The existence of a cyclic sieving phenomenon for permutations via a bound on the number of border strip tableaux and invariant theory

joint work with
Per Alexandersson, Stephan Pfannerer and Joakim Uhlin

8.9.2020

A mystery group action

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Part -I

preliminaries

Group actions and representations

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  $$g \cdot \phi(\vec{v}) = \phi(g \cdot \vec{v}).$$

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  A *morphism $\phi : \mathcal{X} \to \mathcal{Y}$ of group actions* is a map with
  $$g \cdot \phi(x) = \phi(g \cdot x).$$

Brauer's Permutation Lemma

- the *character of a representation* $\rho : G \to \mathrm{End}(V)$ is the map

$$\chi_\rho : \text{(conjugacy classes of) } G \to \mathbb{C}, \qquad g \mapsto \operatorname{tr} \rho(g)$$

- the *character of a group action* $\rho : G \to \mathfrak{S}_{\mathcal{X}}$ is the character of the associated 'permutation representation':

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Lemma

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Lemma

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Lemma (Brauer)

Two cyclic group actions are isomorphic if and only if they are isomorphic as linear representation.

Cyclic sieving

Given

- a finite set $\mathcal{X}$
- a cyclic group $\langle c \rangle$ of order r acting on $\mathcal{X}$
- a polynomial $f \in \mathbb{N}[q]$ such that for any $d \in \mathbb{N}$

$$f(\xi^d) = \mathrm{fix}\left(c^d\right)$$

(ξ a primitive r-th root of unity)

Then $(\mathcal{X}, \langle c \rangle, f)$ *exhibits the cyclic sieving phenomenon*.

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Then $(\mathcal{X}, \langle c \rangle, f)$ *exhibits the cyclic sieving phenomenon*.
Note that

- $f(1) = |\mathcal{X}|$
- $f(q) \mod (q^r - 1)$ is the character of the group action
- mostly, one is interested in 'nice' f

Part O
summary of results

Let $\mathrm{BST}(\lambda/\mu, k)$ be the set of border strip tableaux of shape λ/μ with strips of size k.

Theorem

$\#\mathrm{BST}(\lambda/\mu, k) \geqslant \sum_{d>1} \#\mathrm{BST}(\lambda/\mu, kd)$, *if* $\#\mathrm{BST}(\lambda/\mu, k) \geqslant 2$

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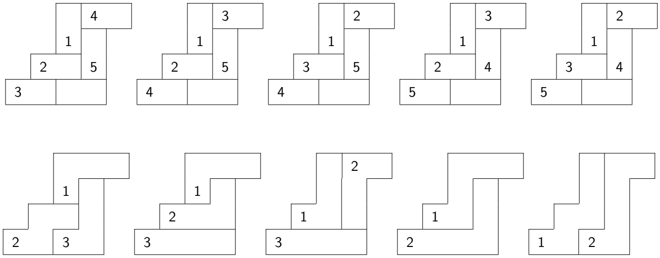
Example

$\lambda/\mu = (5, 4^3)/(2^2, 1)$, $k = 2$, English notation.



Let $\mathrm{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of shape λ/μ and let $S_{\lambda/\mu} = \bigoplus_\nu S_\nu^{\oplus c_{\mu,\nu}^\lambda}$ be the corresponding representation.

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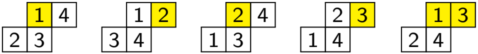
Example

Let $\lambda/\mu = (3,2)/(1)$. The character of $S_{\lambda/\mu} \otimes S_{\lambda/\mu} \downarrow_{\langle(1,\ldots,|\lambda/\mu|)\rangle}$ is



$$\Big(\sum_{T\in\mathrm{SYT}(\lambda/\mu)} q^{\mathrm{maj}(T)}\Big)^2 = (q + q^2 + q^2 + q^3 + q^4)^2$$

$$\equiv 7 + 6q + 6q^2 + 6q^3 \mod (q^4 - 1)$$

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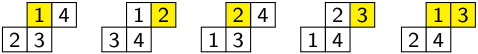

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which means that the group action has one orbit of size 1 and six orbits of size 4.

Let rot be the rotation of the chord diagram of a permutation.

Theorem

$\exists s : \mathfrak{S}_r \to$ integer partitions of size r

- $s \circ \mathrm{rot} = s$
- *s is equidistributed with the Robinson-Schensted shape*
- rot *restricted to $\mathfrak{S}_r^\lambda = \{\sigma \in \mathfrak{S}_r | s(\sigma) = \lambda\}$ has character* $f^\lambda(q)^2 = \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}\, T}\right)^2$

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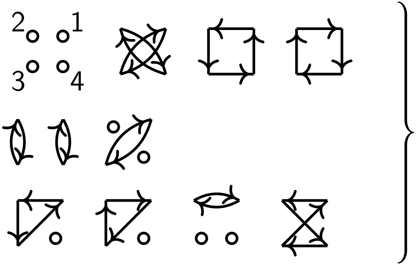

(4) :	once	[1,2,3,4]
$(3,1)$:	nine times	[1,2,4,3],...
(2^2) :	four times	[2,1,4,3],...
$(2,1^2)$:	nine times	[1,4,3,2],...
(1^4) :	once	[4,3,2,1]

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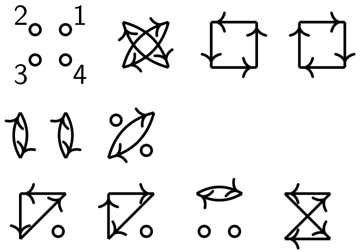

	orbit sizes	
(4) :	1	[1,2,3,4]
$(3,1)$:	4, 4, 1	[1,2,4,3],...
(2^2) :	2, 2	[2,1,4,3],...
$(2,1^2)$:	4, 4, 1	[1,4,3,2],...
(1^4) :	1	[4,3,2,1]

Part I
invariant theory

or: why bother?

Invariants of the adjoint representation of GL_n

Let V be the vector representation of GL_n:

$$\mathrm{GL}_n \to \mathrm{End}(V)$$
$$T \cdot \vec{v} = T\vec{v}$$

Let $\mathfrak{gl}_n \cong V \otimes V^*$ be the adjoint representation of GL_n:

$$\mathrm{GL}_n \to \mathrm{End}(\mathfrak{gl}_n)$$
$$T \cdot A = TAT^{-1}$$

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GL_n acts diagonally on $\mathfrak{gl}_n^{\otimes r}$:

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The *invariants* of this representation are

$$\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n} = \{\mathbf{w} \in \mathfrak{gl}_n^{\otimes r} \mid \forall\, T \in \mathrm{GL}_n : T \cdot \mathbf{w} = \mathbf{w}\}$$

They are hard to describe explicitely.

Rotation and Promotion

- let $\mathcal{A}_r^{(n)}$ be the set of *$\mathfrak{gl}_n$-highest weight words of weight zero*: sequences $(0 = \mu^0, \mu^1, \ldots, \mu^{2r} = 0)$ of vectors in $\mathbb{Z}^n$ such that
  - each vector has weakly decreasing entries
  - $\underset{(-)}{+}\mu^{i+1} \underset{(+)}{-} \mu^i$ is a unit vector for i even (odd)
- $\mathcal{A}_r^{(n)}$ is a natural indexing set for a basis of $\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n}$

e.g.

$$
\begin{array}{lll}
r = 2 & n = 1: & (0, 1, 0, 1, 0) \\
 & n = 2: & (00, 10, 00, 10, 00) \\
 & & (00, 10, 1\bar{1}, 10, 00) \\
r = 3 & n = 1: & (0, 1, 0, 1, 0, 1, 0) \\
 & n = 2: & (00, 10, 00, 10, 00, 10, 00) \\
 & & (00, 10, 1\bar{1}, 2\bar{1}, 1\bar{1}, 10, 00) \\
 & & \ldots \\
 & n = 3: & \ldots \\
 & & (000, 100, 10\bar{1}, 11\bar{1}, 10\bar{1}, 100, 000) \\
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- $\mathcal{A}_r^{(n)}$ is a natural indexing set for a basis of $\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n}$
- *promotion* is a natural (but complicated) operation on $\mathcal{A}_r^{(n)}$, isomorphic to *rotation of tensor positions* in $\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n}$

  [Westbury]
- for $n \geqslant r$, a variant of the Robinson-Schensted correspondence yields an isomorphism between promotion and *rotation of permutations as chord diagrams* [Pfannerer-R.-Westbury]

Rotation and Promotion

Theorem (Pfannerer-R.-Westbury)

There is an explicit bijection
$\mathcal{P} : \mathcal{A}_r^{(r)} \to \mathfrak{S}_r$ *with* $\mathcal{P} \circ \mathrm{pr} = \mathrm{rot} \circ \mathcal{P}$.

We want the same for $\mathcal{A}_r^{(n)}$ with $n < r$!

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Invariants of the adjoint representation of GL_n

The symmetric group $\mathfrak{S}_r$ acts on $\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n}$ permuting positions:

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Schur-Weyl duality yields the $\mathfrak{S}_r$-character of this action:

Proposition

$$\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n} \cong \bigoplus_{\substack{\lambda \vdash r \\ \ell(\lambda) \leqslant n}} S_\lambda \otimes S_\lambda,$$

where S_λ is the Specht module corresponding to λ.

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Corollary

The character of pr *on $\mathcal{A}_r^{(n)}$ is* $\displaystyle\sum_{\substack{\lambda \vdash r \\ \ell(\lambda) \leqslant n}} \Bigg(\sum_{T \in \mathrm{SYT}(\lambda)} q^{majT} \Bigg)^2.$

Invariants of the adjoint representation of GL_n

Proof.

Let V be the vector representation of GL_n

$$\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n} \quad \cong \quad \left(\mathrm{End}(V)^{\otimes r}\right)^{\mathrm{GL}_n}$$

□

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$$\begin{aligned}\left(\mathfrak{gl}_n^{\otimes r}\right)^{\mathrm{GL}_n} &\cong \left(\mathrm{End}(V)^{\otimes r}\right)^{\mathrm{GL}_n} \\ &\cong \mathrm{End}_{\mathrm{GL}_n}(V^{\otimes r})\end{aligned}$$

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Invariants of the adjoint representation of GL_n

Proof.

Let V be the vector representation of GL_n and let V_λ be the irreducible representation of GL_n in Schur-Weyl duality with S_λ.

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A combinatorial mystery

Theorem

$\exists\, \mathsf{s} : \mathfrak{S}_r \to$ *integer partitions of size r*

- $s \circ \mathrm{rot} = s$
- s *is equidistributed with the Robinson-Schensted shape* sh
- rot *restricted to* $\mathfrak{S}_r^\lambda = \{\sigma \in \mathfrak{S}_r \,|\, \mathsf{s}(\sigma) = \lambda\}$ *has character* $f^\lambda(q)^2 := \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}\, T}\right)^2$

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There is a group action ρ_λ on $\mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ with character $f^\lambda(q)^2$

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Proof.

There is a group action ρ_λ on $\mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ with character $f^\lambda(q)^2$, so $\rho := \bigoplus_{\lambda \vdash r} \rho_\lambda$ has character $\sum_{\lambda \vdash r} \left(f^\lambda(q)\right)^2$.

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By Brauer's lemma there is an isomorphism $\phi : (\mathfrak{S}_r, \mathrm{rot}) \cong (\mathfrak{S}_r, \rho)$.

Define $\mathsf{s}(\sigma) := \mathrm{sh}(\phi(\sigma))$ □

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Proof.

rot on $\mathfrak{S}_r^{(n)} := \bigcup_{\substack{\lambda \vdash r \\ \ell(\lambda) \leqslant n}} \mathfrak{S}_r^{\lambda}$ has character $\sum_{\substack{\lambda \vdash r \\ \ell(\lambda) \leqslant n}} \Big(\sum_{T \in \mathrm{SYT}(\lambda)} q^{majT} \Big)^2$.

Invariants of the adjoint representation of GL_n

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pr on $\mathcal{A}_r^{(n)}$ has the same character.

By Brauer's lemma there is an isomorphism $\mathcal{P}$ of group actions. □

Part II

border strip tableaux and the main theorems

How to recognise a cyclic group action

Proposition (Alexandersson-Amini)

Given

- *a polynomial* $f \in \mathbb{N}[q]$ *such that* $f(\xi^d) \in \mathbb{N}$ *for any* $d \in \mathbb{N}$
- $\mathcal{X}$ *any set of size* $f(1)$

Then there exists an action of $\mathbb{Z}_r$ *on* $\mathcal{X}$ *such that* $(\mathcal{X}, \mathbb{Z}_r, f)$ *exhibits the cyclic sieving phenomenon, if and only if*

$$S_k = \sum_{d|k} \mu(k/d) f(\xi^d) \geqslant 0 \quad \text{for every } k|r$$

where

$$\mu(m) = \begin{cases} (-1)^{\#\text{prime factors of } m} & \text{if } m \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

is the Möbius function.
In this case, S_k *is the number of elements in orbits of length* k*.*

How to recognise a cyclic group action

Proposition (Alexandersson-Amini)

A polynomial $f \in \mathbb{N}[q]$ such that $f(\xi^d) \in \mathbb{N}$ for any $d \in \mathbb{N}$ is the character of a cyclic group action, if and only if

$$S_k = \sum_{d|k} \mu(k/d) f(\xi^d) \geqslant 0 \quad \text{for every } k|r$$

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In this case, S_k is the number of elements in orbits of length k.

Border strip tableaux and the main theorems

Theorem

$S_{\lambda/\mu} \otimes S_{\lambda/\mu}\downarrow_{\langle(1,\ldots,|\lambda/\mu|)\rangle}$ *is isomorphic to a cyclic group action.*

We have to show that

$$\sum_{d|k} \mu(k/d) f^{\lambda}(\xi^{d})^{2} \geqslant 0 \quad \text{for every } k|r.$$

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Theorem

For a skew shape λ/μ of size r, $f^{\lambda/\mu}(q) := \sum_{T \in \mathrm{SYT}(\lambda/\mu)} q^{\mathrm{maj}\, T}$, and $d|r$ we have $|f^{\lambda/\mu}(\xi^d)| = \#\mathrm{BST}(\lambda/\mu, r/d)$.

Border strip tableaux and the main theorems

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If $\#\mathrm{BST}(\lambda/\mu, k) \geqslant 2$, then $\#\mathrm{BST}(\lambda/\mu, k) \geqslant \sum_{d>1} \#\mathrm{BST}(\lambda/\mu, kd)$.

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(therefore also $\#\mathrm{BST}(\lambda/\mu, k)^2 \geqslant \sum_{d>1} \#\mathrm{BST}(\lambda/\mu, kd)^2$)

The base case

Theorem (Fomin-Lulov)

$$\#\mathrm{BST}(\lambda, d) \leqslant \left(\frac{d^r}{\binom{r}{r/d,\ldots,r/d}} \#\mathrm{SYT}(\lambda) \right)^{1/d}$$

Lemma

$$\#\mathrm{BST}(\lambda, 1) \geqslant \sum_{d>1} \#\mathrm{BST}(\lambda, d) \qquad \textit{unless } \lambda = (r) \textit{ or } \lambda = (1^r)$$

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Proof strategy.

- check hooks and computer check partitions λ with $|\lambda| \leqslant 8$
- use Fomin-Lulov to turn inequality into a function $B_r\big(\#\mathrm{SYT}(\lambda)\big) = \sum_{d|r} Q_{r,d} \#\mathrm{SYT}(\lambda)^{\frac{1}{d}-1}$
- $B_r(x)$ is strictly decreasing in x
- $B_r\left(\frac{r^2}{3}\right) \leqslant 2$ and $\#\mathrm{SYT}(\lambda) \geqslant \frac{r^2}{3}$ for non-hooks

□

A reduction using the abacus

Lemma (essentially James-Kerber)

Let $\mu_1, \ldots, \mu_k$ be the k-quotient of λ. Then

$$\#\mathrm{BST}(\lambda, kd) = \binom{r/kd}{|\mu_1|/d, \ldots, |\mu_k|/d} \prod_{j=1}^{k} \#\mathrm{BST}(\mu_j, d).$$

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Thus, $\#\mathrm{BST}(\lambda, k) = \binom{r/k}{|\mu_1|, \ldots, |\mu_k|} \prod_{j} \#\mathrm{BST}(\mu_j, 1)$

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Thus,
$$\begin{aligned}\#\mathrm{BST}(\lambda, k) &= \binom{r/k}{|\mu_1|, \ldots, |\mu_k|} \prod_j \#\mathrm{BST}(\mu_j, 1) \\ &\geqslant \binom{}{\bullet} \prod_j \sum_{d>1} \#\mathrm{BST}(\mu_j, d)\end{aligned}$$

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\end{aligned}$$

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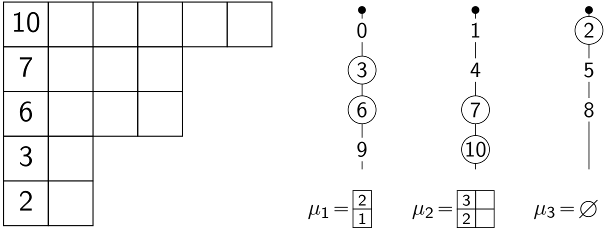
&\geqslant \left(\bullet \right) \prod_j \sum_{d>1} \#\mathrm{BST}(\mu_j, d)/(\tau(|\mu_j|) - 1) \\
&\geqslant \left(\bullet \right) \sum_{d>1} \prod_j \#\mathrm{BST}(\mu_j, d)/(\tau(|\mu_j|) - 1) \\
&\geqslant \left(\bullet \right) \sum_{d>1} \frac{\#\mathrm{BST}(\lambda, kd)}{\prod_j (\tau(|\mu_j|) - 1)} \binom{r/kd}{|\mu_1|/d, \ldots, |\mu_k|/d}^{-1}
\end{aligned}$$

A reduction using the abacus

Idea of proof for $d = 1$.



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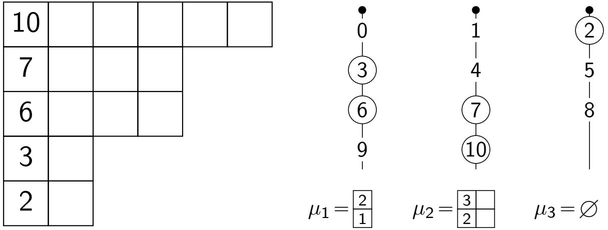
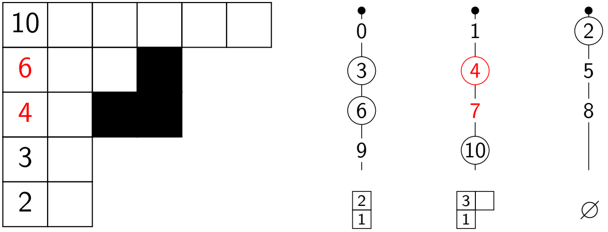


moving a bead up is the same as removing a border strip:



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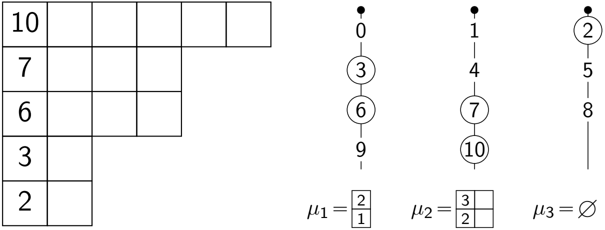

there are $\#\mathrm{SYT}(\mu_j)$ many possibilities to move beads up

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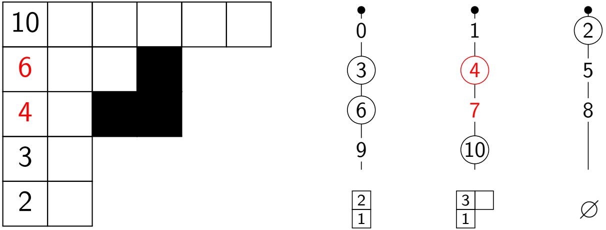

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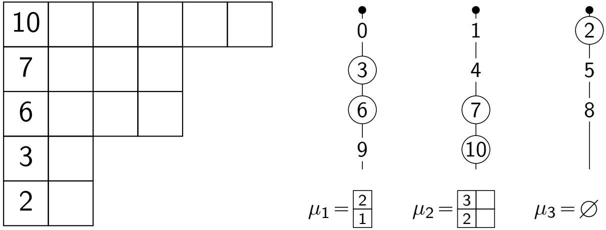

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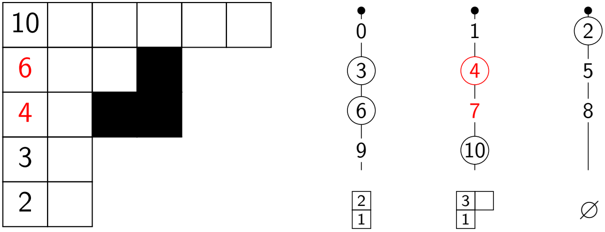

there are $\binom{r/k}{|\mu_1|,\ldots,|\mu_k|}$ possibilities to distribute numbers