Hook type tableaux and Partition identities

Koustav Banerjee

RISC, JKU, Linz

85th Séminaire Lotharingien De Combinatoire

SLC 85

Strobl, Austria

(Joint work with Manosij Ghosh Dastidar, Pondicherry University)

07 September, 2020

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

RISC
RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

JKU
JOHANNES KEPLER UNIVERSITY LINZ

FWF
Der Wissenschaftsfonds.

Outline

- Introduction

Outline

- Introduction
- Two partition identities

Outline

- Introduction
- Two partition identities
- Box stacking principle (BSP)

Outline

- Introduction
- Two partition identities
- Box stacking principle (BSP)
- Generalization of Stanley's theorem

Outline

- Introduction
- Two partition identities
- Box stacking principle (BSP)
- Generalization of Stanley's theorem
- BSP in color partitions

Outline

- Introduction
- Two partition identities
- Box stacking principle (BSP)
- Generalization of Stanley's theorem
- BSP in color partitions
- Generalization of Stanley's theorem in color context

Outline

- Introduction
- Two partition identities
- Box stacking principle (BSP)
- Generalization of Stanley's theorem
- BSP in color partitions
- Generalization of Stanley's theorem in color context
- Extension of Andrews' identity

Introduction

- **Partitions:** A partition of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers whose sum is n. We denote this by writing $\lambda \vdash n$. The empty partition () is the unique partition of 0; i.e., $p(0) = 1$.
  Now, $P(n) := \{\lambda : \lambda \vdash n\}$ and $p(n) = |P(n)|$
  For example, $p(4) = 5$.

- **Partitions:** A partition of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers whose sum is n. We denote this by writing $\lambda \vdash n$. The empty partition () is the unique partition of 0; i.e., $p(0) = 1$.
  Now, $P(n) := \{\lambda : \lambda \vdash n\}$ and $p(n) = |P(n)|$
  For example, $p(4) = 5$.
- **Conjugate of a partition:** If $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, define a new partition $\lambda^{'} = (\lambda_1^{'}, \ldots, \lambda_m^{'}) \vdash n$ (where m is the largest part of λ) by choosing $\lambda_i^{'}$ as the number of parts of λ that are $\geq i$. The resulting partition $\lambda^{'}$ is called the conjugate of λ.
  Ex: if $\lambda = (6, 3, 3, 2, 1)$, the conjugate of λ is $\lambda^{'} = (5, 4, 3, 1, 1, 1)$.

Introduction

- **Partitions:** A partition of a positive integer n is a weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers whose sum is n. We denote this by writing $\lambda \vdash n$. The empty partition () is the unique partition of 0; i.e., $p(0) = 1$.
  Now, $P(n) := \{\lambda : \lambda \vdash n\}$ and $p(n) = |P(n)|$
  For example, $p(4) = 5$.
- **Conjugate of a partition:** If $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, define a new partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m) \vdash n$ (where m is the largest part of λ) by choosing λ'_i as the number of parts of λ that are $\geq i$. The resulting partition λ' is called the conjugate of λ.
  Ex: if $\lambda = (6, 3, 3, 2, 1)$, the conjugate of λ is $\lambda' = (5, 4, 3, 1, 1, 1)$.
- **Color partitions:** For a positive integer $\ell \geq 2$, $P^{(\ell)}(n)$ is the set of all partitions of n where parts multiple of ℓ comes up with 2 colors. We denote $p^{(\ell)}(n) = |P^{(\ell)}(n)|$ and $p^{(\ell)}(0) := 1$.
  Ex: for $\ell = 3$ and $n = 4$, $p^{(3)}(4) = 7$.
  The generating function of $p^{(\ell)}(n)$ is

$$\sum_{n=0}^{\infty} p^{(\ell)}(n) q^n = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)(1-q^{\ell j})}.$$

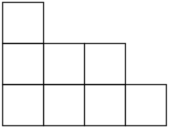
Introduction

- **Young diagram:** To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ. In this context, we prefer the representation to be 'right side up'.

Introduction

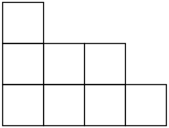
- **Young diagram:** To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ. In this context, we prefer the representation to be 'right side up'.

  Ex: For $\lambda = (4, 3, 1) \vdash 8$, Y_λ is given by

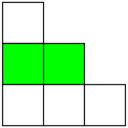


- **Young diagram:** To each partition $\lambda \vdash n$ we associate Y_λ, the celebrated graphical representation called the Young diagram of λ. In this context, we prefer the representation to be 'right side up'.

  Ex: For $\lambda = (4, 3, 1) \vdash 8$, Y_λ is given by



and for $\lambda = (1_1, 2_2, 3_1) \in P^{(2)}(6)$, the associated colored Young diagram Y_λ is;



Introduction

- **Hook length and Hook type tableaux :** For each box v in Y_λ, we define the hook length of v as $h = a + \ell + 1$, where a is the arm length and ℓ the leg length. The ordered pair (a, ℓ) is called hook type of the chosen box in the Young tableaux.

Introduction

- **Hook length and Hook type tableaux :** For each box v in Y_λ, we define the hook length of v as $h = a + \ell + 1$, where a is the arm length and ℓ the leg length. The ordered pair (a, ℓ) is called hook type of the chosen box in the Young tableaux.

Ex: For $\lambda = (4, 3, 1) \vdash 8$, hook length for each boxes in Y_λ is given by;

1			
4	2	1	
6	4	3	1

Introduction

- **Hook length and Hook type tableaux :** For each box v in Y_λ, we define the hook length of v as $h = a + \ell + 1$, where a is the arm length and ℓ the leg length. The ordered pair (a, ℓ) is called hook type of the chosen box in the Young tableaux.

Ex: For $\lambda = (4, 3, 1) \vdash 8$, hook length for each boxes in Y_λ is given by;

1			
4	2	1	
6	4	3	1

Ex: For $\lambda = (3, 1) \vdash 4$, hook type for each boxes in Y_λ is

$(0, 0)$		
$(2, 1)$	$(1, 0)$	$(0, 0)$

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

- $Q_k(n) := \sum_{\lambda \vdash n} \delta_k(\lambda)$, where $\delta_k(\lambda)$ denotes total number of occurences of the part k in λ.

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

- $Q_k(n) := \sum_{\lambda \vdash n} \delta_k(\lambda)$, where $\delta_k(\lambda)$ denotes total number of occurences of the part k in λ.
- $V_k(n) := \sum_{\lambda \vdash n} \nu_k(\lambda)$, where $\nu_k(\lambda)$ denotes the number of parts occurring k or more times λ.

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

- $Q_k(n) := \sum_{\lambda \vdash n} \delta_k(\lambda)$, where $\delta_k(\lambda)$ denotes total number of occurences of the part k in λ.
- $V_k(n) := \sum_{\lambda \vdash n} \nu_k(\lambda)$, where $\nu_k(\lambda)$ denotes the number of parts occurring k or more times λ.
- $S(n) := \sum_{\lambda \vdash n} \mu(\lambda)$, where $\mu(\lambda)$ denotes the number of distinct parts in λ.

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

- $Q_k(n) := \sum_{\lambda \vdash n} \delta_k(\lambda)$, where $\delta_k(\lambda)$ denotes total number of occurences of the part k in λ.
- $V_k(n) := \sum_{\lambda \vdash n} \nu_k(\lambda)$, where $\nu_k(\lambda)$ denotes the number of parts occurring k or more times λ.
- $S(n) := \sum_{\lambda \vdash n} \mu(\lambda)$, where $\mu(\lambda)$ denotes the number of distinct parts in λ.
- $Q_k^{(\ell)}(n) :=$ Number of occurences of parts k_1 and k_2 in $P^{(\ell)}(n)$ when k is a multiple of ℓ; otherwise the number of occurences of the part k_1 in $P^{(\ell)}(n)$. In short we say, $Q_k^{(\ell)}(n)$ is the number of occurences of part k.

Introduction

Throughout the talk, we shall use following notations: for n, k and $\ell \geq 2$ positive integers;

- $Q_k(n) := \sum_{\lambda \vdash n} \delta_k(\lambda)$, where $\delta_k(\lambda)$ denotes total number of occurences of the part k in λ.
- $V_k(n) := \sum_{\lambda \vdash n} \nu_k(\lambda)$, where $\nu_k(\lambda)$ denotes the number of parts occurring k or more times λ.
- $S(n) := \sum_{\lambda \vdash n} \mu(\lambda)$, where $\mu(\lambda)$ denotes the number of distinct parts in λ.
- $Q_k^{(\ell)}(n) :=$ Number of occurences of parts k_1 and k_2 in $P^{(\ell)}(n)$ when k is a multiple of ℓ; otherwise the number of occurences of the part k_1 in $P^{(\ell)}(n)$. In short we say, $Q_k^{(\ell)}(n)$ is the number of occurences of part k.
- For $\lambda \vdash n$ and $p, q \in \mathbb{Z}_{\geq 0}$, $B_{(p,q)}(\lambda) :=$ Number of boxes with hook type (p, q) in Y_λ. $B_{(p,q)}(n) := \sum_{\lambda \vdash n} B_{(p,q)}(\lambda)$.

Introduction

Theorem (Bessenrodt (1998), Bacher-Manivel (2002))

Let $1 \leq k \leq n$ be two integers. Then, for every positive $j < k$, the total number of occurrences of the part k among all partitions of n is equal to the total number of boxes whose hook type is $(j, k-j-1)$; i.e., $Q_k(n) = B_{(j,k-j-1)}(n)$.

Theorem (Bessenrodt (1998), Bacher-Manivel (2002))

Let $1 \leq k \leq n$ be two integers. Then, for every positive $j < k$, the total number of occurrences of the part k among all partitions of n is equal to the total number of boxes whose hook type is $(j, k-j-1)$; i.e., $Q_k(n) = B_{(j,k-j-1)}(n)$.

Note : For $k = 1$, j has to be 0 and for $k > 1$ without loss of generality, one can choose particularly $j = k - 1$.

Two partition identities

Theorem (Stanley (1972))

The total number of 1's in all partitions of a positive integer n is equal to the sum of the numbers of distinct parts of all partitions of n; i.e., $S(n) = Q_1(n)$.

Two partition identities

Theorem (Stanley (1972))

The total number of 1's in all partitions of a positive integer n is equal to the sum of the numbers of distinct parts of all partitions of n; i.e., $S(n) = Q_1(n)$.

For $n = 4$,

$P(4)$			
$\lambda \vdash 4$	$\delta_1(\lambda)$	$\mu(\lambda)$	$B_{(0,0)}(\lambda)$
4	0	1	1
$3+1$	1	2	2
$2+2$	0	1	1
$2+1+1$	2	2	2
$1+1+1+1$	4	1	1
Total	$Q_1(4) = 7$	$S(4) = 7$	$B_{(0,0)}(n) = 7$

.

Two partition identities

Proof sketch:

Two partition identities

Proof sketch:

- We will show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0,0)$; i.e., $\mu(\lambda) = B_{(0,0)}(\lambda)$.

Two partition identities

Proof sketch:

- We will show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0,0)$; i.e., $\mu(\lambda) = B_{(0,0)}(\lambda)$.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and suppose $\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_k}$ are all the distinct parts of λ with respective multiplicities $m_1, m_2, \ldots, m_k$ where $1 \leq a_i \leq r$ and $a_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Without loss of generality assume $\lambda_{a_1} > \lambda_{a_2} > \cdots > \lambda_{a_k}$.

Two partition identities

Proof sketch:

- We will show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0,0)$; i.e., $\mu(\lambda) = B_{(0,0)}(\lambda)$.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and suppose $\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_k}$ are all the distinct parts of λ with respective multiplicities $m_1, m_2, \ldots, m_k$ where $1 \leq a_i \leq r$ and $a_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Without loss of generality assume $\lambda_{a_1} > \lambda_{a_2} > \cdots > \lambda_{a_k}$.
- Next, we note that, the boxes with hook-type $(0,0)$ appear exactly once in Y_λ corresponding to the part λ_{a_m} subject to the condition that the immediate next part λ_{a_n} with $m \neq n$.

Two partition identities

Proof sketch:

- We will show that number of distinct parts of a partition $\lambda \vdash n$ is equal to the number of boxes in Y_λ with hook-type $(0,0)$; i.e., $\mu(\lambda) = B_{(0,0)}(\lambda)$.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ and suppose $\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_k}$ are all the distinct parts of λ with respective multiplicities $m_1, m_2, \ldots, m_k$ where $1 \leq a_i \leq r$ and $a_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Without loss of generality assume $\lambda_{a_1} > \lambda_{a_2} > \cdots > \lambda_{a_k}$.
- Next, we note that, the boxes with hook-type $(0,0)$ appear exactly once in Y_λ corresponding to the part λ_{a_m} subject to the condition that the immediate next part λ_{a_n} with $m \neq n$.
- Therefore, the number of boxes with hook-type $(0,0)$ equals the number of distinct parts of λ. Now, summing over all $\lambda \vdash n$ we get the Stanley's theorem.

Two partition identities

Theorem (Elder (1984))

The total number of occurences of an integer k among all partitions of n is equal to the number of occasions that a part occurs greater or equal k times in $P(n)$; i.e., $Q_k(n) = V_k(n)$.

Two partition identities

Theorem (Elder (1984))

The total number of occurences of an integer k among all partitions of n is equal to the number of occasions that a part occurs greater or equal k times in $P(n)$; i.e., $Q_k(n) = V_k(n)$.

For $n = 4$ and $k = 2$,

$P(4)$					
$\lambda \vdash 4$	$\delta_2(\lambda)$	$\nu_2(\lambda)$	$B_{(1,0)}(\lambda)$	$\lambda' \vdash 4$	$\nu_2(\lambda')$
4	0	0	1	1+1+1+1	1
$3+1$	0	0	1	$2+1+1$	1
$2+2$	2	1	1	$2+2$	1
$2+1+1$	1	1	0	$3+1$	0
1+1+1+1	0	1	0	4	0
Total	$Q_2(4) = 3$	$V_2(4) = 3$	$B_{(1,0)}(4) = 3$	-	$V_2(4) = 3$

.

Two partition identities

Proof sketch:

Two partition identities

Proof sketch:

- We need only to show that the number of boxes with hook type $(k-1, 0)$, $k > 1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ.

Proof sketch:

- We need only to show that the number of boxes with hook type $(k-1, 0)$, $k > 1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ.
- Now, a box with hook-type $(k-1, 0)$ in Y_λ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ precisely describes that there are $k-1$ boxes on the right to it but having no box above.

Two partition identities

Proof sketch:

- We need only to show that the number of boxes with hook type $(k-1,0)$, $k>1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ.
- Now, a box with hook-type $(k-1,0)$ in Y_λ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ precisely describes that there are $k-1$ boxes on the right to it but having no box above.
- When transforming λ to it's conjugate $\lambda^{'}$ it is clear that after conjugation, the box with hook-type $(k-1,0)$ transforms into the box with hook-type $(0,k-1)$. This shows that there are total at least k verticals stacks of boxes (including the box itself); i.e., there exists a part that occurs at least k times in that conjugate partition.

Two partition identities

Proof sketch:

- We need only to show that the number of boxes with hook type $(k-1,0)$, $k>1$, in a partition $\lambda \vdash n$ is equal to the number of parts that occur k or more times in λ.
- Now, a box with hook-type $(k-1,0)$ in Y_λ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ precisely describes that there are $k-1$ boxes on the right to it but having no box above.
- When transforming λ to it's conjugate $\lambda^{'}$ it is clear that after conjugation, the box with hook-type $(k-1,0)$ transforms into the box with hook-type $(0,k-1)$. This shows that there are total at least k verticals stacks of boxes (including the box itself); i.e., there exists a part that occurs at least k times in that conjugate partition.
- So, corresponding to each box with hook-type $(k-1,0)$ there exists a part that occurs at least k times and hence summing over all partitions of n we have Elder's statement.

Box Stacking Principle (BSP)

The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n+k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:

Box Stacking Principle (BSP)

The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n + k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:

For $k = 1$:

Box Stacking Principle (BSP)

The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n+k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:

For $k = 1$:

- We add one box to all permissible places in Y_λ. One can trivially add one box in two ways: (i) Add to the bottom row of Y_λ. (ii) Stack the box on the above of the top row of Y_λ. Also, we can add one box to a row in Y_λ if and only if the difference between the number of boxes in the chosen row and its immediate next is at least 1.

Box Stacking Principle (BSP)

The BSP consists of a set of rules to produce from all partitions of n a new set of partitions of $n+k$ where k is a positive integer. Given a partition $\lambda \vdash n$, the new partitions are produced by adding k boxes as follows:

For $k = 1$:

- We add one box to all permissible places in Y_λ. One can trivially add one box in two ways: (i) Add to the bottom row of Y_λ. (ii) Stack the box on the above of the top row of Y_λ. Also, we can add one box to a row in Y_λ if and only if the difference between the number of boxes in the chosen row and its immediate next is at least 1.
- Explicitly, for $\lambda := (\lambda_1, \ldots, \lambda_r) \vdash n$, following rule (i) the trivial addition of one box corresponds to $\mu := ((\lambda_1 + 1), \ldots, \lambda_r) \vdash n+1$ whereas by rule (ii) we have $\mu := (\lambda_1, \ldots, \lambda_r, 1) \vdash n+1$. Nontrivial addition of one box can be done if and only if for any two consecutive part say, λ_i and λ_j ($\lambda_i \geq \lambda_j$), we have $\lambda_i - \lambda_j \geq 1$.

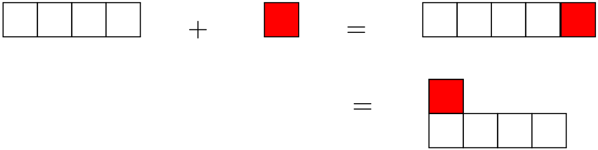
Box Stacking Principle (BSP)

For example, to all partitions of 4 and applying the stacking principle for adding one box to the Young diagram gives:

Box Stacking Principle (BSP)

For example, to all partitions of 4 and applying the stacking principle for adding one box to the Young diagram gives:

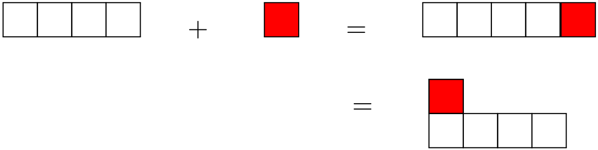
I. $\lambda = 4$:



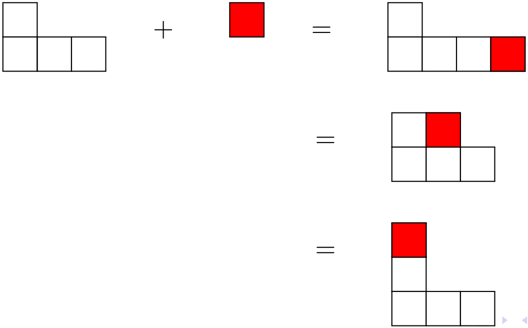
Box Stacking Principle (BSP)

For example, to all partitions of 4 and applying the stacking principle for adding one box to the Young diagram gives:

I. $\lambda = 4$:

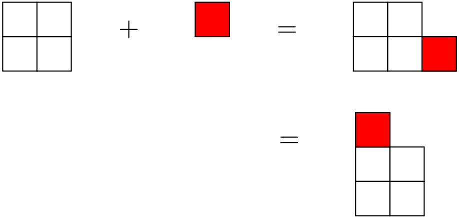


II. $\lambda = 3 + 1$:



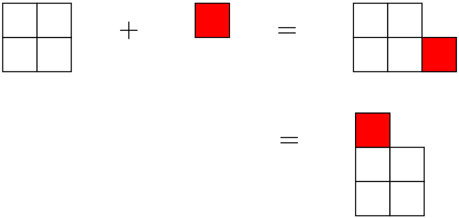
Box Stacking Principle (BSP)

III. $\lambda = 2 + 2$:

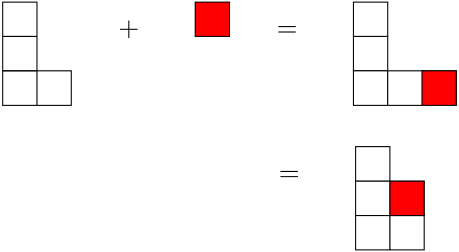


Box Stacking Principle (BSP)

III. $\lambda = 2 + 2$:

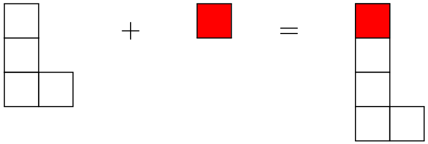


IV. $\lambda = 2 + 1 + 1$:



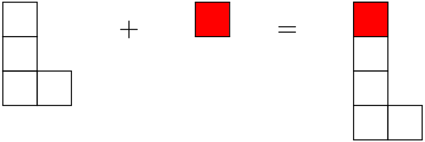
Box Stacking Principle (BSP)

IV. $\lambda = 2 + 1 + 1$:

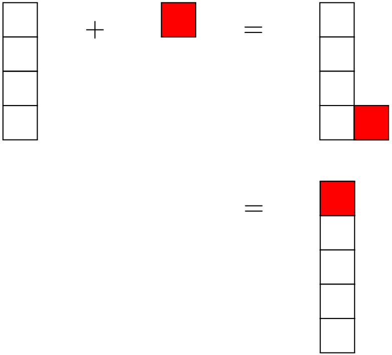


Box Stacking Principle (BSP)

IV. $\lambda = 2 + 1 + 1$:



V. $\lambda = 1 + 1 + 1 + 1$:



Box Stacking Principle (BSP)

For $k > 1$:

Box Stacking Principle (BSP)

For $k > 1$:

- Here we consider the addition of k boxes as a 'packet of k boxes', instead of adding 'k single boxes'. Again one can trivially add a 'packet of k boxes' to the bottom row of Y_λ with $\lambda := (\lambda_1, \ldots, \lambda_r)$. By adding 'packet of k boxes', we mean that adding k to λ_1 so that the resulting partition $\mu := ((\lambda_1 + k), \ldots, \lambda_r) \vdash n + k$.

Box Stacking Principle (BSP)

For $k > 1$:

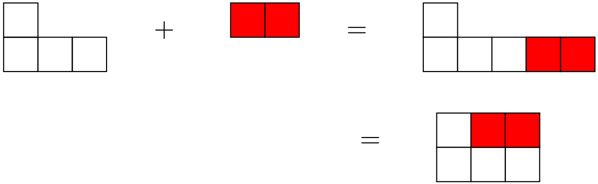
- Here we consider the addition of k boxes as a 'packet of k boxes', instead of adding 'k single boxes'. Again one can trivially add a 'packet of k boxes' to the bottom row of Y_λ with $\lambda := (\lambda_1, \ldots, \lambda_r)$. By adding 'packet of k boxes', we mean that adding k to λ_1 so that the resulting partition $\mu := ((\lambda_1 + k), \ldots, \lambda_r) \vdash n + k$.
- A nontrivial addition of a packet of k boxes to Y_λ can be done if and only if for any two consecutive part say, λ_i and λ_j $(\lambda_i \geq \lambda_j)$, we have $\lambda_i - \lambda_j \geq k$.

Box Stacking Principle (BSP)

For $k > 1$:

- Here we consider the addition of k boxes as a 'packet of k boxes', instead of adding 'k single boxes'. Again one can trivially add a 'packet of k boxes' to the bottom row of Y_λ with $\lambda := (\lambda_1, \dots, \lambda_r)$. By adding 'packet of k boxes', we mean that adding k to λ_1 so that the resulting partition $\mu := ((\lambda_1 + k), \dots, \lambda_r) \vdash n + k$.
- A nontrivial addition of a packet of k boxes to Y_λ can be done if and only if for any two consecutive part say, λ_i and λ_j ($\lambda_i \geq \lambda_j$), we have $\lambda_i - \lambda_j \geq k$.

For stacking of $k = 2$ boxes with $\lambda = (3, 1) \vdash 4$ following BSP,



What we do not allow:

What we do not allow:

- We do not consider the addition of 'k single boxes' which means that we do not allow the cases $\mu_1 := (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \vdash n + k$ and $\mu_2 := (\lambda_1, \ldots, (\lambda_{j_1} + 1), \ldots, (\lambda_{j_2} + 1), \ldots, , (\lambda_{j_k} + 1), \ldots, \lambda_r) \vdash n + k$.

Box Stacking Principle (BSP)

What we do not allow:

- We do not consider the addition of 'k single boxes' which means that we do not allow the cases $\mu_1 := (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \vdash n + k$ and $\mu_2 := (\lambda_1, \ldots, (\lambda_{j_1} + 1), \ldots, (\lambda_{j_2} + 1), \ldots, , (\lambda_{j_k} + 1), \ldots, \lambda_r) \vdash n + k$.

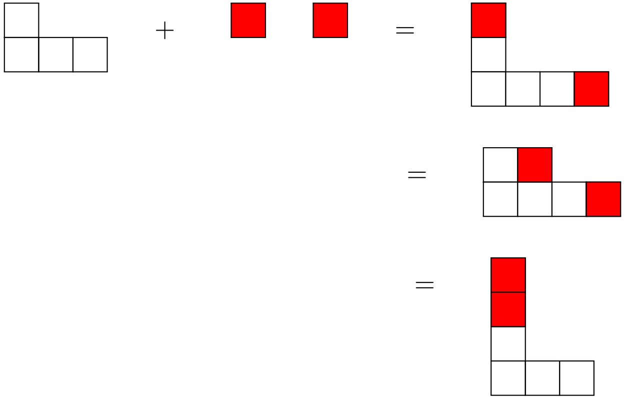
For stacking of $k = 2$ boxes with $\lambda = (3, 1) \vdash 4$, following situations will be regarded as violating our rules;

Box Stacking Principle (BSP)

What we do not allow:

- We do not consider the addition of 'k single boxes' which means that we do not allow the cases $\mu_1 := (\lambda_1, \ldots, \lambda_r, 1, \ldots, 1) \vdash n + k$ and $\mu_2 := (\lambda_1, \ldots, (\lambda_{j_1} + 1), \ldots, (\lambda_{j_2} + 1), \ldots, , (\lambda_{j_k} + 1), \ldots, \lambda_r) \vdash n + k$.

For stacking of $k = 2$ boxes with $\lambda = (3, 1) \vdash 4$, following situations will be regarded as violating our rules;



Generalization of Stanley's theorem

Theorem (Dastidar, Sengupta (2013))

For positive integers n and k,

$$S(n) = Q_k(n) + Q_k(n+1) + Q_k(n+2) + \cdots + Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

Generalization of Stanley's theorem

Theorem (Dastidar, Sengupta (2013))

For positive integers n and k,

$$S(n) = Q_k(n) + Q_k(n+1) + Q_k(n+2) + \cdots + Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

- Ex: For $n = 5$ and $k = 3$; we have $S(5) = 12$, $Q_3(5) = 2$, $Q_3(6) = 4$, $Q_3(7) = 6$. So, $S(5) = Q_3(5) + Q_3(5+1) + Q_3(5+(3-1))$.

Generalization of Stanley's theorem

Theorem (Dastidar, Sengupta (2013))

For positive integers n and k,

$$S(n) = Q_k(n) + Q_k(n+1) + Q_k(n+2) + \cdots + Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

- Ex: For $n = 5$ and $k = 3$; we have $S(5) = 12$, $Q_3(5) = 2$, $Q_3(6) = 4$, $Q_3(7) = 6$. So, $S(5) = Q_3(5) + Q_3(5+1) + Q_3(5+(3-1))$.
- In order to prove the theorem it is enough to prove the following lemma.

Generalization of Stanley's theorem

Theorem (Dastidar, Sengupta (2013))

For positive integers n and k,

$$S(n) = Q_k(n)+Q_k(n+1)+Q_k(n+2)+\cdots+Q_k(n+k-1) = \sum_{j=0}^{k-1} Q_k(n+j).$$

- Ex: For $n = 5$ and $k = 3$; we have $S(5) = 12$, $Q_3(5) = 2$, $Q_3(6) = 4$, $Q_3(7) = 6$. So, $S(5) = Q_3(5) + Q_3(5+1) + Q_3(5+(3-1))$.
- In order to prove the theorem it is enough to prove the following lemma.

Lemma (B., Dastidar (2019))

Stacking k boxes to the Young diagrams corresponding to all partitions of n following the BSP generates as many new partitions as there are occurences of k in all partitions of $n+k$.

Proof sketch:

Generalization of Stanley's theorem

Proof sketch:

- **Trivial Stacking:** We can always add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ and immediately observe that the total number of generated new partition is $p(n)$.

Proof sketch:

- **Trivial Stacking:** We can always add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ and immediately observe that the total number of generated new partition is $p(n)$.
- **Non-trivial Stacking:** Adding k-boxes to a Young diagram Y_λ following BSP is possible if and only if there exists a box in Y_λ with hook-type $(k-1, 0)$. On the other hand, to place a packet of k boxes in the diagram without violating the BSP and structure of Y_λ there must exist a k-consecutive empty places; i.e., a box with hook-type $(k-1, 0)$.

Generalization of Stanley's theorem

Proof sketch:

- **Trivial Stacking:** We can always add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ and immediately observe that the total number of generated new partition is $p(n)$.
- **Non-trivial Stacking:** Adding k-boxes to a Young diagram Y_λ following BSP is possible if and only if there exists a box in Y_λ with hook-type $(k-1, 0)$. On the other hand, to place a packet of k boxes in the diagram without violating the BSP and structure of Y_λ there must exist a k-consecutive empty places; i.e., a box with hook-type $(k-1, 0)$.

This explicitly shows the one to one correspondence between the number of permissible ways of non-trivial addition of packet of k boxes and the number of boxes with hook-type $(k-1, 0)$ in Y_λ.

Proof sketch:

- **Trivial Stacking:** We can always add a packet of k boxes to the largest part of a partition $\lambda \vdash n$ and immediately observe that the total number of generated new partition is $p(n)$.
- **Non-trivial Stacking:** Adding k-boxes to a Young diagram Y_λ following BSP is possible if and only if there exists a box in Y_λ with hook-type $(k-1, 0)$. On the other hand, to place a packet of k boxes in the diagram without violating the BSP and structure of Y_λ there must exist a k-consecutive empty places; i.e., a box with hook-type $(k-1, 0)$.

This explicitly shows the one to one correspondence between the number of permissible ways of non-trivial addition of packet of k boxes and the number of boxes with hook-type $(k-1, 0)$ in Y_λ.

- Following BSP, the total of new generated partition is $p(n) + Q_k(n)$ and it is immediate that $p(n) + Q_k(n) = Q_k(n+k)$.

BSP in color partitions

The BSP in color partitions context consists of a set of rules to produce from all color partitions λ of n a new set of color partitions of $n+k$. In this context we have to take care about the color of a 'packet of k boxes'. If k is not a multiple of ℓ, without loss of generality, we always add a 'packet of k boxes' prescribed by white color. The set of rules are as follows:

BSP in color partitions

The BSP in color partitions context consists of a set of rules to produce from all color partitions λ of n a new set of color partitions of $n+k$. In this context we have to take care about the color of a 'packet of k boxes'. If k is not a multiple of ℓ, without loss of generality, we always add a 'packet of k boxes' prescribed by white color. The set of rules are as follows:

- Let $\lambda := (\lambda_{1_{i_1}}, \lambda_{2_{i_2}}, \ldots, \lambda_{r_{i_r}}) \in P^{(\ell)}(n)$ with $i_k \in \{1, 2\}$ and $1 \leq k \leq r$, $k \in \mathbb{Z}_{>0}$. So when we say $\lambda_{1_{i_1}}$ is the largest part of λ, it means that $\lambda_1 \geq \cdots \geq \lambda_r$. First, we will look at the index of the largest part $\lambda_{1_{i_1}}$.

BSP in color partitions

The BSP in color partitions context consists of a set of rules to produce from all color partitions λ of n a new set of color partitions of $n+k$. In this context we have to take care about the color of a 'packet of k boxes'. If k is not a multiple of ℓ, without loss of generality, we always add a 'packet of k boxes' prescribed by white color. The set of rules are as follows:

- Let $\lambda := (\lambda_{1_{i_1}}, \lambda_{2_{i_2}}, \dots, \lambda_{r_{i_r}}) \in P^{(\ell)}(n)$ with $i_k \in \{1,2\}$ and $1 \leq k \leq r$, $k \in \mathbb{Z}_{>0}$. So when we say $\lambda_{1_{i_1}}$ is the largest part of λ, it means that $\lambda_1 \geq \cdots \geq \lambda_r$. First, we will look at the index of the largest part $\lambda_{1_{i_1}}$.
- If $i_1 = 1$, then trivially we add the packet of k boxes to $\lambda_{1_{i_1}}$; i.e., to bottom row of Y_λ so that the resulting partition $\mu := ((\lambda_{1_{i_1}} + k_1), \dots, \lambda_{r_{i_r}}) \in P^{(\ell)}(n+k)$.

BSP in color partitions

- If $i_1 = 2$, then two cases will arise:

- If $i_1 = 2$, then two cases will arise:

  A. If $\lambda_1 \geq k$, then we consider following two cases:

BSP in color partitions

- If $i_1 = 2$, then two cases will arise:

A. If $\lambda_1 \geq k$, then we consider following two cases:

(i) If there exist any two consecutive parts say $\lambda_{s_{i_s}}$ and $\lambda_{t_{i_t}}$ $(\lambda_s \geq \lambda_t)$ with $i_t = 1$ and $\lambda_s - \lambda_t \geq k$, then we add a packet of k boxes to the row corresponding to the part $\lambda_{t_{i_t}}$ in Y_λ.

BSP in color partitions

- If $i_1 = 2$, then two cases will arise:

A. If $\lambda_1 \geq k$, then we consider following two cases:

(i) If there exist any two consecutive parts say $\lambda_{s_{i_s}}$ and $\lambda_{t_{i_t}}$ $(\lambda_s \geq \lambda_t)$ with $i_t = 1$ and $\lambda_s - \lambda_t \geq k$, then we add a packet of k boxes to the row corresponding to the part $\lambda_{t_{i_t}}$ in Y_λ.

(ii) If there does not exists any two consecutive parts with the condition given in (i), then we simply insert the packet of k-boxes as a new row into Y_λ.

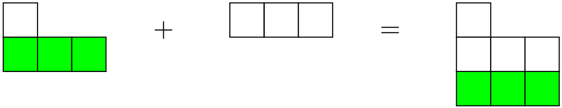
- If $i_1 = 2$, then two cases will arise:

A. If $\lambda_1 \geq k$, then we consider following two cases:

(i) If there exist any two consecutive parts say $\lambda_{s_{i_s}}$ and $\lambda_{t_{i_t}}$ $(\lambda_s \geq \lambda_t)$ with $i_t = 1$ and $\lambda_s - \lambda_t \geq k$, then we add a packet of k boxes to the row corresponding to the part $\lambda_{t_{i_t}}$ in Y_λ.

(ii) If there does not exists any two consecutive parts with the condition given in (i), then we simply insert the packet of k-boxes as a new row into Y_λ.

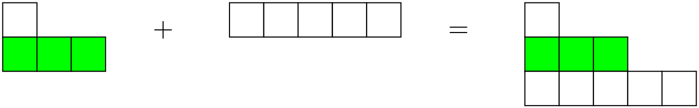
For example, if we consider the addition of a packet of 3 boxes to the partition $\lambda = (3_2, 1_1) \in P^{(3)}(4)$, then:



BSP in color partitions

B. If $\lambda_1 < k$, then we adjoin the packet of k boxes to the below of the bottom row of Y_λ so that resulting partition is $\mu := (k_1, \lambda_{1_{i_1}}, \ldots, \lambda_{r_{i_r}}) \in P^{(\ell)}(n+k)$.

B. If $\lambda_1 < k$, then we adjoin the packet of k boxes to the below of the bottom row of Y_λ so that resulting partition is $\mu := (k_1, \lambda_{1_{i_1}}, \ldots, \lambda_{r_{i_r}}) \in P^{(\ell)}(n+k)$.
For example, if we consider the addition of a packet of 5 boxes to the partition $\lambda = (3_2, 1_1) \in P^{(3)}(4)$, then:



BSP in color partitions

Exclusion Rule:

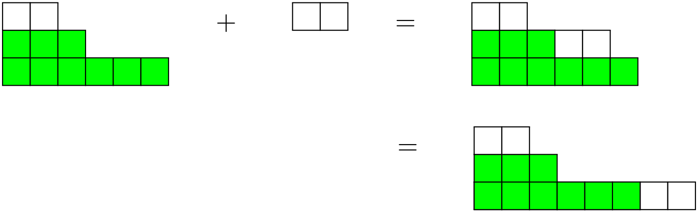
BSP in color partitions

Exclusion Rule:
Here index of parts in the partition $\lambda \in P^{(\ell)}(n)$ is important. For any part of λ, say $\lambda_{m_{i_m}}$ with $i_m = 2$, we do not allow the addition of a packet of k boxes to the row corresponding to the part $\lambda_{m_{i_m}}$ in Y_λ. In short, if the color of the row corresponding to the part with index 2 is green, we do not allow the addition of a packet of k boxes to it.

Exclusion Rule:
Here index of parts in the partition $\lambda \in P^{(\ell)}(n)$ is important. For any part of λ, say $\lambda_{m_{i_m}}$ with $i_m = 2$, we do not allow the addition of a packet of k boxes to the row corresponding to the part $\lambda_{m_{i_m}}$ in Y_λ. In short, if the color of the row corresponding to the part with index 2 is green, we do not allow the addition of a packet of k boxes to it.
For instance, for $n = 11$, $\ell = 3$, $k = 2$ and $\lambda = (6_2, 3_2, 2_1) \in P^{(3)}(11)$:

BSP in color partitions

Exclusion Rule:
Here index of parts in the partition $\lambda \in P^{(\ell)}(n)$ is important. For any part of λ, say $\lambda_{m_{i_m}}$ with $i_m = 2$, we do not allow the addition of a packet of k boxes to the row corresponding to the part $\lambda_{m_{i_m}}$ in Y_λ. In short, if the color of the row corresponding to the part with index 2 is green, we do not allow the addition of a packet of k boxes to it.
For instance, for $n = 11$, $\ell = 3$, $k = 2$ and $\lambda = (6_2, 3_2, 2_1) \in P^{(3)}(11)$:

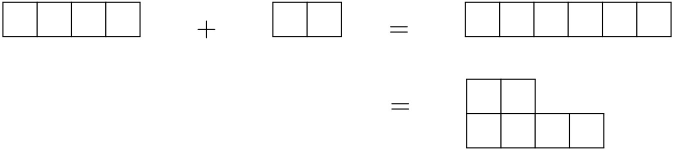


BSP in color partitions

Ex 1. We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 2 boxes to the Young diagram gives:

BSP in color partitions

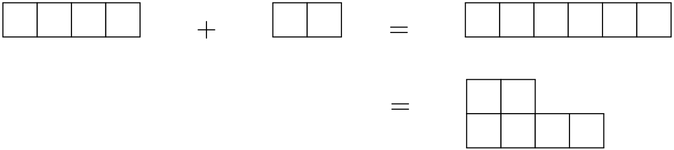
Ex 1. We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 2 boxes to the Young diagram gives:
I. 4_1 :



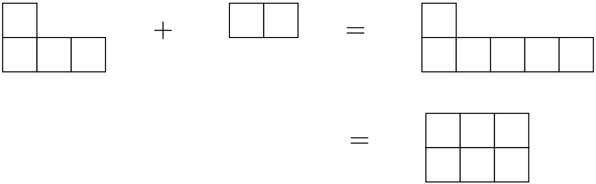
BSP in color partitions

Ex 1. We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 2 boxes to the Young diagram gives:

I. 4_1 :



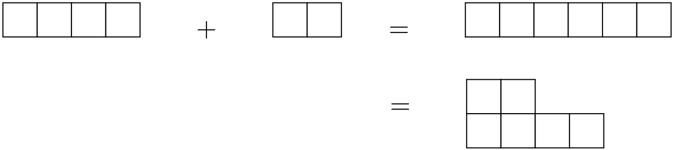
II. $3_1 + 1_1$:



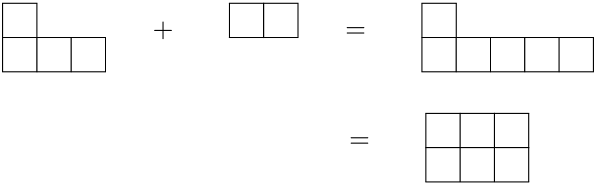
BSP in color partitions

Ex 1. We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 2 boxes to the Young diagram gives:

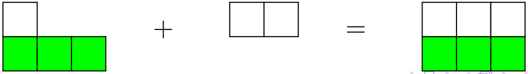
I. 4_1 :



II. $3_1 + 1_1$:

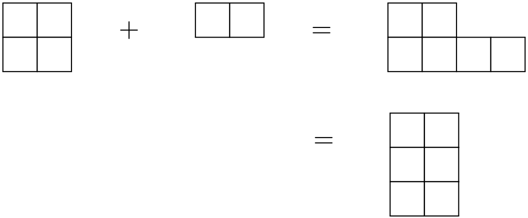


III. $3_2 + 1_1$:



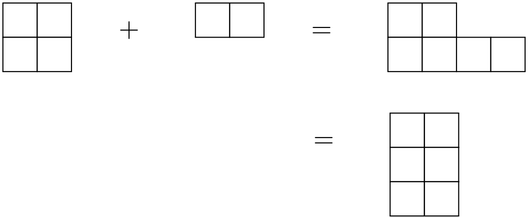
BSP in color partitions

IV. $2_1 + 2_1$:

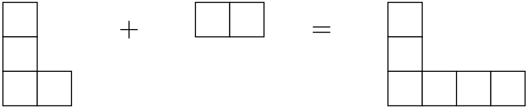


BSP in color partitions

IV. $2_1 + 2_1$:

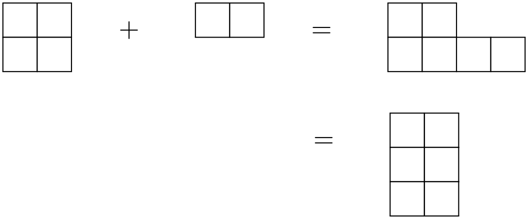


V. $2_1 + 1_1 + 1_1$:

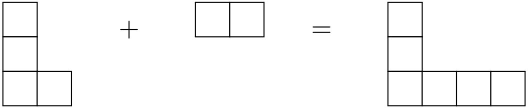


BSP in color partitions

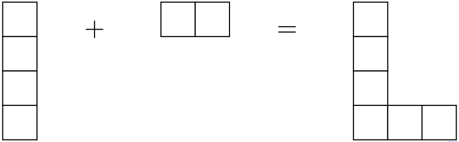
IV. $2_1 + 2_1$:



V. $2_1 + 1_1 + 1_1$:



VI. $1_1 + 1_1 + 1_1 + 1_1$:



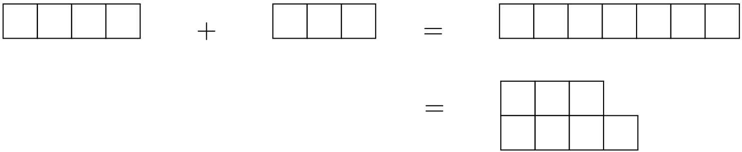
BSP in color partitions

Ex 2(a). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

BSP in color partitions

Ex 2(a). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

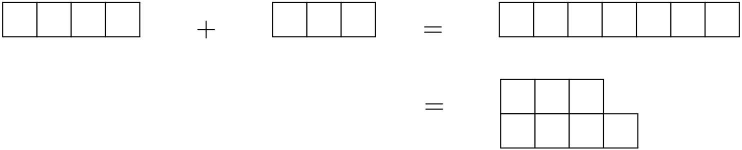
I. 4_1 :



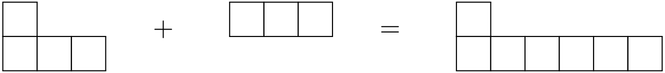
BSP in color partitions

Ex 2(a). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

I. 4_1 :



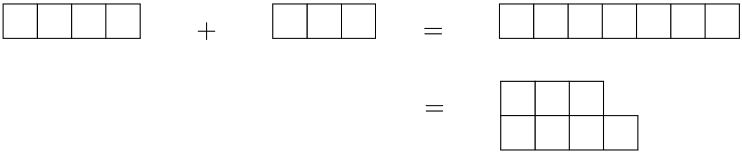
II. $3_1 + 1_1$:



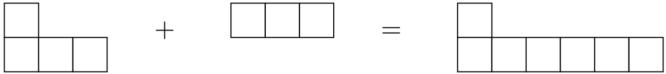
BSP in color partitions

Ex 2(a). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

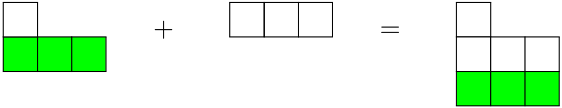
I. 4_1 :



II. $3_1 + 1_1$:

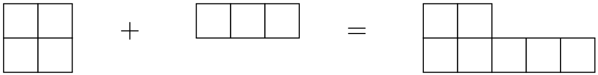


III. $3_2 + 1_1$:



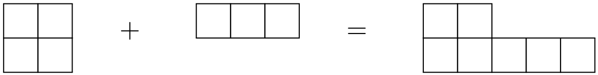
BSP in color partitions

IV. $2_1 + 2_1$:

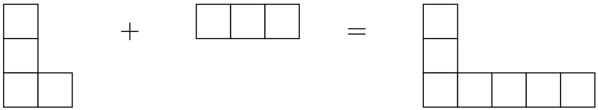


BSP in color partitions

IV. $2_1 + 2_1$:

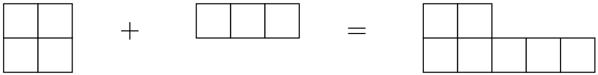


V. $2_1 + 1_1 + 1_1$:

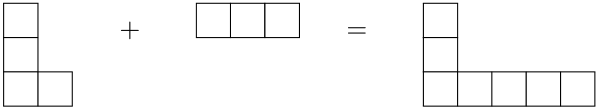


BSP in color partitions

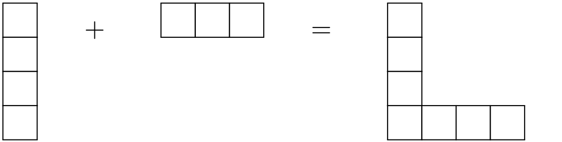
IV. $2_1 + 2_1$:



V. $2_1 + 1_1 + 1_1$:



VI. $1_1 + 1_1 + 1_1 + 1_1$:



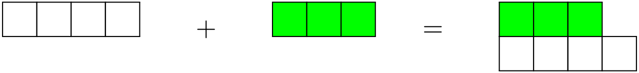
BSP in color partitions

Ex 2(b). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

BSP in color partitions

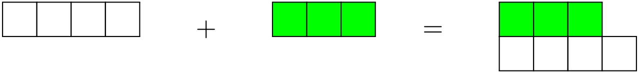
Ex 2(b). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

I. 4_1 :

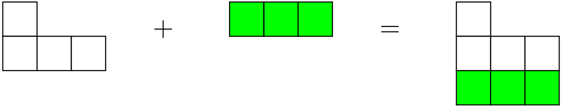


Ex 2(b). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

I. 4_1 :



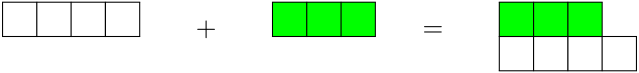
II. $3_1 + 1_1$:



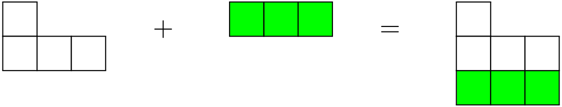
BSP in color partitions

Ex 2(b). We consider all 3 color partitions of 4 and applying the color BSP for adding a packet of 3 boxes to the Young diagram gives:

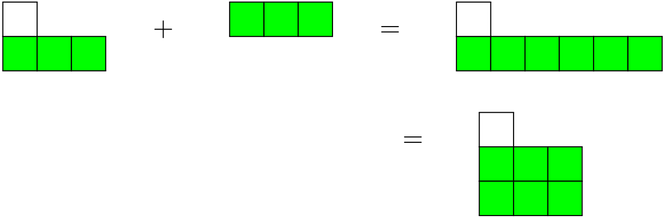
I. 4_1 :



II. $3_1 + 1_1$:

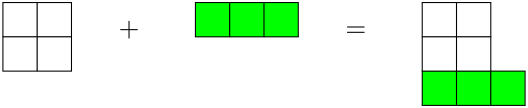


III. $3_2 + 1_1$:



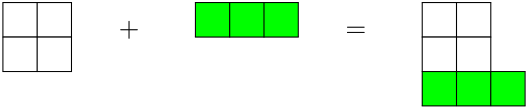
BSP in color partitions

IV. $2_1 + 2_1$:

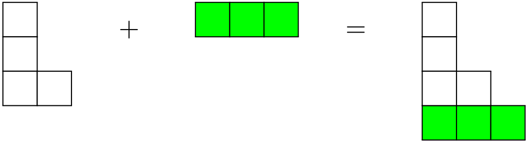


BSP in color partitions

IV. $2_1 + 2_1$:

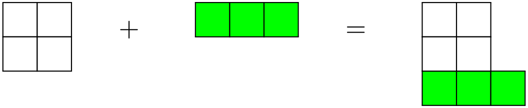


V. $2_1 + 1_1 + 1_1$:

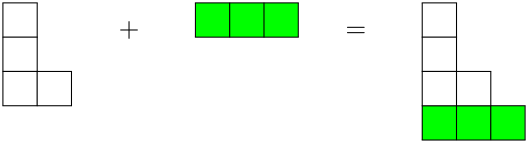


BSP in color partitions

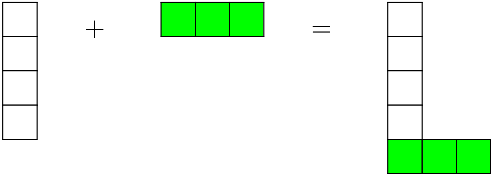
IV. $2_1 + 2_1$:



V. $2_1 + 1_1 + 1_1$:



VI. $1_1 + 1_1 + 1_1 + 1_1$:



Generalization of Stanley's theorem in color context

Theorem (B., Dastidar (2019))

For positive integers $\ell \geq 2, k$ *and* $n \in \mathbb{Z}_{>0}$,

$$Q_1^{(\ell)}(n) = \begin{cases} (Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1))/2, & \text{if } \ell \mid k, \\ Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1), & \text{if } \ell \nmid k. \end{cases}$$

Theorem (B., Dastidar (2019))

For positive integers $\ell \geq 2, k$ *and* $n \in \mathbb{Z}_{>0}$,

$$Q_1^{(\ell)}(n) = \begin{cases} (Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1))/2, & \text{if } \ell \mid k, \\ Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1), & \text{if } \ell \nmid k. \end{cases}$$

- Ex: For $n = 4$, $\ell = 2$ and $k = 2$; we have $Q_1^{(2)}(4) = 9$, $Q_2^{(2)}(4) = 8$ and $Q_2^{(2)}(5) = 10$. On the other hand, with the same example but $k = 3$; we have $Q_1^{(2)}(4) = 9$, $Q_3^{(2)}(4) = 1$, $Q_3^{(2)}(5) = 3$ and $Q_3^{(2)}(6) = 5$.

Generalization of Stanley's theorem in color context

Theorem (B., Dastidar (2019))

For positive integers $\ell \geq 2$, k *and* $n \in \mathbb{Z}_{>0}$,

$$Q_1^{(\ell)}(n) = \begin{cases} (Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1))/2, & \text{if } \ell \mid k, \\ Q_k^{(\ell)}(n) + Q_k^{(\ell)}(n+1) + \cdots + Q_k^{(\ell)}(n+k-1), & \text{if } \ell \nmid k. \end{cases}$$

- Ex: For $n = 4$, $\ell = 2$ and $k = 2$; we have $Q_1^{(2)}(4) = 9$, $Q_2^{(2)}(4) = 8$ and $Q_2^{(2)}(5) = 10$. On the other hand, with the same example but $k = 3$; we have $Q_1^{(2)}(4) = 9$, $Q_3^{(2)}(4) = 1$, $Q_3^{(2)}(5) = 3$ and $Q_3^{(2)}(6) = 5$.
- The above theorem follows from the following recursion: If $\ell \geq 2, k, n \in \mathbb{Z}_{>0}$, then

$$\frac{Q_k^{(\ell)}(n+k)}{2} = p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2}, \quad \text{if } \ell \mid k.$$
$$Q_k^{(\ell)}(n+k) = p^{(\ell)}(n) + Q_k^{(\ell)}(n), \quad \text{otherwise.}$$

Generalization of Stanley's theorem in color context

Lemma (B., Dastidar (2019))

Adding a packet of k boxes to the Young diagrams of $\lambda \in P^{(\ell)}(n)$ following the color BSP generates as many new color partitions as there are occurences of a part k in $P^{(\ell)}(n+k)$ subject to the condition that k is not a multiple of ℓ. But if k is a multiple of ℓ, then adding a packet of k boxes generates as many new color partitions which equals to half of the total number of occurences of the part k in $P^{(\ell)}(n+k)$.

Generalization of Stanley's theorem in color context

Proof sketch:

Proof sketch:

- The trivial addition of a packet of k boxes generates the number of color partitions which equals to $p^{(\ell)}(n)$.

Proof sketch:

- The trivial addition of a packet of k boxes generates the number of color partitions which equals to $p^{(\ell)}(n)$.
- Now, if $\ell \nmid k$, then following rule **A**(i), we conclude that the number of nontrivial addition of a packet of k boxes to Young diagrams is $Q_k^{(\ell)}(n)$. Therefore, total number of new generated color partitions is $p^{(\ell)}(n) + Q_k^{(\ell)}(n)$ and $p^{(\ell)}(n) + Q_k^{(\ell)}(n) = Q_k^{(\ell)}(n+k)$.

Generalization of Stanley's theorem in color context

Proof sketch:

- The trivial addition of a packet of k boxes generates the number of color partitions which equals to $p^{(\ell)}(n)$.
- Now, if $\ell \nmid k$, then following rule **A**(i), we conclude that the number of nontrivial addition of a packet of k boxes to Young diagrams is $Q_k^{(\ell)}(n)$. Therefore, total number of new generated color partitions is $p^{(\ell)}(n) + Q_k^{(\ell)}(n)$ and $p^{(\ell)}(n) + Q_k^{(\ell)}(n) = Q_k^{(\ell)}(n+k)$.
- For $\ell \mid k$, the part k in $\lambda \in P^{(\ell)}(n)$ appears with two colors. Now, adding a packet of k boxes to Young Diagrams enumerate half of the total number of occurrences of k in $P^{(\ell)}(n)$ because we add only a white colored packet of k boxes.
  So in this context, we have to count the total number of occurrences of parts k_1 and k_2 but we have chosen only one representative of k_1 and k_2 in terms of adding only a white colored packet of k boxes. Therefore, the total number of generated color partition is $p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2}$ and $p^{(\ell)}(n) + \frac{Q_k^{(\ell)}(n)}{2} = \frac{Q_k^{(\ell)}(n+k)}{2}$.

Theorem (Andrews (2019))

Let $\mathcal{O}_d(n)$ denote the number of partitions of n in which the odd parts are distinct and each positive odd integer smaller than the largest odd part must appear as a part. Then

$$p_{eu}^{od}(n) = \mathcal{O}_d(n),$$

where $p_{eu}^{od}(n)$ denotes the number of partitions of n in which each even part is less than each odd part and odd parts are distinct.

Extension of Andrews' identity

Theorem (Andrews (2019))

Let $\mathcal{O}_d(n)$ denote the number of partitions of n in which the odd parts are distinct and each positive odd integer smaller than the largest odd part must appear as a part. Then

$$p_{eu}^{od}(n) = \mathcal{O}_d(n),$$

where $p_{eu}^{od}(n)$ denotes the number of partitions of n in which each even part is less than each odd part and odd parts are distinct.

Ex: The 6 partitions enumerated by $\mathcal{O}_d(9)$ are $8+1$, $6+2+1$, $5+3+1$, $4+4+1$, $4+2+2+1$, $2+2+2+2+1$ and those enumerated by $p_{eu}^{od}(9)$ are 9, $7+2$, $5+4$, $5+3+1$, $5+2+2$, $3+2+2+2$.

Definition 1:

$$P^{ou}_{eu}(n) := \left\{ \lambda \vdash n : \begin{array}{c} (1) \text{ all the odd parts of } \lambda \text{ are unrestricted,} \\ (2) \text{ each even part of } \lambda \text{ is less than each odd part of } \lambda \end{array} \right\},$$

Definition 1:

$$P_{eu}^{ou}(n) := \left\{ \lambda \vdash n : \begin{array}{c} (1) \text{ all the odd parts of } \lambda \text{ are unrestricted,} \\ (2) \text{ each even part of } \lambda \text{ is less than each odd part of } \lambda \end{array} \right\},$$

$$p_{eu}^{ou}(n) := \#\{\lambda \vdash n : \lambda \in P_{eu}^{ou}(n)\}.$$

Definition 1:

$$P^{ou}_{eu}(n) := \left\{ \lambda \vdash n : \begin{array}{c} (1) \text{ all the odd parts of } \lambda \text{ are unrestricted,} \\ (2) \text{ each even part of } \lambda \text{ is less than each odd part of } \lambda \end{array} \right\},$$

$$p^{ou}_{eu}(n) := \#\{\lambda \vdash n : \lambda \in P^{ou}_{eu}(n)\}.$$

- For example, $p^{ou}_{eu}(9) = 12$ $(9, 7+2, 7+1+1, 5+4, 5+3+1, 5+2+2, 5+1+1+1+1, 3+3+3, 3+3+1+1+1, 3+2+2+2, 3+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$.

Extension of Andrews' identity

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,
$\mathrm{OMax}(\lambda) :=$ greatest odd part of λ,

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,

$\text{OMax}(\lambda) :=$ greatest odd part of λ,

$$\text{EMax}(\lambda) := \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,

$\mathrm{OMax}(\lambda) :=$ greatest odd part of λ,

$$\mathrm{EMax}(\lambda) := \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

$\mathrm{OEMaxSum}(\lambda) := \mathrm{OMax}(\lambda) + \mathrm{EMax}(\lambda)$,

Extension of Andrews' identity

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,

$\text{OMax}(\lambda) :=$ greatest odd part of λ,

$$\text{EMax}(\lambda) := \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

$\text{OEMaxSum}(\lambda) := \text{OMax}(\lambda) + \text{EMax}(\lambda)$,

$\text{OEMaxDiff}(\lambda) := |\text{OMax}(\lambda) - \text{EMax}(\lambda)|$.

Extension of Andrews' identity

Definition 2: For $\lambda \vdash n$ such that an odd integer must appear as a part of λ,
OMax(λ) := greatest odd part of λ,

$$\text{EMax}(\lambda) := \begin{cases} \text{greatest even part of } \lambda, & \text{if even parts occur in } \lambda, \\ 0, & \text{otherwise} \end{cases}$$

OEMaxSum(λ) := OMax(λ) + EMax(λ),
OEMaxDiff(λ) := |OMax(λ) - EMax(λ)|.

Definition 3:

$$O_{\overline{u}}(n) := \left\{ \lambda \vdash n : \begin{array}{c} \text{(1) for odd k with k} < \text{OMax}(\lambda)\text{; k appears in } \lambda, \\ \text{(2) for } k \text{ odd with } \delta_k(\lambda) \geq 2; \text{ OEMaxSum } (\lambda) \leq n \end{array} \right\},$$

Extension of Andrews' identity

Definition 4:

$$\text{OEMaxDiff}^{*}(\lambda) = \min\left\{\text{OEMaxDiff}\,(\lambda^{'}) : \lambda^{'} \in \mathcal{O}_{\overline{u}}(n)\right\}.$$

Definition 4:

$\text{OEMaxDiff}^*(\lambda) = \min\{\text{OEMaxDiff}(\lambda') : \lambda' \in \mathcal{O}_{\overline{u}}(n)\}$.

$\mathcal{O}^*_{\overline{u}}(n) := \{\lambda \in \mathcal{O}_{\overline{u}}(n) : \text{OEMaxDiff}^*(\lambda)\}$.

Extension of Andrews' identity

Definition 4:

$\text{OEMaxDiff}^*(\lambda) = \min \{\text{OEMaxDiff}\,(\lambda^{'}) : \lambda^{'} \in O_{\overline{u}}(n)\}.$

$O^*_{\overline{u}}(n) := \{\lambda \in O_{\overline{u}}(n) : \text{OEMaxDiff}^*(\lambda)\}.$

- For example, $o^*_{\overline{u}}(9) = 12$ $(8+1, 6+2+1, 5+3+1, 4+4+1, 4+3+1+1, 4+2+2+1, 3+2+1+1+1+1, 2+2+2+2+1, 3+3+1+1+1, 3+1+1+1+1+1+1, 2+2+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$.

Extension of Andrews' identity

Definition 4:

$\text{OEMaxDiff}^*(\lambda) = \min\{\text{OEMaxDiff}(\lambda^{'}) : \lambda^{'} \in O_{\overline{u}}(n)\}$.

$O^*_{\overline{u}}(n) := \{\lambda \in O_{\overline{u}}(n) : \text{OEMaxDiff}^*(\lambda)\}$.

- For example, $o^*_{\overline{u}}(9) = 12$ $(8+1, 6+2+1, 5+3+1, 4+4+1, 4+3+1+1, 4+2+2+1, 3+2+1+1+1+1, 2+2+2+2+1, 3+3+1+1+1, 3+1+1+1+1+1+1, 2+2+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$.
- According to our definition, the partition $\lambda = (6, 1, 1, 1) \notin O^*_{\overline{u}}(9)$ but the partition $(4, 3, 1, 1) \in O^*_{\overline{u}}(9)$.

Definition 4:

$\text{OEMaxDiff}^*(\lambda) = \min\{\text{OEMaxDiff}(\lambda') : \lambda' \in O_{\overline{u}}(n)\}$.

$O^*_{\overline{u}}(n) := \{\lambda \in O_{\overline{u}}(n) : \text{OEMaxDiff}^*(\lambda)\}$.

- For example, $o^*_{\overline{u}}(9) = 12$
  $(8+1, 6+2+1, 5+3+1, 4+4+1, 4+3+1+1, 4+2+2+1, 3+2+1+1+1+1, 2+2+2+2+1, 3+3+1+1+1, 3+1+1+1+1+1+1, 2+2+1+1+1+1+1, 1+1+1+1+1+1+1+1+1)$.

- According to our definition, the partition $\lambda = (6,1,1,1) \notin O^*_{\overline{u}}(9)$ but the partition $(4,3,1,1) \in O^*_{\overline{u}}(9)$.

Theorem (B., Dastidar (2019))

$o^*_{\overline{u}}(n) = p^{ou}_{eu}(n)$

Proof Sketch: $(O^*_{\overline{u}}(n) \longrightarrow P^{ou}_{eu}(n))$

- First, consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in O^*_{\overline{u}}(n)$.

Extension of Andrews' identity

Proof Sketch: $(O^{*}_{\overline{u}}(n) \longrightarrow P^{ou}_{eu}(n))$

- First, consider the Young diagram Y_{λ} for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in O^{*}_{\overline{u}}(n)$.
- We separate λ into $\lambda^{'} = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq \ell$ and $\lambda^{''} = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq o_j \leq l$ according to the odd and even parts, respectively with corresponding Young diagrams $Y_{\lambda'}$ and $Y_{\lambda''}$.

Extension of Andrews' identity

Proof Sketch: $(O_{\bar{u}}^{*}(n) \longrightarrow P_{eu}^{ou}(n))$

- First, consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in O_{\bar{u}}^{*}(n)$.
- We separate λ into $\lambda^{'} = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq \ell$ and $\lambda^{''} = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq o_j \leq l$ according to the odd and even parts, respectively with corresponding Young diagrams $Y_{\lambda'}$ and $Y_{\lambda''}$.
- Next, we join $Y_{\lambda'}$ and $Y_{\lambda''}$ by successively adjoining their rows with respect to the ordering of the parts in $\lambda^{'}, \lambda^{''}$, respectively, starting with the largest one and end with the smallest one with restricting Young diagram, say, $Y_{\lambda'''}$.

Extension of Andrews' identity

Proof Sketch: $(O_{\overline{u}}^{*}(n) \longrightarrow P_{eu}^{ou}(n))$

- First, consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in O_{\overline{u}}^{*}(n)$.
- We separate λ into $\lambda^{'} = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq \ell$ and $\lambda^{''} = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq o_j \leq l$ according to the odd and even parts, respectively with corresponding Young diagrams $Y_{\lambda'}$ and $Y_{\lambda''}$.
- Next, we join $Y_{\lambda'}$ and $Y_{\lambda''}$ by successively adjoining their rows with respect to the ordering of the parts in $\lambda^{'}, \lambda^{''}$, respectively, starting with the largest one and end with the smallest one with restricting Young diagram, say, $Y_{\lambda'''}$.
- Now, we consider the following three cases:

Proof Sketch: $(O^*_{\bar{u}}(n) \longrightarrow P^{ou}_{eu}(n))$

- First, consider the Young diagram Y_λ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in O^*_{\bar{u}}(n)$.
- We separate λ into $\lambda^{'} = (\lambda_{o_1}, \lambda_{o_2}, \ldots, \lambda_{o_r})$ where $1 \leq o_i \leq \ell$ and $\lambda^{''} = (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_t})$ where $1 \leq o_j \leq l$ according to the odd and even parts, respectively with corresponding Young diagrams $Y_{\lambda'}$ and $Y_{\lambda''}$.
- Next, we join $Y_{\lambda'}$ and $Y_{\lambda''}$ by successively adjoining their rows with respect to the ordering of the parts in $\lambda^{'}, \lambda^{''}$, respectively, starting with the largest one and end with the smallest one with restricting Young diagram, say, $Y_{\lambda'''}$.
- Now, we consider the following three cases:
(1) If the number of odd parts is equal to the number of even parts in a partition $\lambda \in O^*_{\bar{u}}(n)$, then $Y_{\lambda'''}$ is with $\lambda^{'''} \in P^{ou}_{eu}(n)$ as for $\lambda^{'} = (\lambda_{o_1}, \ldots, \lambda_{o_r})$ and $\lambda^{''} = (\lambda_{e_1}, \ldots, \lambda_{e_r})$, the resulting partition $\lambda^{'''} = (\lambda_{o_1} + \lambda_{e_1}, \ldots, \lambda_{o_r} + \lambda_{e_r})$.

Extension of Andrews' identity

- The remaining two cases are:

- The remaining two cases are:

  (2) If number of odd parts is greater than the number of even parts in a partition $\lambda \in O^{*}_{\overline{u}}(n)$ and let the difference be t. Then a similar argument shows that the t rows in $Y_{\lambda'}$ remain left after adjoining of rows of $Y_{\lambda'}$ and $Y_{\lambda''}$. Therefore, in the resulting $Y_{\lambda'''}$ with $\lambda^{'''} \in P^{ou}_{eu}(n)$, t rows will be positioned in the same order as in $Y_{\lambda'}$.

Extension of Andrews' identity

- The remaining two cases are:

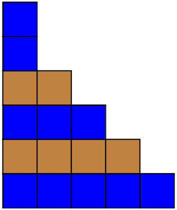
  (2) If number of odd parts is greater than the number of even parts in a partition $\lambda \in O^*_{\bar{u}}(n)$ and let the difference be t. Then a similar argument shows that the t rows in $Y_{\lambda'}$ remain left after adjoining of rows of $Y_{\lambda'}$ and $Y_{\lambda''}$. Therefore, in the resulting $Y_{\lambda'''}$ with $\lambda^{'''} \in P^{ou}_{eu}(n)$, t rows will be positioned in the same order as in $Y_{\lambda'}$.

  (3) Last, if the number of even parts is greater than the number of odd parts in a partition $\lambda \in O^*_{\bar{u}}(n)$ and let the difference be u. Similarly, we see that u rows in $Y_{\lambda''}$ remain left after adjoining the rows of $Y_{\lambda'}$ and $Y_{\lambda''}$ and here u rows will be inserted into $Y_{\lambda'}$ so that the resulting $Y_{\lambda'''}$ with $\lambda^{'''} \in P^{ou}_{eu}(n)$ does not violate the structure of the Young diagram.

For example, given Y_λ with the partition $\lambda = (5, 4, 3, 2, 1, 1) \in O^*_{\overline{u}}(16)$:

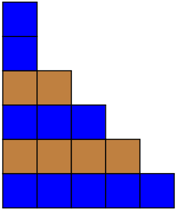
Extension of Andrews' identity

For example, given Y_λ with the partition $\lambda = (5,4,3,2,1,1) \in O_{\bar{u}}^{*}(16)$:



Extension of Andrews' identity

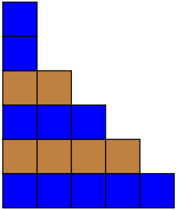
For example, given Y_λ with the partition $\lambda = (5,4,3,2,1,1) \in O_{\overline{u}}^{*}(16)$:



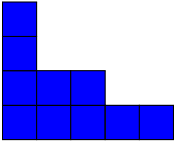
Step 1: Separating Y_λ into the odd and even parts; i.e., into $Y_{\lambda'}$ with $\lambda^{'} = (5,3,1,1)$ and $Y_{\lambda''}$ with $\lambda^{''} = (4,2)$ yields;

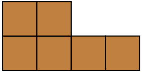
Extension of Andrews' identity

For example, given Y_λ with the partition $\lambda = (5, 4, 3, 2, 1, 1) \in O^*_{\overline{u}}(16)$:



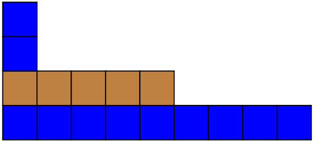
Step 1: Separating Y_λ into the odd and even parts; i.e., into $Y_{\lambda'}$ with $\lambda^{'} = (5, 3, 1, 1)$ and $Y_{\lambda''}$ with $\lambda^{''} = (4, 2)$ yields;





Step 2: Adjoining the rows of $Y_{\lambda'}$ and $Y_{\lambda''}$ gives $Y_{\lambda'''}$ with the partition $\lambda^{'''} = (9, 5, 1, 1) \in P_{eu}^{ou}(16)$;

Step 2: Adjoining the rows of $Y_{\lambda'}$ and $Y_{\lambda''}$ gives $Y_{\lambda'''}$ with the partition $\lambda''' = (9, 5, 1, 1) \in P_{eu}^{ou}(16)$;



Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\overline{u}}^{*}(n))$

- Let $\mu = (\mu_1, \ldots, \mu_s) \in P_{eu}^{ou}(n)$. Separate μ into $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with the odd parts, $\mu_{o_i} \leq \mu_{o_{i-1}} \leq \cdots \leq \mu_{o_1}$ where $\mu_{o_i} \geq \mu_s$, $\mu_{o_1} \leq \mu_1$ and into $\mu^{''}$ with the even parts.

Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\bar{u}}^{*}(n))$

- Let $\mu = (\mu_1, \ldots, \mu_s) \in P_{eu}^{ou}(n)$. Separate μ into $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with the odd parts, $\mu_{o_i} \leq \mu_{o_{i-1}} \leq \cdots \leq \mu_{o_1}$ where $\mu_{o_i} \geq \mu_s$, $\mu_{o_1} \leq \mu_1$ and into $\mu^{''}$ with the even parts.
- We keep aside the even component $Y_{\mu^{''}}$ of Y_{μ}. Next, we consider two cases:

Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\bar{u}}^{*}(n))$

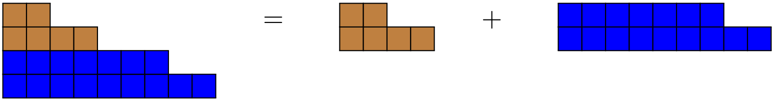
- Let $\mu = (\mu_1, \ldots, \mu_s) \in P_{eu}^{ou}(n)$. Separate μ into $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with the odd parts, $\mu_{o_i} \leq \mu_{o_{i-1}} \leq \cdots \leq \mu_{o_1}$ where $\mu_{o_i} \geq \mu_s$, $\mu_{o_1} \leq \mu_1$ and into $\mu^{''}$ with the even parts.
- We keep aside the even component $Y_{\mu^{''}}$ of Y_{μ}. Next, we consider two cases:
  (1) All odd parts of μ are distinct; i.e., there are i distinct odd values with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$. For all j $(1 \leq j \leq i)$, we extract $2j-1$ boxes from the jth row of $Y_{\mu^{'}}$ and attach $2j-1$ boxes to $Y_{\mu^{'}}$ without violating the structure of the Young diagram $Y_{\mu^{'}}$. Explicitly, we break an odd part μ_{o_t} of the partition $\mu^{'}$ into $(\mu_{o_t} - (2v-1), 2v-1)$ where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu^{'}}$. The Young diagram $Y_{\mu^{'''}}$ obtained from $Y_{\mu^{'}}$ by the above construction and adjoining $Y_{\mu^{''}}$ with it to get the unique resulting Young diagram, say Y_{π} with $\pi \in O_{\bar{u}}^{*}(n)$.

Extension of Andrews' identity

For example, Y_μ with $\mu = (9, 7, 4, 2) \in P_{eu}^{ou}(22)$ breaks into $Y_{\mu'}$ with $\mu^{'} = (9, 7)$ and $Y_{\mu''}$ with $\mu^{''} = (4, 2)$;

For example, Y_μ with $\mu = (9, 7, 4, 2) \in P_{eu}^{ou}(22)$ breaks into $Y_{\mu'}$ with $\mu' = (9, 7)$ and $Y_{\mu''}$ with $\mu'' = (4, 2)$;

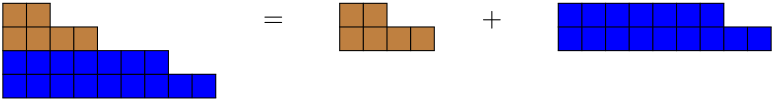
Step 1:



Extension of Andrews' identity

For example, Y_μ with $\mu = (9, 7, 4, 2) \in P_{eu}^{ou}(22)$ breaks into $Y_{\mu'}$ with $\mu' = (9, 7)$ and $Y_{\mu''}$ with $\mu'' = (4, 2)$;

Step 1:

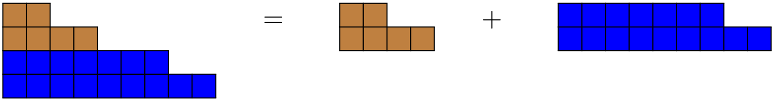


Step 2: Following the above construction, $Y_{\mu'}$ results $Y_{\mu'''}$ with $\mu''' = (1, 3, 6, 6)$;

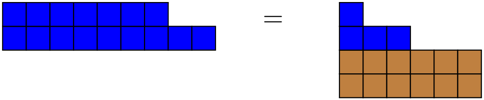
Extension of Andrews' identity

For example, Y_μ with $\mu = (9,7,4,2) \in P_{eu}^{ou}(22)$ breaks into $Y_{\mu'}$ with $\mu' = (9,7)$ and $Y_{\mu''}$ with $\mu'' = (4,2)$;

Step 1:



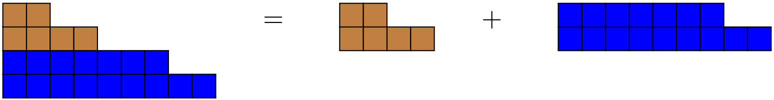
Step 2: Following the above construction, $Y_{\mu'}$ results $Y_{\mu'''}$ with $\mu''' = (1,3,6,6)$;



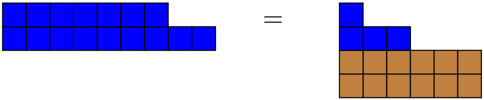
Extension of Andrews' identity

For example, Y_μ with $\mu = (9, 7, 4, 2) \in P^{ou}_{eu}(22)$ breaks into $Y_{\mu'}$ with $\mu^{'} = (9, 7)$ and $Y_{\mu''}$ with $\mu^{''} = (4, 2)$;

Step 1:



Step 2: Following the above construction, $Y_{\mu'}$ results $Y_{\mu'''}$ with $\mu^{'''} = (1, 3, 6, 6)$;

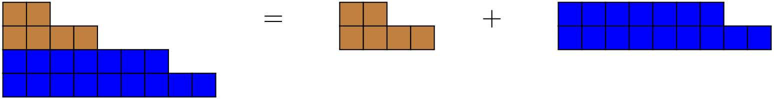


Step 3: Then the resulting diagram Y_π with $\pi = (6, 6, 4, 3, 2, 1) \in O^*_u(22)$ is the unique pre-image of μ;

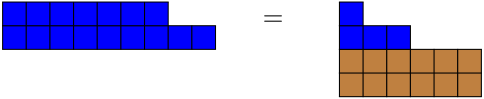
Extension of Andrews' identity

For example, Y_μ with $\mu = (9,7,4,2) \in P_{eu}^{ou}(22)$ breaks into $Y_{\mu'}$ with $\mu' = (9,7)$ and $Y_{\mu''}$ with $\mu'' = (4,2)$;

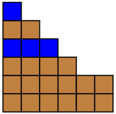
Step 1:



Step 2: Following the above construction, $Y_{\mu'}$ results $Y_{\mu'''}$ with $\mu''' = (1,3,6,6)$;



Step 3: Then the resulting diagram Y_π with $\pi = (6,6,4,3,2,1) \in O_u^*(22)$ is the unique pre-image of μ;



Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\overline{u}}^{*}(n))$

Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\bar{u}}^{*}(n))$

- The remaining case:

Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\overline{u}}^{*}(n))$

- The remaining case:

  (2) Odd parts of μ repeats; i.e., $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$ with the assumption that $\mu_{o_1}, \ldots, \mu_{o_i}$ occurs with multiplicity $k_1, k_2, \ldots, k_i$, respectively. Now, for all $1 \leq t \leq i$, we break the k_t tuple $(\mu_{o_t}, \ldots, \mu_{o_t})$ into $((\mu_{o_t} - (2v-1), 2v-1), \ldots, (\mu_{o_t} - (2v-1), 2v-1))$, where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu^{'}}$. Similar argument shows that the resulting partition, say $\pi \in O_{\overline{u}}^{*}(n)$.

Extension of Andrews' identity

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\bar{u}}^{*}(n))$

- The remaining case:

  (2) Odd parts of μ repeats; i.e., $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$ with the assumption that $\mu_{o_1}, \ldots, \mu_{o_i}$ occurs with multiplicity $k_1, k_2, \ldots, k_i$, respectively. Now, for all $1 \leq t \leq i$, we break the k_t tuple $(\mu_{o_t}, \ldots, \mu_{o_t})$ into $((\mu_{o_t} - (2v-1), 2v-1), \ldots, (\mu_{o_t} - (2v-1), 2v-1))$, where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu^{'}}$. Similar argument shows that the resulting partition, say $\pi \in O_{\bar{u}}^{*}(n)$.

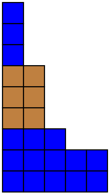
For example, the pre-image of $\mu = (7, 7, 5, 1, 1, 1) \in P_{eu}^{ou}(22)$ is $\pi = (5, 5, 3, 2, 2, 2, 1, 1, 1) \in O_{\bar{u}}^{*}(22)$;

Proof Sketch: $(P_{eu}^{ou}(n) \longrightarrow O_{\overline{u}}^{*}(n))$

- The remaining case:

  (2) Odd parts of μ repeats; i.e., $\mu^{'} = (\mu_{o_1}, \ldots, \mu_{o_i})$ with $\mu_{o_i} < \mu_{o_{i-1}} < \cdots < \mu_{o_1}$ with the assumption that $\mu_{o_1}, \ldots, \mu_{o_i}$ occurs with multiplicity $k_1, k_2, \ldots, k_i$, respectively. Now, for all $1 \leq t \leq i$, we break the k_t tuple $(\mu_{o_t}, \ldots, \mu_{o_t})$ into $((\mu_{o_t} - (2v-1), 2v-1), \ldots, (\mu_{o_t} - (2v-1), 2v-1))$, where the part μ_{o_t} corresponds to the number of boxes in the vth row of $Y_{\mu^{'}}$. Similar argument shows that the resulting partition, say $\pi \in O_{\overline{u}}^{*}(n)$.

For example, the pre-image of $\mu = (7,7,5,1,1,1) \in P_{eu}^{ou}(22)$ is $\pi = (5,5,3,2,2,2,1,1,1) \in O_{\overline{u}}^{*}(22)$;



References

- G.E. Andrews, The Theory of Partitions, Addison-Wesley Pub. Co., NY, 300 pp. (1976). Reissued, Cambridge University Press, New York, 1998.
- C. Bessenrodt, *On hooks of Young diagrams*, Annals of Combinatorics **2** (1998), 103-110.
- R. Bacher and L. Manivel, *Hooks and powers of parts in partitions*, Séminaire Lotharingien de Combinatoire **47** (2002).
- H.C. Chan, *Ramanujan's cubic continued fraction and an analogue of his most beautiful identity*, International Journal of Number Theory **06** (2010), 673–680.
- M.G. Dastidar and S. Sengupta, *Generalization of a few results in integer Partitions*, Notes in Number theory and Discrete Mathematics **19** (2013), 69-76.
- G.-N. Han, *Some conjectures and open problems on partition hook lengths*, Experimental Mathematics **18** (2009), 97-106.
- R. Honsberger, *Mathematical Gems III*, Washington, DC: Math. Assoc. Amer, 1985.

Thank you!