The Geometry of Quantum Algorithms

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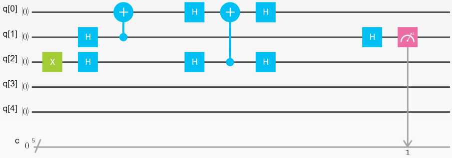
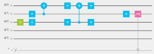
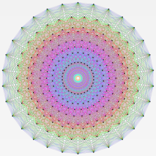


Presentation

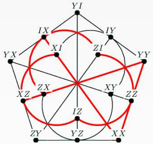
- Part I: Quantum computation, quantum communication: Entanglement, non-locality and contextuality



- Part II: The geometry of quantum states: complex projective geometry, invariants of tensors and projective duality



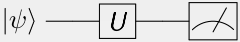
- Part III: The geometry of operators: symplectic geometry over $\mathbb{F}_2$, Kochen-Specker Theorem and Mermin's polynomials



Part I: Quantum compuation, quantum communication: Entanglement, non-locality and contextuality

Axioms of quantum computing

In quantum computation, information is encoded in a **quantum state** $|\psi\rangle \in \mathcal{H}$, it evolves by **unitary transformations** and part of the information can be recovered by **measurement**

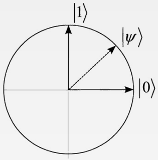


- **Axiom 1** (superposition principle): A quantum state is a unit vector in a complex vector space $\mathcal{H}$ equiped with an inner product
- **Axiom 2** (unitary evolution): The evolution of a quantum state is described by a unitary transformation
- **Axiom 3** (measurement principle): Measuring a quantum state consists in projecting the state to an orthonormal basis of $\mathcal{H}$. Given an orthonormal basis $|e_1\rangle, \dots, |e_n\rangle$ of $\mathcal{H}$ and assuming $|\psi\rangle = \sum a_i |e_i\rangle$, the probability that $|\psi\rangle$ is projected to $|e_i\rangle$ after measurement in the basis $\mathcal{B}$ is $|a_i|^2$. Morevoer after measurement $|\psi\rangle \rightsquigarrow |e_i\rangle$

Example: The qubit (vector representation)

$\mathcal{H} = \mathbb{C}^2$ with standard basis $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A qubit is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. $p(|\psi\rangle = |0\rangle) = |\alpha|^2$, $p(|\psi\rangle = |1\rangle) = |\beta|^2$

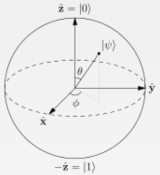


- Transformation $M \in U_2(\mathbb{C})$, example $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Pauli), $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- Example $|\psi\rangle = \frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle$. Measuring $|\psi\rangle$ in the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ basis. $p(|\psi\rangle = |+\rangle) = |\langle\psi|+\rangle|^2 = 1/2$ and $p(|\psi\rangle = |-\rangle) = |\langle\psi|-\rangle|^2 = 1/2$

The qubit (Bloch Sphere representation)

- Global phase are irrelevant

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \sim \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$



- In the Bloch pircutre, unitary transformations correspond to rotations
- An observable $\mathcal{O}$ is an hermitian operator whose eigenvalues are the outcomes of a measurement and its eigenstates are the projected state after measurement $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the observable for measuring the qubit in the Z direction (standard basis) while $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponds to measuring the qubit in the X direction ($|+\rangle, |-\rangle$ basis)

Physical example: The Spin of a particule

The first historical example of a qubit was observed in 1922 with the Stern-Gerlach experiment that shows for the first time that the angular momentum of particules is quantized[1]

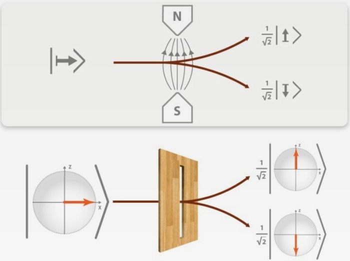


Figure: The Stern-Gerlach experiment: The spin is in a superposition but after measurement in the Z-basis, the state is projected to either “up” or “down”.

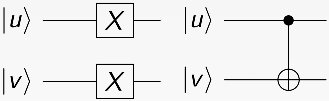
[1] *What we can learn about Quantum Physics from a single qubit*, Dür, Heusler, arxiv.1312.1463

Example: Two-qubit system

- $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$
- $|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|01\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle$
- Examples of unitary transformations

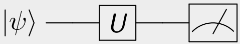
$$X \otimes X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

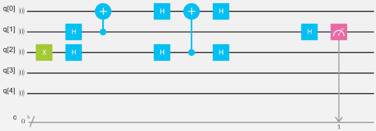
$$|u, v\rangle \mapsto |Xu, Xv\rangle \qquad \begin{array}{rcl} |0, v\rangle & \mapsto & |0, v\rangle \\ |1, v\rangle & \mapsto & |1, Xv\rangle \end{array}$$



- Measurement: $|\psi\rangle \rightsquigarrow |\psi'\rangle$ where $|\psi'\rangle$ is an element of $\mathcal{B}$ a basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$
- A measurement basis can be given as eigenvectors of $\mathcal{O}_1 \otimes \mathcal{O}_2$

Quantum computing and the circuit formalism





Theorem (D. DiVincenzio, 98)

U_2 (2×2 unitary matrices) and CNOT are universal gates for quantum computing

Theorem

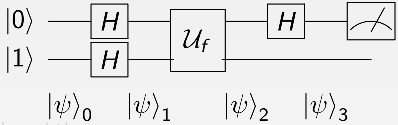
$f : \{0,1\}^n \to \{0,1\}^m$ a classical function. Then there exists a unitary matrix $U_f \in U_{2^n+2^m}$ such that

$$U_f\,|x, y\rangle = |x, y \oplus f(x)\rangle$$

Deutsch's algorithm[2]

Problem: Determine if $f : \{0,1\} \to \{0,1\}$ is constant or not.

- $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (Hadamard gate), $\mathcal{U}_f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$



- $|\psi\rangle_0 = |01\rangle = |0\rangle \otimes |1\rangle$,
- $|\psi\rangle_1 = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$
- $|\psi\rangle_2 = \begin{cases} \pm(\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \otimes (\frac{|0\rangle - |1\rangle}{\sqrt{2}}) & \text{if } f(0) = f(1) \\ \pm(\frac{|0\rangle - |1\rangle}{\sqrt{2}}) \otimes (\frac{|0\rangle - |1\rangle}{\sqrt{2}}) & \text{if } f(0) \neq f(1) \end{cases}$
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[2] D. Deutsch, *Quantum theory, the Church-Turing principle and the universal quantum computer* Proc. R. Soc. Lond 1985

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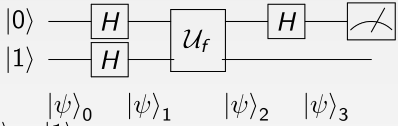

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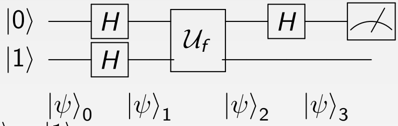

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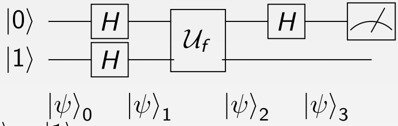

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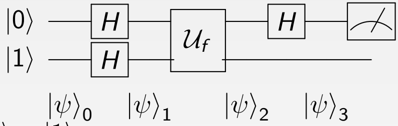

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Entanglement

One of the main resources in quantum computation that is non-classical is *Entanglement*

Example (Entangled state)

$|EPR\rangle = \frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle)$. $p(A = |0\rangle) = 0.5$, $p(B = |1\rangle) = 0.5$ but $p(A = |0\rangle, B = |1\rangle) = 0 \neq p(A = |0\rangle)p(B = |1\rangle)$

Not all two-qubit states are entangled $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is called separable state

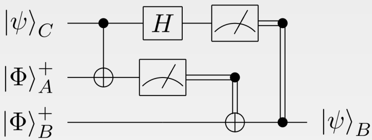
Example (Separable state)

$|\psi_{AB}\rangle = (\frac{1}{2}|0\rangle + i\frac{\sqrt{3}}{2}|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2\sqrt{2}}(|00\rangle + |01\rangle) + \frac{i\sqrt{3}}{2\sqrt{2}}(|10\rangle + |11\rangle)$, then $p(A = |0\rangle) = 1/4$, $p(B = |1\rangle) = 1/2$ and $p(AB = |01\rangle) = 1/8$

Let $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\psi\rangle \text{ is separable iff } a_{00}a_{11} - a_{01}a_{10} = 0 \Leftrightarrow \det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 0$$

Quantum Teleportation



$$|\psi\rangle_C = \alpha|0\rangle + \beta|1\rangle,\ |\Phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\otimes(|00\rangle+|11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle+\alpha|011\rangle+\beta|100\rangle+\beta|111\rangle)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

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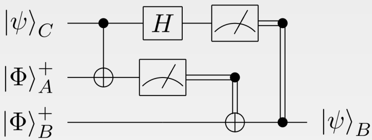

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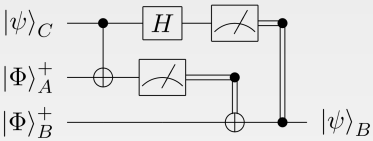

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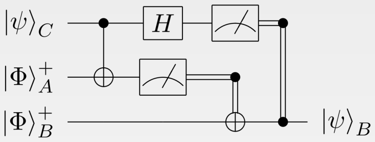

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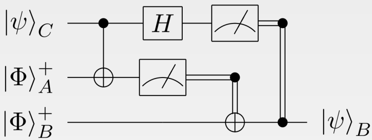

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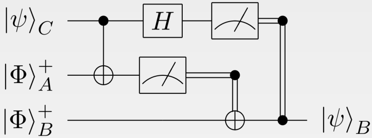
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(\alpha|{+00}\rangle + \alpha|{+11}\rangle + \beta|{-10}\rangle + \beta|{-01}\rangle)$$

$$= 1/2(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$$

$$|\psi_2\rangle = 1/2(|00\rangle\otimes(\alpha|0\rangle + \beta|1\rangle) + |01\rangle\otimes(\alpha|1\rangle + \beta|0\rangle) + |10\rangle\otimes(\alpha|0\rangle - \beta|1\rangle) + |11\rangle\otimes(\alpha|1\rangle - \beta|0\rangle))$$

$$|\psi\rangle_B = \alpha|0\rangle + \beta|1\rangle$$

Quantum Teleportation



$$|\psi\rangle_C = \alpha|0\rangle + \beta|1\rangle,\ |\Phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)\otimes(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(\alpha|{+}00\rangle + \alpha|{+}11\rangle + \beta|{-}10\rangle + \beta|{-}01\rangle)$$

$$= 1/2(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$$

$$|\psi_2\rangle = 1/2(|00\rangle\otimes(\alpha|0\rangle + \beta|1\rangle) + |01\rangle\otimes(\alpha|1\rangle + \beta|0\rangle) + |10\rangle\otimes(\alpha|0\rangle - \beta|1\rangle) + |11\rangle\otimes(\alpha|1\rangle - \beta|0\rangle))$$

$$|\psi\rangle_B = \alpha|0\rangle + \beta|1\rangle$$

Entanglement and Non-locality

When Alice measures her qubit, it fixes Bob's outcomes no matter what the distance is between them: Einstein called it *Spooky action at the distance*[3].

Recall that an **observable** is an hermitian operator that encodes the outcomes of a mesurement. For example the observable $X = |+\rangle\langle+| - |-\rangle\langle-| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ encodes the measurement of a qubit in the x-direction.

Theorem (J. Bell (1960))

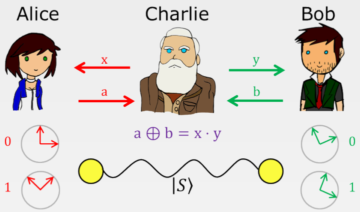
Let us consider the following two-qubit operator

$$Bell = Z \otimes (\frac{X+Z}{\sqrt{2}}) + X \otimes (\frac{X+Z}{\sqrt{2}}) + Z \otimes (\frac{Z-X}{\sqrt{2}}) - X \otimes (\frac{Z-X}{\sqrt{2}})$$

Then $\langle Bell\rangle^{LR} \leq 2$ *and* $\langle Bell\rangle^{QM} \leq 2\sqrt{2}$ *(in fact* $\langle Bell\rangle_{EPR} = 2\sqrt{2}$*)*

[3]Einstein, A., Podolsky, B., & Rosen, N. (1935). Can quantum-mechanical description of physical reality be considered complete?. Physical review, 47(10), 777.

CHSH game[4]

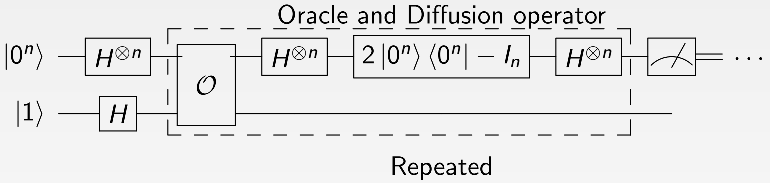


- Alice and Bob win iff $a \oplus b = x.y$
- Classical strategy $p_{AB \text{ wins}} \leq 0.75$, Quantum strategy $p_{AB \text{ wins}} \approx 0.85$:
  - Alice and Bob shares an EPR state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - If $x = 0$ Alice mesures her qubit in the Z-basis, if $x = 1$ she measures in the X-basis. Then she sends $a = 0$ to Charlie if she got $|0\rangle$ or $|+\rangle$ and she sends $a = 1$ if she got $|1\rangle$ or $|-\rangle$
  - If $y = 0$ Bob measures his qubit in the $\frac{Z+X}{\sqrt{2}}$-basis, if $y = 1$ he measures in the $\frac{Z-X}{\sqrt{2}}$-basis and sends b the result of his measurement

[4] Clauser, Horne, Shimony and Holt, *Proposed experiment to test local hidden-variable theories*. Phys. Rev. Letters 1969

Grover's quantum algorithm[5]

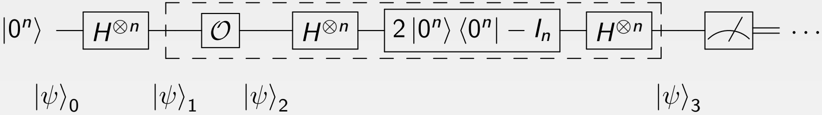
Problem: Find a "marked" element in an unsorted database of $N = 2^n$ items.
Let f a classical function that recognizes the marked element $|x_0\rangle$



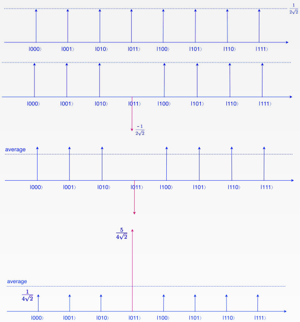
- The gate $\mathcal{O}$ (for Oracle) signs the marked element $\mathcal{O}(|x\rangle|y\rangle) = |x\rangle|y \oplus f(x)\rangle$ and thus $\mathcal{O}(|x\rangle|-\rangle) = (-1)^{f(x)}|x\rangle|-\rangle$
- The Diffusion operator symmetrizes the amplitudes of the state with respect to the mean value of the amplitudes

[5] L. Grover, *A fast quantum mechanical algorithm for database search*. Proc. of the 28th annual ACM 1996

Grover's quantum algorithm

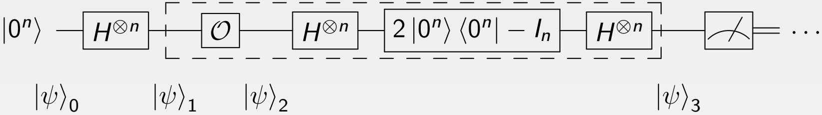


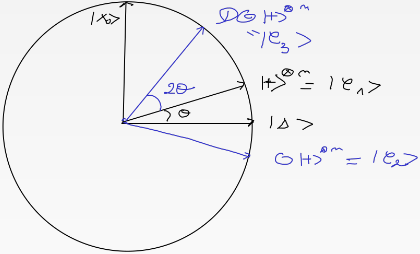
Evolution of Grover's algorithm



- $|\psi\rangle_k = \alpha|x_0\rangle + \beta\sum_{x\in\{0,..,N-1\}\backslash x_0}|x\rangle$ with $|\alpha|^2 > |\beta|^2$
- After enough rounds we have $|\alpha|^2 \approx 1 >> |\beta|^2$.

Grover's quantum algorithm: complexity





- $|s\rangle = \frac{1}{\sqrt{N-1}} \sum_{x=0, x\neq x_0}^{N-1} |x\rangle$
- For $N = 2^n$ large $\sin(\theta) = 1/\sqrt{N}$, i.e. $\theta \approx 1/\sqrt{N}$
- Oracle+Diffusion=rotation by angle 2θ
- After k iterations $(D\mathcal{O})^k |+^{\otimes n}\rangle = \cos((2k+1)\theta)|s\rangle + \sin((2k+1)\theta)|x_0\rangle$
- $\sin((2k+1)\theta) \approx 1 \Leftrightarrow (2k+1)\theta \approx \pi/2$

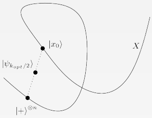
Conclusion $k \approx \frac{\pi\sqrt{N}}{4}$, complexity in $\mathcal{O}(\sqrt{N})$ compared to $\mathcal{O}(N)$ (classical)

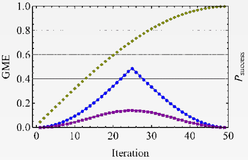
Grover's quantum algorithm

A little bit of geometry

$$\begin{aligned} |\psi\rangle_k &= \alpha_k |x_0\rangle + \beta_k \sum_{x\in\{0,\dots,N-1\}\setminus x_0} |x\rangle \\ &= (\alpha_k - \beta_k)|x_0\rangle + \beta_k \sum_{x\in\{0,\dots,N-1\}} |x\rangle = \tilde{\alpha}_k |x_0\rangle + \tilde{\beta}_k |+\rangle^{\otimes n} \end{aligned}$$

For one marked element, the states generated by Grover's algorithm $|\psi\rangle_k$ are rank[6] two tensors[7]





Running Grover's algorithm means moving on a secant line from $|+\rangle^{\otimes n}$ to the marked element. It gives qualitative interpretation of numerical results[8]

[6]Brylinsky, J.-L. *Algebraic measures of entanglement* in Mathematics of Quantum Computation. Comput. Math. Ser. Chapman and Hall (2002)

[7]H-, Jaffali, Nounouh *Grover's algorithm and the Secant varieties*, Quant. Inf. Proc. 2016

[8]Rossi, M., D. Bruß, and C. Macchiavello. *Scale invariance of entanglement dynamics in Grover's quantum search algorithm*. Physical Review A 87.2 (2013): 022331.

Studying quantum algorithms

Quantum algorithms are given by evolution of quantum states. To get a better understanding, one can ask the following questions:

- How can we characterize geometrically the states generated by a given quantum algorithm ?
- What about the non-local properties of the states generated ?

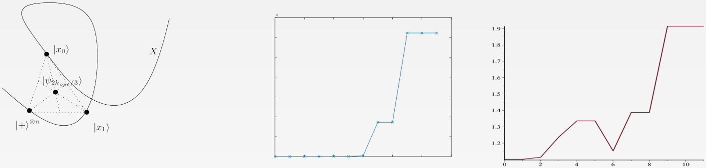


Figure: Left: *Grover's algorithm and the Secant varieties*, H-, Jaffali, Nounouh, Quantum Information Processing 2016. Middle: *Quantum Entanglement involved in Grover and Shor's algorithm: The four-qubit case*, Jaffali, H-, Quantum Information Processing 2019. Right: *Mermin Polynomials for Non-locality and Entanglement Detection in Grover's algorithm and Quantum Fourier Transform*, de Boutray, Jaffali, H-, Masson, Giorgetti, submitted.

Contextuality

A context is a set of compatible measurements, i.e. a set of mutually commuting observables

Theorem (Kochen-Specker (1967))

There is no non-contextual deterministic theory that reproduces the outcomes of quantum physics

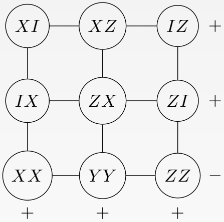
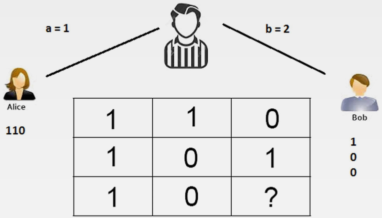


Figure: Peres-Mermin Magic square

- Each node is a two qubit operators with eigenvalues $\{-1, 1\}$
- A deterministic theory that can assign the eigenvalues and satisfy the row/column constrains should be context dependent

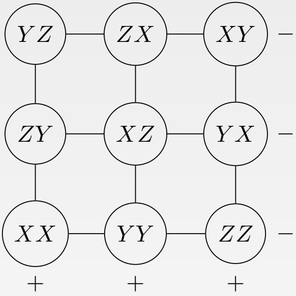
Quantum pseudo-telepathy game



1	1	0
1	0	1
1	0	?

- Charlie sends $a \in \{1,2,3\}$ to Alice and $b \in \{1,2,3\}$ to Bob
- Alice and Bob send back triplets (x_1, x_2, x_3), (y_1, y_2, y_3) , $x_i, y_j \in \{-1, 1\}$
- Alice and Bob win the game iff
  - Alice sends an odd number of -1
  - Bob sends an even number of -1
  - $x_a = y_b$
- There is no classical strategy that win the game with certainty
- However there is a quantum strategy that win the game with $p = 1$

Pseudo-telepathy game (quantum strategy)



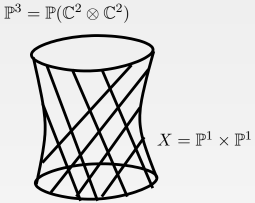
- Alice and Bob share the following quantum state

$$|EPR\rangle_{AB} \otimes |EPR\rangle_{AB} = \frac{1}{2}(|0_A 0_B 0_A 0_B\rangle + |0_A 0_B 1_A 1_B\rangle + |1_A 1_B 0_A 0_B\rangle + |1_A 1_B 1_A 1_B\rangle)$$

- Then Alice measures her two qubit system using the context corresponding to row a and Bob uses the context given by column b

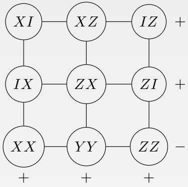
Recap: Entanglement, non-locality and contextuality

Geometry of quantum states

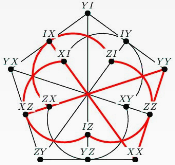


- $\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2$
- Entangled $\Leftrightarrow$ det $\neq 0$
- Separable $\Leftrightarrow$ det $= 0$

Geometry of operators



- Pauli observables
- $\mathcal{W}(3,2)$, the symplectic polar space of rank 2



Part II: The geometry of quantum states: complex projective geometry, invariants of tensors and projective duality

Entanglement of pure quantum states

- $\mathcal{H} = \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, Hilbert space for a n-partite system
- $|\psi\rangle \in \mathbb{P}(\mathcal{H})$, a quantum pure state
- SLOCC $= SL_{d_1}(\mathbb{C}) \times \cdots \times SL_{d_n}(\mathbb{C})$, reversible local operations
- LU $= U_{d_1}(\mathbb{C}) \times \dots U_{d_n}(\mathbb{C})$, local unitary transformations
- $X_{Sep} = \{|\psi\rangle \in \mathbb{P}(\mathcal{H}), |\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle\}$, the variety of separable states
- $\mathbb{P}(\mathcal{H}) \setminus X_{Sep}$, the set of entangled states

The set X_{Sep} is well-known to geometers as the Segre variety

$$Seg : \begin{array}{ccc} \mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_n-1} & \mapsto & \mathbb{P}(\mathcal{H}) \\ (v_1, \dots, v_n) & \mapsto & [v_1 \otimes \cdots \otimes v_n] \end{array} \tag{1}$$

$Seg(\mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_n-1})$ is a SLOCC closed orbit, the orbit of rank one tensors

What Mathematics (representation theory, invariant theory, geometry...) tell us about entanglement ?

How to study entanglement ?

- Representation Theory perspective: study SLOCC-orbits of $\mathcal{H}$ (Parfenov, Nurmiev Vinberg, Verstraete, Djokovic) $\rightsquigarrow$ classes of entanglement.
- Geometric perspective: study auxiliary varieties in $\mathbb{P}(\mathcal{H})$. The variety X of separable states is a SLOCC-closed orbit in $\mathbb{P}(\mathcal{H})$ all varieties built geometrically from X (secant, tangent) are SLOCC-invariant (Landsberg, Ottaviani, Heydari, H-) $\rightsquigarrow$ geometric interpretation of classes of entanglement
- Invariant theory: compute the algebras invariants/covariants (Luque, Thibon, Briand, Verstraete) $\rightsquigarrow$ algorithms to identify a given state with his class of entanglement
- Hyperdeterminant: combine geometry and invariant theory perspectives to study specific invariants (Gelfand-Kapranov-Zelevinski, Miyake, Lévay, Duff, Borsten, H-, Luque, Thibon, Jaffali, Oeding)

Three qubit classification

In 2000, the physicists Dür, Vidal and Cirac[9] proved the existence of 6 distinguished classes of entanglement under SLOCC

- The orbit of separable state $\text{SLOCC}.|000\rangle$
- Three orbits of biseparable states $\text{SLOCC}.\frac{1}{\sqrt{2}}(|100\rangle + |111\rangle)$, $\text{SLOCC}.\frac{1}{\sqrt{2}}(|010\rangle + |111\rangle)$, $\text{SLOCC}.\frac{1}{\sqrt{2}}(|001\rangle + |111\rangle)$
- The W state orbit $\text{SLOCC}.\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$
- The GHZ state orbit $\text{SLOCC}.\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

This paper got a lot of attention in the quantum physics literature (citations=3338)

[9] Dür, W., Vidal, G. and Cirac, J.I., (2000). Three qubits can be entangled in two inequivalent ways. Physical Review A, 62(6), p.062314.

Secant and tangential varieties

Let $X \subset \mathbb{P}(V)$ a projective variety. Define the secant and tangential varieties

- The secant variety $\sigma(X) = \overline{\cup_{x,y}\mathbb{P}^1_{xy}}$
- The tangential variety $\tau(X) = \cup_{x\in X} T_x X$

Example

$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ then $\sigma(X) = \mathbb{P}(\overline{\text{SLOCC}.|GHZ\rangle})$ $\tau(X) = \mathbb{P}(\overline{\text{SLOCC}.|W\rangle})$

Theorem (Fulton-Hansen (1979))

Let $X \subset \mathbb{P}(V)$ a projective variety of dimension d, then one of the two following situations holds

- *either $dim(\sigma(X)) = 2d+1$ and $\tau(X) \subsetneq \sigma(X)$ and $dim(\tau(X)) = 2d$,*
- *or $dim(\sigma(X)) < 2d+1$ and $\sigma(X) = \tau(X)$*

Remark

More secant varieties can be defined $\sigma_k(X) = \overline{\cup_{x_1,\dots,x_k\in X}\mathbb{P}^{k-1}_{x_1,\dots,x_k}}$

Three qubit classiciation (revisited I)

The fact that $|GHZ\rangle$ and $|W\rangle$ define two distinguished classes of entanglement follows from Fulton-Hansen's Theorem

$$\mathbb{P}(\overline{\mathrm{SLOCC}.|GHZ\rangle}) = \sigma_2(X_{\mathrm{Sep}}) = \mathbb{P}^7$$

$$\mathbb{P}(\overline{\mathrm{SLOCC}.|W\rangle}) = \tau(X_{\mathrm{Sep}})$$

$$\mathbb{P}(\overline{\mathrm{SLOCC}.|B_1\rangle}) = \mathbb{P}^1 \times \mathbb{P}^3 \qquad \mathbb{P}(\overline{\mathrm{SLOCC}.|B_2\rangle}) \qquad \mathbb{P}(\overline{\mathrm{SLOCC}.|B_3\rangle}) = \mathbb{P}^3 \times \mathbb{P}^1$$

$$\mathbb{P}(\mathrm{SLOCC}.|000\rangle) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

Le Paige in 1881[10] computed the algebra of covariants of trilinear forms in binary variables.
With its result, one can separate[11] the different orbits by evaluating the vector $\langle [B_x], [B_y], [B_z], [C], [\Delta_{222}] \rangle$ where Δ_{222} is the quartic invariant of $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2]^{\mathrm{SLOCC}}$, C is a covariant polynomial of degree 3 and B_x, B_y and B_z are covariants of degree 2.

[10] Le Paige, C. (1881). Bull. Acad. Roy. Sci. Belgique (3), bf, 2, 40-53.

[11] H-, Luque and Thibon, 2012. Geometric descriptions of entangled states by auxiliary varieties. Journal of mathematical physics, 53(10), p.102203.

Hyperdeterminants

The quartic invariant Δ_{222} is Cayley hyperdeterminant. Hyperdeterminants generalize the notion of determinant and can be defined geometrically,

Definition

Let $X \subset \mathbb{P}(V)$ be a projective variety, the dual of X is the variety of $\mathbb{P}(V^*)$ defined by

$$X^\vee = \overline{\{H \in \mathbb{P}(V^*), \exists x \in X_{\text{smooth}},\ T_x X \subset H\}} \quad (2)$$

When $X^\vee$ is a hypersurface, its defining equation is called the X-discriminant Δ_X.

Example

If $X = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_n} \subset \mathbb{P}^{(d_1+1)\dots(d_n+1)-1}$ with $d_i \leq \sum_{j\neq i} d_j$ then $X^\vee$ is an irreducible hypersurface and we call $\Delta_X = \text{Det}_{d_1+1,\dots,d_n+1}$ the hyperdeterminant of format $(d_1+1)\times(d_2+1)\times\cdots\times(d_n+1)$

Hyperdeterminant (the $2 \times 2 \times 2$)

In 1845 Cayley computed the hyperdeterminant for a hypermatrix of format $2 \times 2 \times 2$

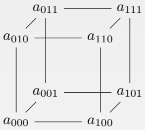


Figure: A $2 \times 2 \times 2$ matrix $A = (a_{ijk})$

$$
\begin{aligned}
\mathrm{Det}(A) \quad &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 \\
&-2a_{000}a_{001}a_{110}a_{111} - 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} \\
&-2a_{001}a_{010}a_{101}a_{110} - 2a_{001}a_{011}a_{110}a_{100} - 2a_{010}a_{011}a_{101}a_{100} \\
&+4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}
\end{aligned}
\tag{3}
$$

This polynomial is irreducible, SLOCC invariant and so is its singular locus

Three qubit classification (revisited II)

Miyake[12] points out that this classification can be obtained by considering Det_{222} and its singular locus which was studied by Weyman and Zelevinsky[13]

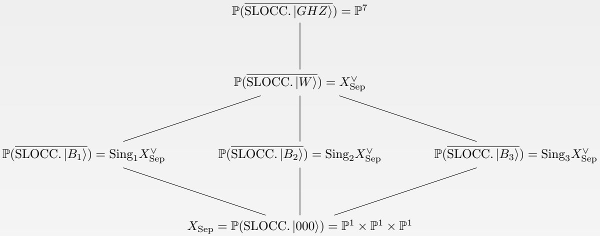


Figure: Stratification of $\mathbb{P}^7 = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$

[12] Miyake, A. (2003). Physical Review A, 67(1), 012108.
[13] Weyman, J., & Zelevinsky, A. (1996). Annales de l'institut Fourier (Vol. 46, No. 3, pp. 591-644).

More Hyperdeterminants and more Quantum information

Finding explicit equations of dual varieties is difficult in general. Interestingly a lot of them have interpretation in terms of quantum information

$\mathcal{H}$	G	Δ	Computable	Authors	Quantum Inf
$\text{Sym}^m(\mathbb{C}^n)$	SL_n	Discr.	$n = 2$	Sylvester	Symmetric qubits
$\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$	$SL_{d_1} \times \cdots \times SL_{d_n}$	HDet	(m, m)		Multiqubit
			$(2, 2, 2)$	Cayley	systems of format
			$(2, 2, 3)$		$(d_1, \ldots, d_n)$
			$(3, 3, 3)$	Schläfli/Bremner-Hu-Oeding	
			$(2, 2, 2, 2)$	Schläfli/Luque-Thibon	
$\bigwedge^k \mathbb{C}^n$	SL_n	HPfaff	$(k, n) = (2, n)$		k-fermions
			$(3, 6)$		with n single-particle states
			$(3, 7)$	Lévay-Sarosi	
			$(3, 8)$	Lévay-Sarosi	
			$(3, 9)$	H-, Oeding	
			$(4, 8)$	H-, Oeding	
V_{56}	E_7	quartic		Cartan Duff-Ferrara/Lévay	Tripartite entanglement of seven qubits

HPfaff and fermionic systems

Fermions are indistinguishable particles that are skew-symmetric, i.e. the wave function describing the system of two (or more) fermions picks up a phase factor $(-)$ when we exchange two particles. For pure system with n-single particle states one has

- $\mathcal{H} = \bigwedge^k \mathbb{C}^n$
- SLOCC=SL_n
- The Grassmannian variety $G(k,n) \subset \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is the set of separable fermions

Recall that the Grassmannian variety is defined by its Plücker embedding,

$$\{v_1, \ldots, v_k\} \longmapsto [v_1 \wedge v_2 \wedge \cdots \wedge v_k] \quad (4)$$

The dual of $G(k,n)$, when it is a hypersurface, is given by an invariant polynomial: The hyperpfaffian $\text{HPfaff}_{k,n}$

The duals of $G(3,9)$ and $G(4,8)$

With Luke Oeding[14], we computed $\mathrm{HPfaff}_{3,9}$ and $\mathrm{HPfaff}_{4,8}$ the defining equations of the duals of $G(3,9)$ and $G(4,8)$. Those computations were possible because

- The rings of invariant polynomials are finitely generated[15] $\mathbb{C}[\bigwedge^3 \mathbb{C}^9]^{SL_9(\mathbb{C})} = \mathbb{C}[f_{12}, f_{18}, f_{24}, f_{30}]$ and $\mathbb{C}[\bigwedge^4 \mathbb{C}^8]^{SL_8(\mathbb{C})} = \mathbb{C}[f_2, f_6, f_8, f_{10}, f_{12}, f_{14}, f_{18}]$
- In both cases ($\bigwedge^3 \mathbb{C}^9$ and $\bigwedge^4 \mathbb{C}^8$) the number of orbits is infinite but there is classification depending on parameters. In particular a description of a Cartan subspace of semi-simple elements is known in both cases[16]
- The degree of both duals are not too big[17] $\deg(\mathrm{HPfaff}_{3,9}) = 120$ and $\deg(\mathrm{HPfaff}_{4,8}) = 126$

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[14] H-, Oeding. (2018). Hyperdeterminants from the E_8 discriminant. arxiv.1810.05857

[15] Katanova, A.A., 1992. Explicit form of certain multivector invariants. Lie groups, their discrete subgroups, and invariant theory, Adv. Soviet Math, 8, pp.87-93.

[16] L. V. Antonyan,Classification of four-vectors of an eight-dimensional space, Trudy Sem. Vektor. Tenzor.Anal.20(1981), 144–161. - E. B. Vinberg and A. G. Èlasvili, A classification of the three-vectors of nine-dimensional space, Trudy Sem. Vektor. Tenzor. Anal.18(1978), 197–233.

[17] Alain Lascoux,Degree of the dual of a Grassmann variety, Comm. Algebra (1981), no. 11, 1215–1225

Dual of $G(3,9)$ in terms of the fundamental invariants

$$
\begin{aligned}
\mathrm{HPfaff}_{3,9} =& f_{12}^{10} - \frac{188875}{1526823} f_{12}^{8} f_{24} - \frac{4494021876517227046 3}{2232199994248855116} f_{12}^{7} f_{18}^{2} + \frac{522717082571600510}{5022449987059924011} f_{12}^{6} f_{18} f_{30} + \frac{156259946875}{27974261679948} f_{12}^{6} f_{24}^{2} \\
&+ \frac{20955843759677134000}{15067349961179772033} f_{12}^{5} f_{18}^{2} f_{24} + \frac{113325967730636958495085217}{100918096569989 8771226274} f_{12}^{4} f_{18}^{4} - \frac{8007699664851700}{4520204988353931 6099} f_{12}^{5} f_{30}^{2} \\
&- \frac{951594557840795000}{135606149650617948297} f_{12}^{4} f_{18} f_{24} f_{30} - \frac{37339826093750}{327991224631970313} f_{12}^{4} f_{24}^{3} - \frac{4631798176278228432974860}{4541314345649544470518233} f_{12}^{3} f_{18}^{3} f_{30} \\
&- \frac{4338109872429427187 5}{244091069371112306934 6} f_{12}^{3} f_{18}^{2} f_{24}^{2} - \frac{480987578992750926 25}{15067349961179772033} f_{12}^{2} f_{18}^{4} f_{24} - \frac{1151884590176865103 9}{329340982758027804} f_{12} f_{18}^{6} \\
&+ \frac{1392403335812500}{135606149650617948297} f_{12}^{3} f_{24} f_{30}^{2} + \frac{668635746252714792530 0}{151377144854984815683 9411} f_{12}^{2} f_{18}^{2} f_{30}^{2} + \frac{140973248590625000}{12204553468555615346 73} f_{12}^{2} f_{18} f_{24}^{2} f_{30} \\
&+ \frac{351718750000}{327991224631970313} f_{12}^{2} f_{24}^{4} + \frac{2133816827644645000}{135606149650617948297} f_{12} f_{18}^{3} f_{24} f_{30} - \frac{198339133437500}{741017211205562559} f_{12} f_{18}^{2} f_{24}^{3} \\
&+ \frac{456915743822635 90}{741017211205562559} f_{18}^{5} f_{30} - \frac{32778366465625}{48591292538069676} f_{18}^{4} f_{24}^{2} - \frac{14445540571041712000}{151377144854984815683 9411} f_{12} f_{18} f_{30}^{3} \\
&- \frac{216716472500000}{12204553468555615346 73} f_{12} f_{24}^{2} f_{30}^{2} - \frac{2371961791512500}{135606149650617948297} f_{18}^{2} f_{24} f_{30}^{2} + \frac{10890275000000}{20007464702550189093} f_{18} f_{24}^{3} f_{30} \\
&- \frac{1250000000}{327991224631970313} f_{24}^{5} + \frac{34328756109890000}{4541314345649544470518233} f_{30}^{4}.
\end{aligned}
\tag{5}
$$

Dual of $G(4,8)$ in terms of the fundamental invariants

The expression of $\mathrm{HPfaff}_{4,8}$ is made of 15,942 monomials and looks like

$$\begin{aligned}\mathrm{HPfaff}_{4,8} = & -(1122855063416382069258273636706506680023766222775944934559 8\\ & 8613743812708107015862353 92/19003599762623464544744484198090 74\\ & 8804840887634298311679396814662046046877707311584472 65625)\, f_2^{63}\\ & + \cdots + (3/169051466416875407082142917861 8909)\, f_{18}^{7},\end{aligned} \quad (6)$$

Proof

The proof is based on interpolation. Let $V = \bigwedge^3 \mathbb{C}^9$ or $\bigwedge^4 \mathbb{C}^8$ and $G = SL_9(\mathbb{C})$ or $SL_8(\mathbb{C})$

1. Notice that if f in G-invariant, then $f(x) = f(x_s)$ where x_s is the semi-simple part of x
2. Choose a Cartan subspace of V,
3. Restrict the fundamental invariants to a Cartan subspace of semi-simple elements
4. Define a generic f polynomial of the degree of $X^\vee$ in terms of the fundamental invariants
5. Choose points x_s in $X^\vee$ to obtain enough equations $f(x_s) = 0$,
6. Solve the system of equation

Remark

The $G(3,9)$ case can be worked out on a regular Laptop. The $G(4,8)$ case, because the large number of monomials to consider, requires to work in modulo p arithmetic to avoid memory issues because of the large coefficient. A rationnal reconstruction was therefore necessary to obtain the coefficients

Interpretation

Once the equations $\text{HPfaff}_{3,9}$ and $\text{HPfaff}_{4,8}$ are known, we can establish an interesting connection with the E_8-discriminant

- Consider Lie algebra $\mathfrak{e}_8$ and the adjoint action of the corresponding Lie group E_8. The adjoint variety $X_{E_8} \subset \mathbb{P}(\mathfrak{e}_8)$ is the unique closed orbit (the highest weight orbit) and its dual is a hypersurface. We denote its polynomial equation by Δ_{E_8}. Let us denote by $\mathfrak{h} \subset \mathfrak{e}_8$ a Cartan subalgebra, then it is known that the expression of Δ_{E_8} restricted to $\mathfrak{h}$ is

$$\Delta_{E_8}(x_s) = \Pi_{\alpha \in R}\alpha(x_s) \qquad (7)$$

- A realization of $\mathfrak{e}_8$ is $\mathfrak{e}_8 = \bigwedge^3 \mathbb{C}^{9^*} \oplus \mathfrak{sl}_9 \oplus \bigwedge^3 \mathbb{C}^9$
- Check that $\text{HPfaff}^2_{3,9_{|\mathfrak{C}}} = \pi_{\mathfrak{C}}(\Delta_{E_{8}|\mathfrak{h}})$ where $\mathfrak{C}$ is a Cartan subspace of $\bigwedge^3 \mathbb{C}^9$ $\rightsquigarrow$ an expression of $\text{HPfaff}_{3,9}$ on semi-simple elements
- Similarly $\Delta_{E_7} = \Pi_{\alpha \in R}\alpha$ and $\mathfrak{e}_7 = \mathfrak{sl}_8 \oplus \bigwedge \mathbb{C}^8 \rightsquigarrow \text{HPfaff}_{4,8}$

E_8-discriminant

The projection argument allows us to recover from the E_8 discriminant most of the known (non-trivial) duals of homogeneous varieties. They all have a quantum physics interpretation

$$
\begin{array}{ccccccc}
\Delta_{E_8} & \longrightarrow & \Delta_{E_7}|\Delta_{E_8} & \longrightarrow & \Delta_{E_6}|\Delta_{E_7} & \longrightarrow & \Delta_{SO(8)}|\Delta_{E_6} \\
\downarrow \pi_{((\bigwedge^3 \mathbb{C}^9)^*)^\perp} & & \downarrow \pi_{((\bigwedge^4 \mathbb{C}^8)^*)^\perp} & & \downarrow \pi_{(\mathbb{C}^3)^{\otimes 3}} & & \downarrow \pi_{((\mathbb{C}^2)^{\otimes 4 *})^\perp} \\
\mathrm{HPfaff}^2_{3,9}|\pi(\Delta_{E_8}) & & \mathrm{HPfaff}_{4,8}|\pi(\Delta_{E_7}) & & \Delta^2_{(\mathbb{P}^2)^{\times 3}}|\pi(\Delta_{E_6}) & & \Delta_{(\mathbb{P}^1)^{\times 4}}|\pi(\Delta_{SO(8)}) \\
\downarrow \pi_{((\mathbb{C}^3)^{\otimes 3 *})^\perp} & & \downarrow \pi_{((\mathbb{C}^2)^{\otimes 4 *})^\perp} & & \downarrow \pi_{(Sym^3(\mathbb{C}^3)^*)^\perp} & & \downarrow \pi_{(Sym^4(\mathbb{C}^2)^*)^\perp} \\
\Delta_{(\mathbb{P}^2)^{\times 3}}|\pi(\mathrm{HPfaff}_{3,9}) & & \Delta_{(\mathbb{P}^1)^{\times 4}}|\pi(\mathrm{HPfaff}_{4,8}) & & \Delta_{v_3(\mathbb{P}^2)}|\Delta_{(\mathbb{P}^2)^{\times 3}} & & \Delta_{v_4(\mathbb{P}^1)}|\Delta_{(\mathbb{P}^1)^{\times 4}} \\
\downarrow \pi_{(Sym^3(\mathbb{C}^3)^*)^\perp} & & \downarrow \pi_{(Sym^4(\mathbb{C}^2)^*)^\perp} & & & & \\
\Delta_{v_3(\mathbb{P}^2)}|\pi(\Delta_{(\mathbb{P}^2)^{\times 3}}) & & \Delta_{v_4(\mathbb{P}^1)}|\pi(\Delta_{(\mathbb{P}^1)^{\times 4}}) & & & &
\end{array} \tag{8}
$$

Application 1: Measuring entanglement

- Gour and Wallach[18] used the absolute value of the $2 \times 2 \times 2 \times 2$ Hyperdeterminant as a measure of entanglement. They conjectured numerically the maximally 4-qubit entangled states and their result was proved analytically by Chen and Djokovic [19]. They showed that the states that maximize $|\text{Det}_{2222}|$ also maximize the α-Tsallis entropy
- We have started some numerical search[20] to maximize $|\text{Det}_{333}|, |\text{HPfaff}_{3,9}|$ and $|\text{HPfaff}_{4,8}|$
  - Can we confirm analytically the results ?
  - Is there a physical meaning with respect to others usual measures of entanglement ?
  - Do the connections between Det_{2222} and $\text{HPfaff}_{4,8}$ and the one between Det_{333} and $\text{HPfaff}_{3,9}$ manifest when we look for maximally entangled states ?

[18] Gour, G., & Wallach, N. R. (2010). All maximally entangled four-qubit states. Journal of Mathematical Physics, 51(11), 112201.

[19] Chen, L., & Đoković, D. Ž. (2013). Proof of the Gour-Wallach conjecture. Physical Review A, 88(4), 042307.

[20] Jaffali, H-, Oeding. Entanglement of Fermionic systems from Hyperpfaffian. In preparation

Application 2: Classifying entanglement by singularities

In his paper on hyperdeterminant and entanglement, Miyake noticed that the singular locus of $X^\vee$ defines SLOCC invariants subvarieties of $X^\vee$ with two main components (the cusp and node component). But we can go further and analyze the type of singularities of the hyperplane section[21]

- $H \in X^\vee_{sm} \Leftrightarrow X \cap H$ has a unique Morse singularity
- $H \in X^\vee_{cusp} \Leftrightarrow X \cap H$ has a singularity which is not Morse
- $H \in X^\vee_{node} \Leftrightarrow X \cap H$ has (at least) two Morse singularities
- $H \in X^\vee_{T} \Leftrightarrow X \cap H$ has a singularity of type (at least) T

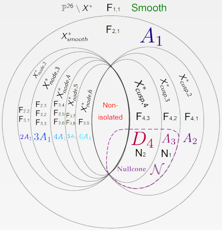


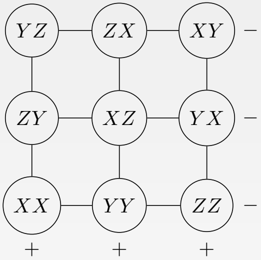
Figure: Stratification of the 3 qutrit Hilbert space by singularities

[21] H-, Luque, & Planat (2014). Singularity of type D4 arising from four-qubit systems. Journal of Physics A: Mathematical and Theoretical, 47(13), 135301 – H-., & Jaffali (2016). Three-qutrit entanglement and simple singularities. Journal of Physics A: Mathematical and Theoretical, 49(46), 465301.

Part III: The geometry of operators: symplectic geometry over $\mathbb{F}_2$, Kochen-Specker Theorem and Mermin polynomials

The Mermin square and the Kochen Specker Theorem (contextuality)

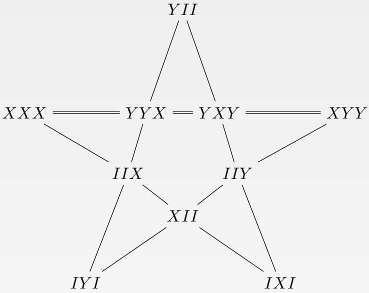
The Pauli matrices satisfy $X^2 = Y^2 = Z^2 = I$, $XY = iZ$, $YZ = iX$, $ZX = iY$



1. Lines: set of mutually commuting operators
2. There is an odd number of lines whose product gives $-I \otimes I$
3. Eigenvalues $\{+1, -1\}$ at each node of the grid
4. Impossibility to pre-assign values to the 9 observables which satisfy the signs
5. $\rightsquigarrow$ Any "Hidden variables" theory should be contextual (there is no such theory which is non-contextual)

The Mermin pentagram

Similarly with three qubits, the Mermin pentagram is a contextual configuration



The Mermin square and Mermin pentagram[22] are the smallest configurations[23] providing observables based proofs of contextuality (Kochen Specker Theorem).

[22] Mermin, N. D. (1993). Hidden variables and the two theorems of John Bell. Reviews of Modern Physics, 65(3), 803.

[23] H- & Saniga (2017). Contextuality with a small number of observables. International Journal of Quantum Information, 15(04), 1750026.

The generalized N-qubit Pauli group

One considers the subgroup P_N of $GL(2^N, \mathbb{C})$ generated by the tensor products of Pauli matrices,

$$A_1 \otimes A_2 \otimes \cdots \otimes A_N = A_1 A_2 \dots A_N$$

with $A_i \in \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$.

1. $Z(P_N) = \{\pm I, \pm iI\}$
2. $V_N = P_N/Z(P_N)$ is an Abelian group

To any class $\overline{p} \in V_N = P_N/Z(P_N)$ corresponds a unique element in $\mathbb{Z}_2^{2N}$. More precisely for any $p \in P_N$ we have $p = sZ^{\mu_1}X^{\nu_1} \otimes \dots Z^{\mu_N}X^{\nu_N}$ with $s \in \{\pm 1, \pm i\}$ and $(\mu_1, \nu_1, \dots, \mu_N, \nu_N) \in \mathbb{Z}_2^{2N}$.

Thus V_N is a $2N$ dimensional vector space over $\mathbb{Z}_2$ and we can associate to any $p \in P_N \setminus I^N$ a unique point in the projective space $PG(2N-1, 2)$

Example

For single qubit on have $I \leftrightarrow (0,0)$, $X \leftrightarrow (0,1)$, $Y \in\leftrightarrow (1,1)$ *and* $Z \leftrightarrow (1,0)$ *and we have the projective line* $PG(1,2) = \{X, Y, Z\}$.

Commutation relations

According to the previous slide for $p \in P_N$ we have $p = sZ^{\mu_1}X^{\nu_1} \otimes \dots Z^{\mu_N}X^{\nu_N}$ with $s \in \{\pm 1, \pm i\}$ and $(\mu_1, \nu_1, \dots, \mu_N, \nu_N) \in \mathbb{Z}_2^{2N}$.
Thus for $p, p' \in P_N$ we have

$$pp' = (ss'(-1)^{\sum_{j=1}^N \mu_j'\nu_j}, \mu_1 + \mu_1', \nu_1 + \nu_1', \dots, \mu_N + \mu_N', \nu_N + \nu_N').$$

Therefore:

two Pauli elements of P_N commute if and only if $\sum_{j=1}^N (\mu_j\nu_j' + \mu_j'\nu_j) = 0$.

We equiped V_N with the symplectic form

$$\langle \overline{p}, \overline{q} \rangle = \sum_{j=1}^N (\mu_j\nu_j' + \mu_j'\nu_j)$$

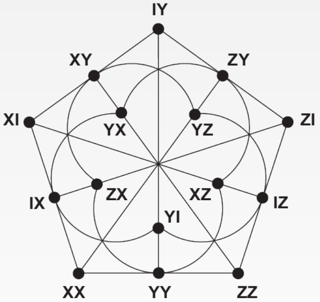
and let us denote by $\mathcal{W}(2N-1, 2)$, the symplectic polar space of rank N, the set of totally isotropic subspaces of $(PG(2N-1, 2), \langle , \rangle)$.

$W(3,2)$ aka The Doily

For two qubits Pauli group, $II \leftrightarrow (0,0,0,0)$, $XI \leftrightarrow (0,1,0,0)$, $IX \leftrightarrow (0,0,0,1)$, $XX \leftrightarrow (0,1,0,1)$, etc...

The symplectic polar space $\mathcal{W}(3,2)$ accomodates the commutation relations of the two qubit Pauli group[24]. It contains

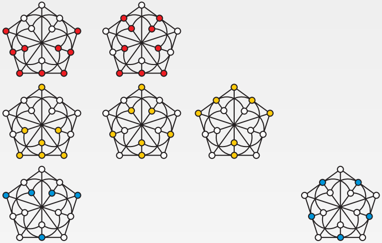
- 15 points
- 15 lines



[24] Planat, M., & Saniga, M. (2007). On the Pauli graphs of N-qudits. arXiv preprint quant-ph/0701211.

The doily and its hyperplanes

The hyperplanes of $\mathcal{W}(3,2)$ are subsets H such that all lines of the configuration is either contained in H or has a unique intersection with H



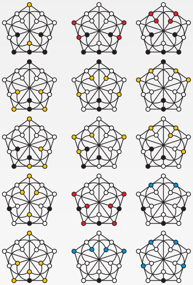
Three kinds of hyperplanes[25]:

- 10 grids (Mermin squares $\simeq GQ(1,2) \simeq Q^+(3,2)$)
- 15 perp-set
- 6 ovoïds

[25] Saniga, M., Planat, M., Pracna, P., & Havlicek, H. (2007). The Veldkamp space of two-qubits. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 3, 075.

The doily and its Veldkamp space

Let H_1, H_2 two hyperplanes, one defines $H_3 = H_1 \boxplus H_2 = \overline{H_1 \Delta H_2}$
When a geometry $\mathcal{G}$ has hyperplanes one can associate its Veldkamp space $\mathcal{V}(\mathcal{G})$, i.e. the set of its hyperplanes. One says that three hyperplanes H_1, H_2 and H_3 make a Veldkamp line iff $H_1 \boxplus H_2 = H_3$



$\mathcal{V}(\mathcal{W}(3,2))$ comprises:

- 31 points splitting in 3 orbits
- 155 lines splitting in 5 different types

One can show that $\mathcal{V}(\mathcal{W}(3,2)) \simeq PG(4,2)$

Three qubits case: $\mathcal{W}(5,2)$ and the split Cayley Hexagon

The symplectic polar space $\mathcal{W}(5,2)$ contains 63 points, 315 lines, 135 Fano planes, 12096 Magic pentagrams. There is also an embedding[26] of the split Cayley Hexagon which accomodates the 63 three-qubit operators (63 points, 63 lines)

[26] Lévay, P., Saniga, M., & Vrana, P. (2008). Three-qubit operators, the split Cayley hexagon of order two, and black holes. Physical Review D, 78(12), 124022.

Quadrics in $\mathcal{W}(5,2)$

A three qubit observables (up to $\pm 1, \pm i$) $p = Z^{\mu_1}X^{\nu_1}Z^{\mu_2}X^{\nu_2}Z^{\mu_3}X^{\nu_3}$ corresponds $(\mu_1,\nu_1,\mu_2,\nu_2,\mu_3,\nu_3) \in V_3 = \mathbb{Z}_2^6$. We define a quadratic form on V_3

$$Q_0(p) = \sum \mu_j \nu_j \quad (9)$$

One says that p is symmetric iff $Q_0(p) = 0$ (even number of Y's) and p is skew-symmetric iff $Q_0(p) = 1$ (odd number of Y's).
One gets 63 alternative quadratic forms on V_3 by considering

$$Q_q(p) = Q_0(p) + \langle q, p \rangle^2 \quad (10)$$

Two types of quadrics, hyperbolic and elliptic

$$\mathcal{Q}^+(5,2) = \{\text{quadrics parametrized by symmetric elements}\} \quad (11)$$

$$\mathcal{Q}^-(5,2) = \{\text{quadrics parametrized by skew-symmetric elements}\} \quad (12)$$

In $\mathcal{W}(5,2)$ one finds 36 hyperbolic quadrics and 28 ellipic quadrics. Each hyperbolic quadric contains 35 points and each elliptic quadrics contains 28 points

Veldkamp lines in $\mathcal{W}(2N-1,2)$

It can be proven[27] that in general hyperplanes of the symplectic polar spaces can be described as

$$C_q = \{p \in \mathcal{W}(2N-1,2), \langle p, q\rangle = 0\} \quad (13)$$

This set corresponds to "perp-set" i.e. it is the set of elements commuting with q
Or

$$H_q = \{p \in \mathcal{W}(2N-1,2), Q_q(p) = 0\} \quad (14)$$

where $Q_q(p) = Q_0(p) + \langle q, p\rangle$ with $Q_0(p) = \langle p, p\rangle$

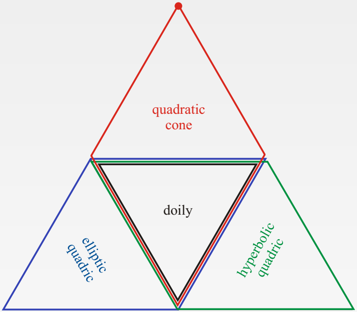
The set H_q represents the set of observables either symmetric (containing an even number of Y's) and commuting with q or anti-symmetric and anticommuting with q

One also have $C_p \boxplus C_q = C_{p+q}$, $H_p \boxplus H_q = C_{p+q}$ and $C_p \boxplus H_q = H_{p+q}$
This leads to five types of Veldkamp lines

[27] Vrana, P., & Lévay, P. (2010). The Veldkamp space of multiple qubits. Journal of Physics A: Mathematical and Theoretical, 43(12), 125303.

The Magic Veldkamp line of three qubit

Once we consider the Veldkamp line $\{H_{III}, H_{YYY}, C_{YYY}\}$, one gets the following partition of $\mathcal{W}(5,2)$,

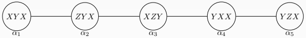


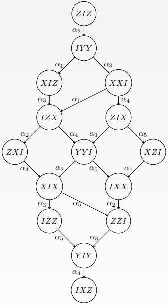
From Quantum Information to representation theory

The core set of the Veldkamp line is the set of elements commuting with YYY (they belong to C_{YYY}), symmetric (they belong to H_{III}). An explicit list of those elements is given by:

$$\begin{array}{cccccccc} YYI & YIY & IYY & ZZI & ZIZ & IZZ & XXI & XIX \\ IXX & ZXI & ZIX & IZX & XZI & XIZ & IXZ & \end{array} \quad (15)$$

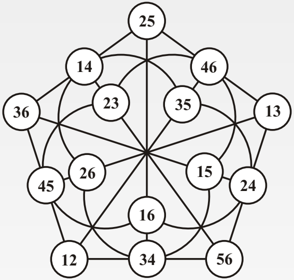
This set of operator form a Doily that encapsulates the weight diagram of the 2nd fundamental representation of A_5





From Quantum Information to representation theory

This core set also encapsulates the Pfaffian of 6×6 skew-symmetric matrices which is the invariant of the 15-irreducible representation of A_5 (SL_6)

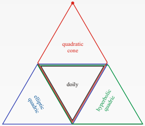


Consider the observable $\Omega = \sum_{1\leq i<j\leq 6} a_{ij}\mathcal{O}_{ij}$ where $\mathcal{O}_{ij}$ is a three qubit observable located at (ij). Then the polynomial $Tr(\Omega^3)$ is proportional to $Pf(A)$ where $A = (a_{ij})_{1\leq i<j\leq 6}$ is a skew symmetric matrix

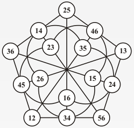
The magic Veldkamp Line and representation theory[28]

Geometry	Representation	Branching
Doily, $\mathcal{H} \cap \mathcal{E}$	15 irrep of A_5	
Quadratic cone	$1 \oplus 15 \oplus 15$ rep of A_5	
Elliptic Quadric, $\mathcal{E}$	27 irrep of E_6	$27 = 15 \oplus 6 \oplus 6$ for $A_5 \subset E_6$
Hyperbolic Quadric, $\mathcal{H}$	35 irrep of A_6	$35 = 15 \oplus 20$ for $A_5 \subset A_6$
$\mathcal{E} \Delta \mathcal{H}$	32 irrep of D_6	$32 = 20 \oplus 6 \oplus 6$ for $A_5 \subset D_6$

- A description of the extended line as a 64 dimensional irrep. of D_7
- Finite geometric description in terms of extended generalized quadrangle



- Associated invariants



$\Omega = \sum_{1 \leq i < j \leq 6} a_{ij} \mathcal{O}_{ij}$ and $A = (a_{ij})$
$Pf(A) = Tr(\Omega^3)$
$Pf(A)$ has 15 variables (nodes), 15 monomials (lines)

[28] Lévay, P., H-, & Saniga, M. (2017). Magic three-qubit Veldkamp line: A finite geometric underpinning for form theories of gravity and black hole entropy. Physical Review D, 96(2), 026018.

Bell's inequality and Mermin's polynomials

David Mermin proposes[29] in 1990 an inductive generalization of Bell's inequalities. Let us denote by $a_1, a_2, \ldots, a_k, \ldots, a'_1, a_2, dots, a'_k, \ldots$ two families of one qubit observable. Then Mermin's polynomials are defined by:

- $M_1 = a_1$
- $M_n = M_{n-1} \otimes (a_n + a'_n) + M'_{n-1} \otimes (a_n - a'_n)$

where the prime operator interchanges primed and unprimed operators.

Example

For $a_1 = Z, a_2 = X, a'_1 = \frac{X+Z}{\sqrt{2}}, a'_2 = \frac{Z-X}{\sqrt{2}}$ one gets

$$Bell = Z \otimes (\frac{X+Z}{\sqrt{2}}) + X \otimes (\frac{X+2}{\sqrt{2}}) + Z \otimes (\frac{Z-X}{\sqrt{2}}) - X \otimes (\frac{Z-X}{\sqrt{2}})$$

Mermin's inequalities:

$$\langle M_n \rangle^{LR} \leq 2^{n-1} \qquad \langle M_n \rangle^{QM} \leq 2^{3(n-1)/2}$$

[29] Mermin, N. D. (1990). Extreme quantum entanglement in a superposition of macroscopically distinct states. Physical Review Letters, 65(15), 1838.

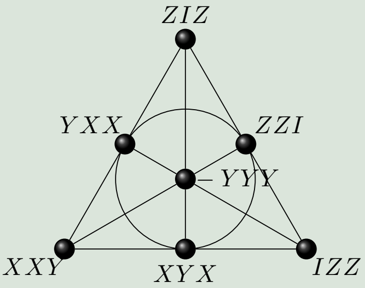
Mermin's polynomials for $n = 3$

When we restrict the choice of observables to Pauli matrices (X, Y, Z, I) the monomials of Mermin's polynomials M_3 span a Fano plane in $\mathcal{W}(5, 2)$

$$M_3 = 2(a_1 a_2 a'_3 + a_1 a'_2 a_3 + a'_1 a_2 a_3 - a'_1 a'_2 a'_3)$$

Example

Let $a_i = X$ and $a'_i = Y$
$M_3 = 2(XXY + XYX + YXX - YYY)$



This Fano plane is an isotropic subspace of maximal dimension in $\mathcal{W}(5, 2)$
The product of operator on each line gives $+Id$ and there exists a unique eigenvector $|\psi\rangle = \frac{|000\rangle + i|111\rangle}{\sqrt{2}}$ of eigenvalue $+1$ for all seven operators leading to

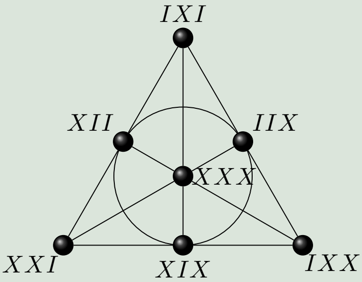
$$\langle\psi|M_3|\psi\rangle = 8 > 4$$

Mermin's polynomials for $n = 3$

Polynomials can be associated to each Fano planes of $\mathcal{W}(5,2)$ but not all of them are useful to prove non-locality

Example

Another isotropic Fano plane of $\mathcal{W}(5,2)$ is



From this Fano plane one can generate for instance the polynomial $P_3 = 2(XXI + XIX + IXX + XXX)$
Because $|\psi\rangle = |+++\rangle$ is the unique common eigenvector of eigenvalue $+1$ for the Fano plane one gets

$$\langle\psi|P_3|\psi\rangle = 8$$

But because of the signs one also sees that

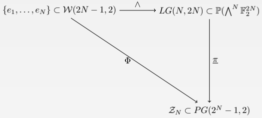
$$\langle P_3\rangle^{LR} = 8$$

The Lagrangian's mapping

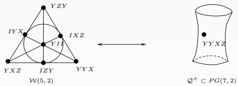
How to study isotropic plane of $\mathcal{W}(5,2)$?

Theorem (H-, Saniga, Lévay SIGMA (2014))

There exists a bijection between the set of generators of $\mathcal{W}(2N-1,2)$ and the variety of principal minors of symmetric matrices over $\mathbb{F}_2$, $\mathcal{Z}_N \subset PG(2^N-1,2)$



Sketch of the proof: Isotropic N-planes are mapped to the Lagrangian variety. The Lagrangian variety is parametrized by minors of symmetric matrices and can be projected to the variety of principal minors. Over $\mathbb{F}_2$ this map is a bijection
Planes of mutually commuting three-qubit operators $\leftrightarrow$ four-qubit symmetric operators



Orbits in PG($2^n - 1, 2$)

Bremner and Stravou [30] gave a computer classification of the $SL_2(\mathbb{F}_2)^{M\times N} \rtimes \sigma_N$ orbits of $PG(2^N - 1, 2)$ for $N = 3, 4$. In particular we have

- $\mathcal{Q}^+(7,2) = \mathcal{Z}_3$ contains 3 orbits
- $\mathcal{Z}_4$ contains 6 orbits

Leading to

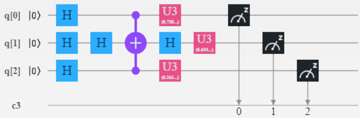
- 3 distinguish types of polynomials in the three qubit case (only one is relevant to prove non-locality)
- 6 distinguish types of polynomials of operators in the four-qubit case (probably more interesting polynomials for studying non-locality, work in progress)

[30] Bremner, M. R., & Stavrou, S. G. (2013). Canonical forms of 2× 2× 2 and 2× 2× 2× 2 arrays over 2 and 3. Linear and Multilinear Algebra, 61(7), 986-997.

Mermin's polynomials and Quantum computers

Back to quantum computer: Mermin's polynomial can be implemented on Quantum computers to test Mermin's inequalities:

- Test quantum properites of states generated by quantum algorithm[31]
- Exhibit nonlocal properties of specific quantum states[32]



Further direction of research: How to evaluate algebraic invariants with a Quantum Machine ?

[31]de Boutray, Jaffali, H-, Giorgetti & Masson, (2020). Mermin Polynomials for Entanglement Evaluation in Grover's algorithm and Quantum Fourier Transform. arXiv preprint arXiv:2001.05192.

[32]Amouzou, Boffelli, Jaffali, Atchonouglo, H-. Entanglement and Non-locality of four-qubit hypergraph states. In preparation.