculations significantly cooled down our laptops.

The individual products in (14) and (15) (and, similarly, the products in (21), (25), and in (22), (26)) are not modular but, possibly, can be paired to form vector-valued modular forms (and linked to mock theta functions).

The main obstacle for using the powerful (mock) modular structure in proving the Kanade–Russell conjectures and their reflected counterparts is the difficulty in establishing the modular behavior for the sum sides of the identities.

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$$\sum_{m,n\ge0}\frac{q^{m^2+2mn+2n^2}}{(q;q)_m(q^2;q^2)_n}=\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty},$$

$$\sum_{m,n\ge0}\frac{q^{m^2+2mn+2n^2+m+2n}}{(q;q)_m(q^2;q^2)_n}=\frac{(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}$$

are instances of Bressoud's identities from [7].

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Appendix A. Combinatorial construction of the finite versions of the Kanade–Russell–Kurşungöz style double series

We give a brief description and considerations that go into identifying the polynomial analogues (16)–(20) of (6)–(10). To that end, we present how we *combinatorially* construct the refinement of (9). Recall that (9) was shown by Kurşungöz [14] to be the generating function for the number of partitions that satisfy the gap conditions prescribed in the I_4 conjecture of [11]. Here we prove that the finite analogue (19) is the generating function for the number of partitions into parts $\le N$ that satisfy the gap conditions prescribed in the I_4 conjecture — the conditions are given as (a)–(d) in the Introduction. We will be mimicking the constructions of [5, 19, 20], and invite the interested reader to examine these references to see longer expositions of this technique.

Recall conditions (b) and (c) from the second page. The I_4 conjecture considers partitions in which 1 does not appear as a part, the difference between parts is ≥ 3 at distance 2 such that, if two successive parts differ by ≤ 1, then their sum is congruent to 2 (mod 3). First we would like to explain how to interpret, partition-theoretically, the pieces of (9),

$$\sum_{m,n\geq 0} \frac{q^{m^2+3mn+3n^2+m+2n}}{(q;q)_m(q^3;q^3)_n},$$

as the generating function for these partitions. This is done in the spirit of Kurşungöz's construction in [14] using the vocabulary of [5, 19, 20]. For the rest of this discussion, we consider a partition to be a finite sequence of *non-decreasing* positive integers. The q-factor $q^{m^2+3mn+3n^2+m+2n}$ is the size of the partition

$$\pi_{m,n} = (\underline{2,3}, \underline{5,6}, \dots, \underline{3n-1,3n}, 3n+2, 3n+4, \dots, 3n+2m).$$

We call the underlined consecutive parts of $\pi_{m,n}$ *pairs* and the rest of the terms *singletons*. Notice that $\pi_{m,n}$ is a partition that satisfies the gap conditions of I_4. Moreover, it is not hard to observe that $\pi_{m,n}$ is the partition with the smallest possible size that satisfies the gap conditions of I_4 into $2n+m$ parts, where the minimal distance condition (if two successive parts differ by ≤ 1, then their sum is congruent to 2 (mod 3)) appears exactly n times. We call $\pi_{m,n}$ a *minimal configuration* with $2n+m$ parts and n minimal gaps.

If one adds any non-negative integer value r_m to the largest part $3n+2m$ of $\pi_{m,n}$, the outcome partition still satisfies the gap conditions of I_4. Similarly, after adding r_m to the largest part, if one adds some non-negative integer $r_{m-1} \leq r_m$ to the second largest value of $\pi_{m,n}$, the outcome partition satisfies the gap conditions of I_4. Repeating this process, we can conclude that if one adds $(r_1, \dots, r_m)$, where $0 \leq r_1 \leq r_2 \leq \dots \leq r_m$, to the m largest parts (the singletons) of $\pi_{m,n}$, the outcome partition π_n still satisfies the gap conditions of I_4. Moreover, this addition can be reversed. Given a partition π_n, where the singletons are possibly not at their original locations while the pairs are left untouched, we can easily recover the list $(r_1, \dots, r_m)$ and the minimal configuration $\pi_{n,m}$ that gives rise to π_n. Finally, we note that the lists $(r_1, \dots, r_m)$, where $0 \leq r_1 \leq \dots \leq r_m$, are in one-to-one correspondence to partitions into $\leq m$ parts. The generating function for the partitions into $\leq m$ parts is $(q;q)_m^{-1}$. Hence, now we can conclude that

$$\frac{q^{m^2+3mn+3n^2+m+2n}}{(q;q)_m}$$

is the generating function for the number of all the partitions of the form

$$\pi_n = (\underline{2,3}, \underline{5,6}, \dots, \underline{3n-1,3n}, s_1, s_2, \dots, s_m),$$

where $3n+2 \leq s_1$ and $s_i - s_{i-1} \geq 2$ for $i = 2, \dots, m$.

Now, given a partition π_n with m singletons, assuming that there are no close-by singletons, we can move the largest pair to the next permissible pair and repeat this (reversible) process to generate other partitions with n arbitrary pairs and

m singletons. To be precise, there are two possible (free) forward motions of pairs and these are

$$\underline{3k-1,3k} \mapsto \underline{3k+1,3k+1} \quad \text{and} \quad \underline{3k+1,3k+1} \mapsto \underline{3k+2,3k+3}.$$

In both forward motions of a pair, the total size change of the partition is 3. Similarly to moving singletons, by moving the largest pair, followed by the second largest pair, etc., we never need to worry about pairs crossing each other. However, although the forward motions of singletons were done freely, as we move a pair, we may come close to a singleton and violate the gap conditions of I_4. To avoid this, we define (reversible) cross-over rules of pairs over singletons. The two necessary cross-over rules are as follows:

$$\underline{3k-1,3k}, 3k+2 \mapsto 3k-1, \underline{3k+2,3k+3},$$
$$\underline{3k+1,3k+1}, 3k+4 \mapsto 3k+1, \underline{3k+4,3k+4}.$$

Notice that these motions also add 3 to the total size of the partition. Hence, now given a partition π_n, starting from the largest pair, we can move the pairs forward and make new partitions into $2n+m$ parts that have n pairs which satisfy the gap conditions of I_4. Similarly to the singletons' case, the forward motion lists of the pairs corresponds to partitions into $\leq n$ parts. Since each motion of the pairs adds 3 to the total size of the partition, we instead use $(q^3;q^3)_\infty^{-1}$ here. Therefore, the summand of (9),

$$\frac{q^{m^2+3mn+3n^2+m+2n}}{(q;q)_m(q^3;q^3)_n},$$

is the generating function for the partitions that satisfy the gap conditions of I_4 and have n pairs and m singletons. Summing over all m and n, we obtain the generating function for all the partitions that satisfy the gap conditions of I_4. This finishes the combinatorial construction/interpretation of (9) in the spirit of Kurşungöz.

For the finite analogue, all we need to do is to restrict the forward motions of the terms. If we want to add the new restriction that all the parts of our partitions are $\leq N$ (condition (a) on the second page), the forward motion of the largest singleton must not be free. In this situation r_m must be $\leq N-(3n+2m)$, hence the generating function for partitions into $\leq m$ parts, which represented the reversible forward motion of the singletons, is now replaced by the generating function for the partitions into $\leq m$ parts with each part $\leq N-(3n+2m)$. This generating function is

$$\begin{bmatrix} N-(3n+2m)+m \\ m \end{bmatrix}_q .$$

Similarly, we need to adjust the reversible forward motion of the pairs. At each move of a pair, the midpoint of a pair (the arithmetic mean of the elements in the pair) moves 2/3 steps. From the largest pair $\underline{3n-1,3n}$ to the upper bound N, there are exactly $\lfloor 2(N-3n)/3 \rfloor + \delta_{3|(N-1)}$ many steps that a pair can move forward if all the motions are done via free motion rules. As a pair moves forward, it may cross up to m singletons, and crossing over each singleton also means missing one possible location where the pair could have stopped. Hence, the actual number of steps a pair can move forward is $\lfloor 2(N-3n)/3\rfloor + \delta_{3|(N-1)} - m$. Therefore, we need to replace the generating function $(q^3;q^3)_n^{-1}$ by the q-binomial coefficient

$$\begin{bmatrix} \lfloor 2(N-3n)/3\rfloor + \delta_{3|(N-1)} - m + n \\ n \end{bmatrix}_{q^3}$$

to restrict the forward motion of the n pairs, where all the parts still remain $\le N$ after the motions. Compiling the above discussion we get that

$$\sum_{m,n\ge 0} q^{m^2+3mn+3n^2+m+2n} \begin{bmatrix} N-m-3n \\ m \end{bmatrix}_q \begin{bmatrix} \lfloor 2(N-3n)/3\rfloor + \delta_{3|(N-1)} - m + n \\ n \end{bmatrix}_{q^3}$$

is the generating function for the partitions into parts $\le N$ which satisfy the gap conditions of I_4, and this expression is equal to (19) after simplification of the top argument of the second q-binomial coefficient.

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