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A refinement of the Murnaghan-Nakayama rule by descents for border strip tableaux

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Definition

A *partition* of *n*, $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, is a sequence of positive integers with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ whose sum $|\lambda| := \lambda_1 + \cdots + \lambda_\ell$ is *n*.

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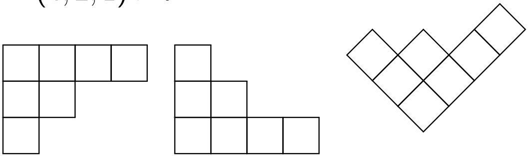
Example: $\lambda = (4, 2, 1) \vdash 7$.



English French Russian

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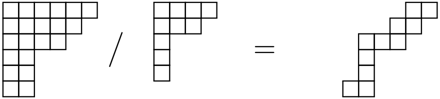
Example: $\lambda = (6, 5, 4, 2, 2, 2)$, $\mu = (4, 3, 1, 1, 1)$



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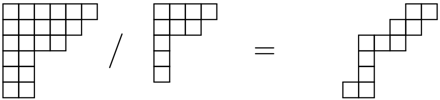

A *border strip* (ribbon, rim-hook) is a connected skew shape that does not contain a 2×2 square. The *height* $\mathrm{ht}(\lambda/\mu)$ is the number of rows the shape spans minus 1.

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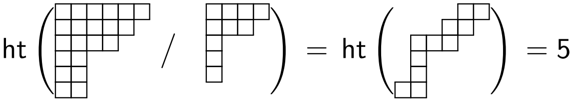

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$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 7 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

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$\mathrm{SYT}(\lambda)$ denotes the set of all standard Young tableaux of shape λ.

Descents and major index for SYT

Definition

The *descent set* $\mathrm{DES}(T)$ of a SYT T, is the set of positive integers i such that $i+1$ lies in a row strictly below the cell containing i.

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The major index generating function or *fake degree polynomial* for λ is

$$f^{\lambda}(q) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}$$

Example

1	4
2	5
3	6

q^{12}

1	3
2	5
4	6

q^{9}

1	2
3	5
4	6

q^{10}

1	3
2	4
5	6

q^{8}

1	2
3	4
5	6

q^{6}

$$f^{(2,2,2)}(q) = \sum_{T \in \mathrm{SYT}(2,2,2)} q^{\mathrm{maj}(T)} = q^{12} + q^{10} + q^{9} + q^{8} + q^{6}$$

Example

1	4	1	3	1	2	1	3	1	2
2	5	2	5	3	5	2	4	3	4
3	6	4	6	4	6	5	6	5	6
q^{12}		q^{9}		q^{10}		q^{8}		q^{6}	

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1	3
2	4
5	6

q^{8}

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3	4
5	6

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prim. root	6^{th}	3^{rd}	$2^{nd} = -1$	$1^{st} = 1$
$f^{(2,2,2)}(\cdot)$	0	2	3	5

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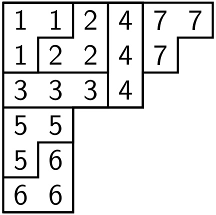

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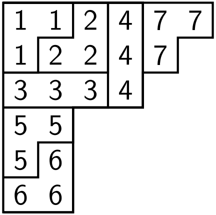

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1	2	2	4	7	
3	3	3	4		
5	5				
5	6				
6	6				

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$\mathrm{BST}(\lambda, k) \neq \emptyset \Leftrightarrow \lambda$ has *empty k-core*

$k = 1$

1	4
2	5
3	6

1	3
2	5
4	6

1	2
3	5
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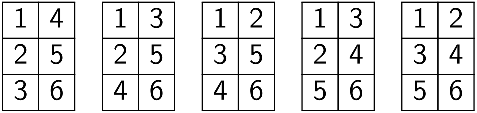
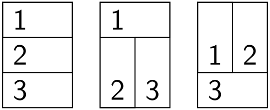


$k = 2$



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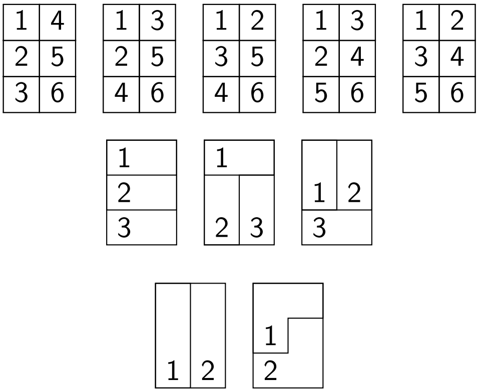

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Theorem (Murnaghan-Nakayama rule)

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Theorem (James, Kerber 1984)

For $\rho = (k^{n/k})$ *Murnaghan-Nakayama rule is cancellation free.*

$$\chi^\lambda(\rho) = (-1)^{\mathrm{ht}(T_0)} |\mathrm{BST}(\lambda, k)|$$

Theorem (Springer 1974)

Let $\lambda \vdash n$ *be a partition and* $k \mid n$. *Let* ξ *a primitive* k*-th root of unity and* $\rho = (k^{n/k})$, *then*

$$f^\lambda(\xi) = \chi^\lambda(\rho).$$

Corollary

Let $\lambda \vdash n$ be a partition and $k \mid n$. Let ξ a primitive k-th root of unity, then for some $\epsilon_{\lambda,k} \in \{\pm 1\}$

$$f^{\lambda}(\xi) = \epsilon_{\lambda,k} \cdot |\mathrm{BST}(\lambda, k)|.$$

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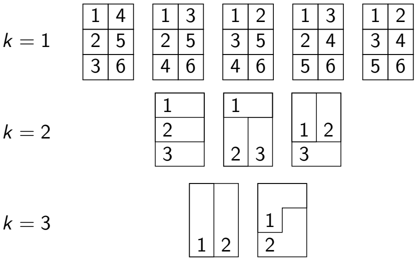

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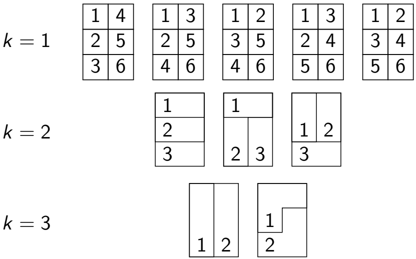

We will refine this!

$$f^{\lambda}(q, t) = \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathsf{maj}(T)} t^{|DES(T)|}$$

Refining the fake degree

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Example:

$$\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} &
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\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \\
q^{12}t^{4} & q^{9}t^{3} & q^{10}t^{3} & q^{8}t^{3} & q^{6}t^{2}
\end{array}$$

$$f^{(2,2,2)}(q, t) = q^{12}t^{4} + (q^{10} + q^{9} + q^{8})t^{3} + q^{6}t^{2}$$

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$$q^{12}t^4 \qquad q^9t^3 \qquad q^{10}t^3 \qquad q^8t^3 \qquad q^6t^2$$

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prim. root	6^{th}	3^{rd}	$2^{nd} = -1$	$1^{st} = 1$
$f^{(2,2,2)}(\cdot, t)$	$\underbrace{t^4 - 2t^3 + t^2}_{\text{6-core not empty}}$	$t^4 + t^2$	$t^4 + t^3 + t^2$	$t^4 + 3t^3 + t^2$

Write the strip labels in the leftmost cell in the last row of each strip.

Example:

$$B = \begin{array}{cccccc} 1 & 1 & 2 & 4 & 7 & 7 \\ 1 & 2 & 2 & 4 & 7 & \\ 3 & 3 & 3 & 4 & & \\ 5 & 5 & & & & \\ 5 & 6 & & & & \\ 6 & 6 & & & & \end{array}$$

Descents for BST

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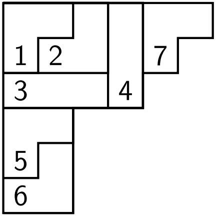

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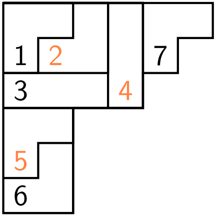
$B =$ 

$$\mathrm{DES}(B) = \{2, 4, 5\}$$

Recall: The *height* ht of a border strip is the number of rows it spans minus 1.

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Definition

Let $B \in \mathrm{BST}(\lambda, k)$ and B^1 be the strip in B containing 1.

$$\mathsf{stat}(B) = k \cdot |\,\mathsf{DES}(B)| + \mathsf{ht}(B^1)$$

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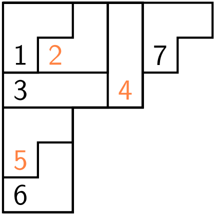
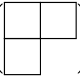
$B =$ 

$$\mathrm{DES}(B) = \{2, 4, 5\}$$

$\mathrm{ht}(B^1) = \mathrm{ht}\Big($  $\Big) = 1$

$$\mathrm{stat}(B) = 3 \cdot 3 + 1 = 10$$

$k = 1$

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$$

$$t^{1\cdot 4+0} \quad t^{1\cdot 3+0} \quad t^{1\cdot 3+0} \quad t^{1\cdot 3+0} \quad t^{1\cdot 2+0}$$

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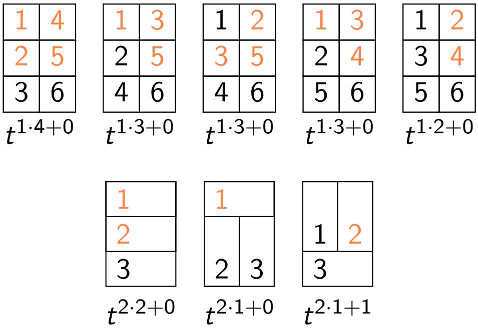

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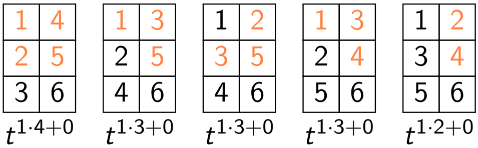

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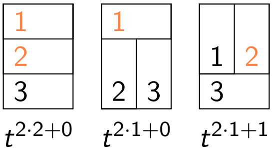

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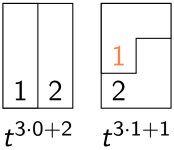

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Theorem (P 2020+)

Let λ be a partition of n with empty k-core. Let ξ a primitive k-th root of unity,

$$f^{\lambda}(\xi, t) = \epsilon_{\lambda,k} \cdot \sum_{B \in \mathrm{BST}(\lambda,k)} t^{\mathsf{stat}(B)}$$

for some $\epsilon_{\lambda,d} \in \{\pm 1\}$.

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Proof ingredients:

Theorem (P 2020+)

Let λ be a partition of n with empty k-core. Let ξ a primitive k-th root of unity,

$$f^{\lambda}(\xi, t) = \epsilon_{\lambda,k} \cdot \sum_{B \in \mathrm{BST}(\lambda,k)} t^{\mathsf{stat}(B)}$$

for some $\epsilon_{\lambda,d} \in \{\pm 1\}$.

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- Partition quotients
- Schur function identities
- A bijection
- Per Alexandersson's symmetric functions catalogue

Proof steps

$$f^\lambda(\xi,t) = \epsilon_{\lambda,k} \cdot \sum_{B \in \mathrm{BST}(\lambda,k)} t^{k\cdot|\mathsf{DES}(B)|+\mathsf{ht}(B^1)}$$

1. $\xrightarrow{k\text{-quotient}}$ standard tableau tuples:

$$\sum_{B \in \mathrm{BST}(\lambda,k)} t^{k\cdot|\mathsf{DES}(B)|+\mathsf{ht}(B^1)} = \sum_{\mathcal{T} \in \text{SYT-tuples}} t^{k\cdot|\mathsf{DES}(\mathcal{T})|+\mathsf{idx}_1(\mathcal{T})}$$

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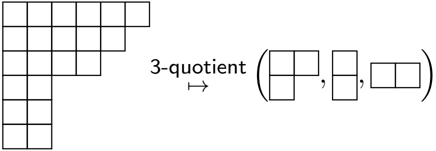
Partition quotient

Theorem (Littlewood 1951)

There exists a bijection that maps a partition λ of n with empty k-core to a tuple of partitions $(\lambda^0, \lambda^1, \dots, \lambda^{k-1})$ with $|\lambda^0| + |\lambda^1| + \cdots + |\lambda^{k-1}| = \frac{n}{k}$. The image is called k-quotient.

Example:



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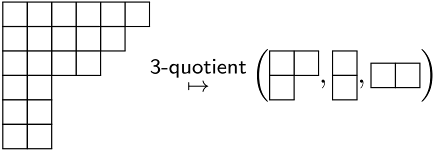
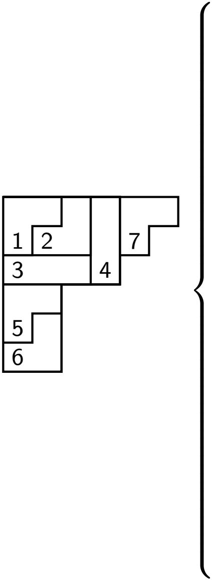


Iterating this bijection one obtains a map between $\mathrm{BST}(\lambda, k)$ and fillings of $(\lambda^0, \lambda^1, \ldots, \lambda^{k-1})$, that we call *standard tableau tuples*.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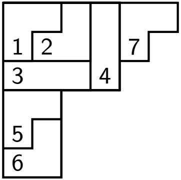
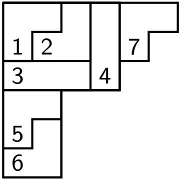


$$\left\{ \emptyset \right.$$



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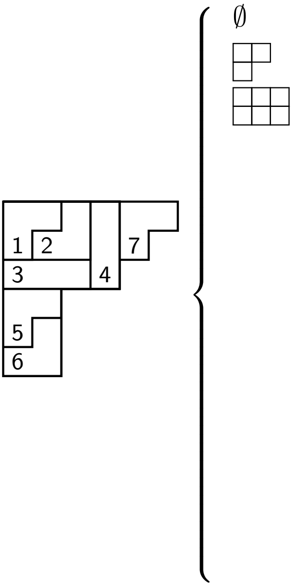
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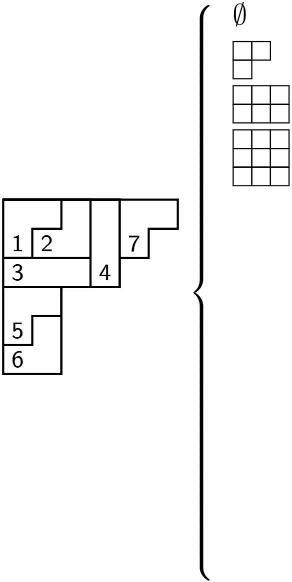
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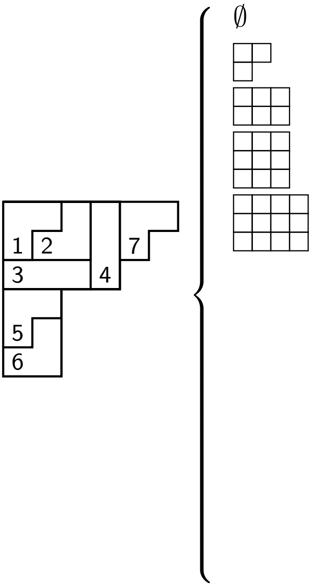

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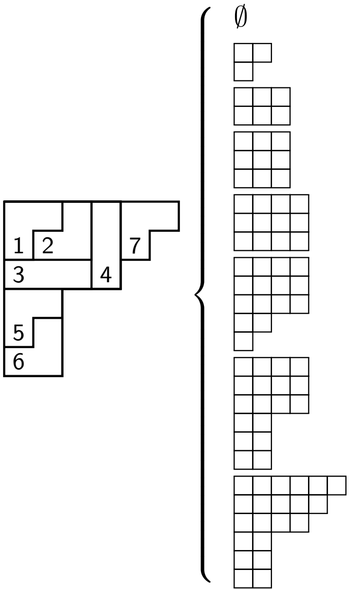
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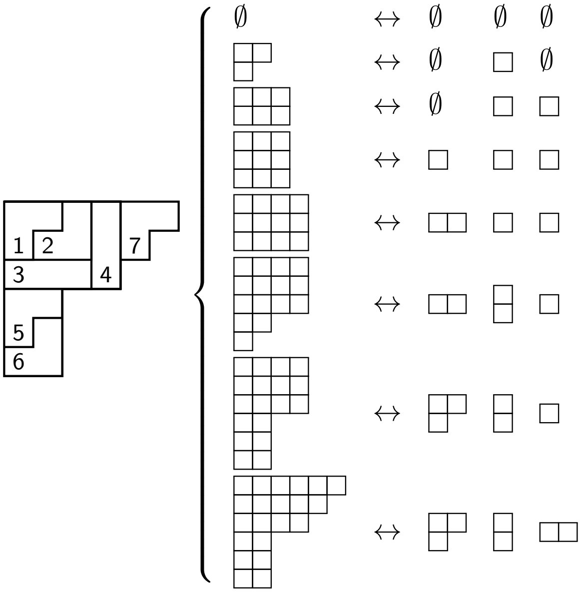
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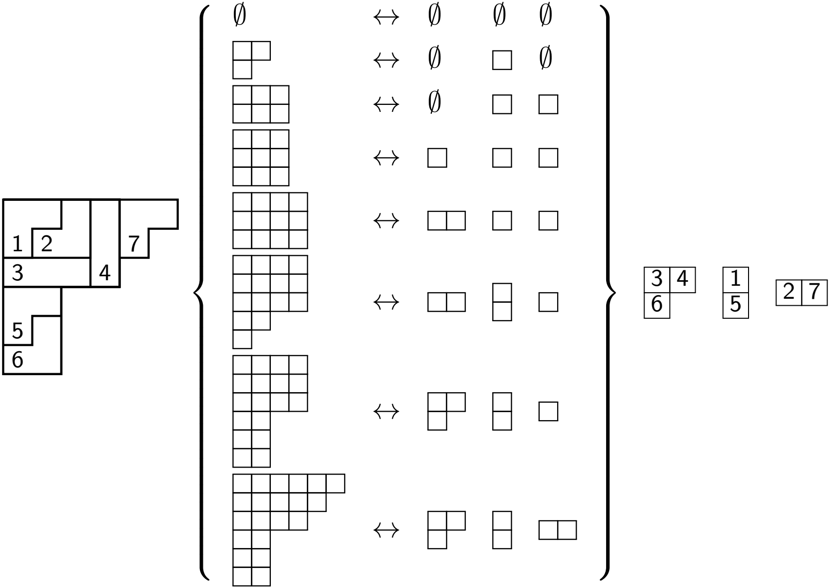
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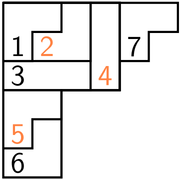
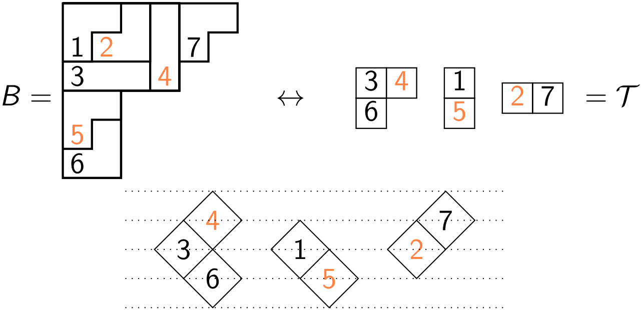
$B =$ 

$$\leftrightarrow \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 6 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 7 \\ \hline\end{array} = \mathcal{T}$$


$B =$
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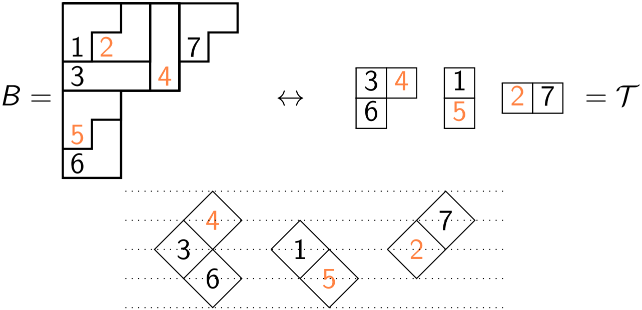
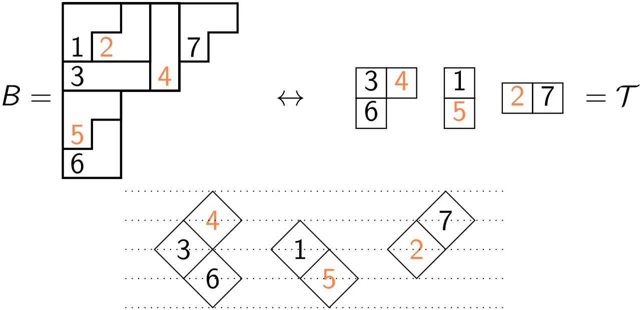


i is a descent, if and only if $(i + 1)$ is in

1. a tableau to the left and weakly below of i, or
2. the same tableau or a tableau to the right and strictly below i.



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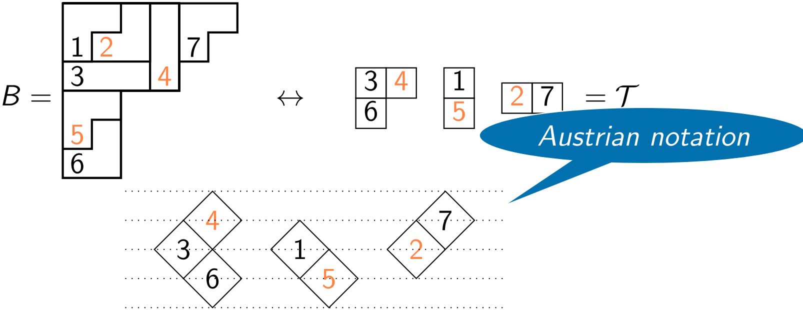
The height $\mathrm{ht}(B^1)$ is given by the index $\mathrm{idx}_1(\mathcal{T})$ of the tableau containing 1.

Recognising descents in a standard tableau tuples



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Schur functions

A *semistandard Young tableau* of shape λ is a filling of the Young diagram with weakly increasing rows and strictly increasing columns. $\mathrm{SSYT}(\lambda)$ is the (infinite) set of all semistandard Young tableaux of shape λ.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \cdots$$

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Definition

$$s_\lambda(x_0, x_1, x_2, \dots) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T$$

is the *Schur function* associated with λ.

$$\begin{array}{ccccccccccccc}
 & & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} & & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} & & \dots \\
s_{(2,1)} & = & x_0^2 x_1 & + & x_0 x_1^2 & + & x_0^2 x_2 & + & x_0 x_1 x_2 & + & x_0 x_1 x_2 & + & \dots
\end{array}$$

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$$s_\lambda(\underbrace{1, \dots, 1}_{m}, 0, 0, \dots) = s_\lambda(1^m) = \#\mathrm{SSYT} \text{ with all entries } \leq m$$

Theorem (Gessel 1993)

Let $\lambda \vdash n$ *and let* $(t;q)_{n+1} = (1-t)(1-tq)\dots(1-tq^n)$ *be the q-Pochhammer-symbol.*

$$\frac{f^\lambda(q,t)}{(t;q)_{n+1}} = \sum_{m=0}^{\infty} t^m s_\lambda(1,q,\dots,q^m)$$

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$$\frac{1}{(1-t^k)^{n/k-1}} f^\lambda(\xi,t) = (1-t)(1-t^k) \sum_{m=0}^{\infty} t^m s_\lambda(1,\xi,\dots,\xi^m)$$

Theorem (Reiner,Stanton,White 2004, P 2020+)

Let $\lambda \vdash n$ with empty k-core, $(\lambda^0, \ldots, \lambda^{k-1})$ its k-quotient and ξ a prim. k-th root of unity. If $m-1 = \ell \cdot k + r$ for $0 \leq k < r$, then

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$$(1-t)(1-t^k) \sum_{m=0}^{\infty} t^m s_\lambda(1, \xi, \ldots, \xi^m) = \epsilon_{\lambda,k} \cdot \sum_{\mathfrak{T} \in \mathrm{SSYT}\text{-tuples}} t^{k \cdot (\max(\mathfrak{T}) - 1) + \mathrm{idx}_1(\mathfrak{T})}.$$

$\max(\mathfrak{T})$ is the maximal entry and $\mathrm{idx}_1(\mathfrak{T})$ is the index of the tableau containing the leftmost 1 in $\mathfrak{T}$.

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can be shown using a bijection

$$(\text{comp. with } n/k - 1 \text{ parts}) \times \mathrm{SYT\text{-}tuples} \leftrightarrow \mathrm{SSYT\text{-}tuples}$$
$$(\alpha, \mathcal{T}) \leftrightarrow \mathfrak{T}$$

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$$\begin{aligned} (\text{comp. with } n/k - 1 \text{ parts}) \times \mathrm{SYT\text{-}tuples} &\leftrightarrow \mathrm{SSYT\text{-}tuples} \\ (\alpha, \mathcal{T}) &\leftrightarrow \mathfrak{T} \end{aligned}$$

such that

$$|\alpha| + |\,\mathrm{DES}(\mathcal{T})| + 1 = \max(\mathfrak{T}) \quad \text{and} \quad \mathrm{idx}_1(\mathcal{T}) = \mathrm{idx}_1(\mathfrak{T})$$

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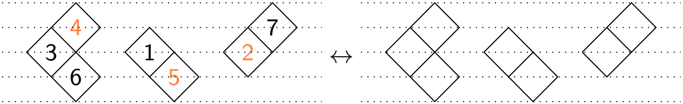

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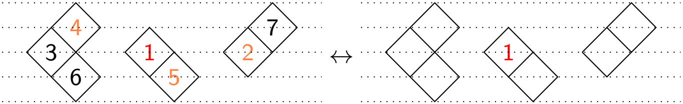
$(0,0,0,1,2,2),$ 

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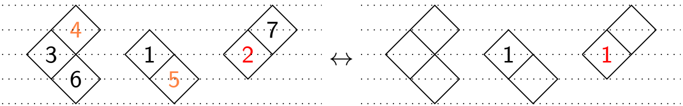
$(0, 0, 0, 1, 2, 2)$, 

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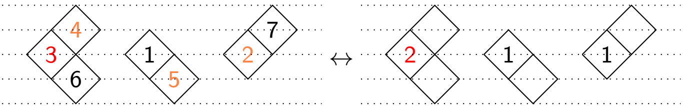

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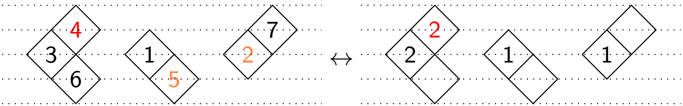
$(0, 0, 0, 1, 2, 2)$, 

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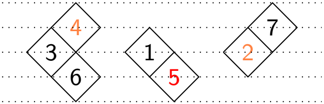
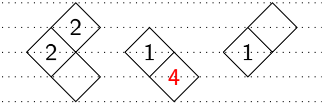
$(0,0,0,1,2,2)$, 

$\leftrightarrow$ 

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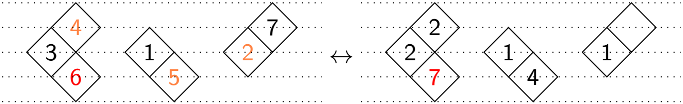

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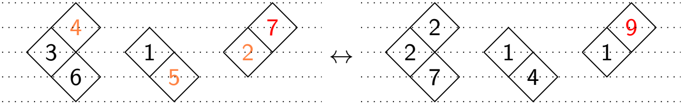

$$f^\lambda(\xi, t) = \epsilon_{\lambda,k} \cdot \sum_{B \in \mathrm{BST}(\lambda,k)} t^{k \cdot |\,\mathsf{DES}(B)| + \mathsf{ht}(B^1)}$$

1. $\xrightarrow{k\text{-quotient}}$ standard tableau tuples:

$$\sum_{B \in \mathrm{BST}(\lambda,k)} t^{k \cdot |\,\mathsf{DES}(B)| + \mathsf{ht}(B^1)} = \sum_{\mathcal{T} \in \mathrm{SYT}\text{-tuples}} t^{k \cdot |\,\mathsf{DES}(\mathcal{T})| + \mathsf{idx}_1(\mathcal{T})}$$

2. $\xrightarrow{\text{Schur function identity}}$ SSYT-tuples:

$$\frac{1}{(1 - t^k)^{n/k-1}} f^\lambda(\xi, t) = \epsilon_{\lambda,k} \cdot \sum_{\mathfrak{T} \in \mathrm{SSYT}\text{-tuples}} t^{k \cdot (\max(\mathfrak{T}) - 1) + \mathsf{idx}_1(\mathfrak{T})}$$

3. (comp. with $n/k - 1$ parts) $\times$ SYT-tuples $\leftrightarrow$ SSYT-tuples

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Let λ be a partition of n with empty k-core. Let ξ a primitive k-th root of unity,

$$f^{\lambda}(\xi, t) = \epsilon_{\lambda,k} \cdot \sum_{B \in \mathrm{BST}(\lambda,k)} t^{\mathsf{stat}(B)}$$

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THANKS!