Where do the maximum absolute q-series coefficients of $(1-q)(1-q^2)(1-q^3)\dots(1-q^{n-1})(1-q^n)$ occur?

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Sep 9, 2020

Partitions

A *partition* π is a finite sequence of *non-decreasing* positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$.

For a given partition $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_{\#(\pi)})$ the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_{\#(\pi)}$ is the *size* of the partition π and it is denoted by $|\pi|$.

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<u>Ex:</u>

- $\pi = (1, 1, 5)$ is a partition of $|\pi| = 7$.
- $\pi = \emptyset$ is the unique partition of 0.

Generating Functions

For a sequence $\{a_n\}_{n=0}^{\infty}$, the series

$$\sum_{n\geq 0} a_n q^n$$

is called a *generating function*.

Let $\mathcal{D}$ be the set of all partitions into non-repeating parts.

$$\sum_{\pi\in\mathcal{D}} q^{|\pi|} = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + 5q^7 + 6q^8 + 8q^9 \dots .$$

$$\begin{array}{c|l|l} \emptyset & (1,2),\ (3) & (1,2,3),\ (1,5),\ (2,4),\ (6) \\ (1) & (1,3),\ (4) & (1,2,4),\ (1,6),\ (2,5),\ (3,4),\ (7) \\ (2) & (1,4),\ (2,3),\ (5) & (1,2,5),\ (1,3,4),\ (1,7),\ (2,6),\ (3,5),\ (8) \end{array}$$

q-Pochhammer Symbol

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i), \text{ and } (a;q)_\infty := \lim_{L\to\infty}(a;q)_L.$$

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where $\mathcal{D}$ is the set of all partitions into non-repeating parts. Similarly,

$$(q;q)_\infty = \sum_{\pi\in\mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|}.$$

Generating Functions

$$\sum_{\pi\in\mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|} = \sum_{\substack{\pi\in\mathcal{D}\\ \#(\pi)\equiv 0 \bmod 2}} q^{|\pi|} - \sum_{\substack{\pi\in\mathcal{D}\\ \#(\pi)\equiv 1 \bmod 2}} q^{|\pi|}.$$

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$$\sum_{\pi\in\mathcal{D}} (-1)^{\#(\pi)} q^{|\pi|} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots.$$

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Theorem (Euler's Pentagonal Number Theorem, 1750s)

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

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$$(q;q)_4 = 1 - q - q^2 + 2q^5 - q^8 - q^9 + q^{10}.$$

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Theorem

For $m \in \mathbb{Z}_{\geq 0}$, $(q;q)_m := \prod_{i=1}^{m}(1-q^m)$ is of Bloch–Pólya type iff $m = 0,\ 1,\ 2,\ 3$ or 5.

A. Berkovich, A. K. Uncu, *On Some Polynomials of Bloch–Pólya type*. Proc. of the Amer. Math. Soc. 146 (2018) 7, 2827-2838.

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(q;q)_0 &= 1,\\
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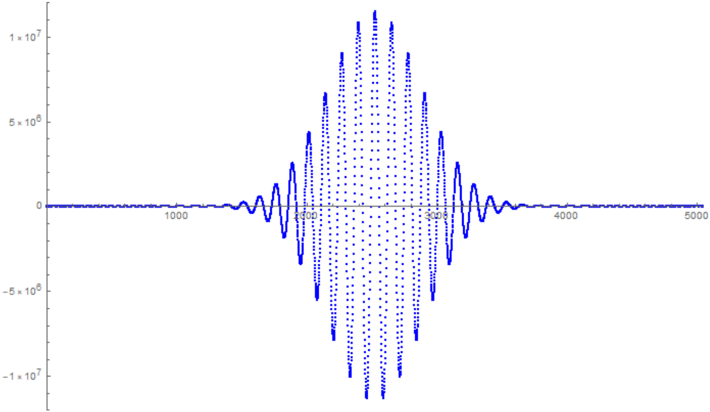
Plot of $(q;q)_{100}$



Maximum absolute coefficient is $a_{2525,100} = 11,493,312.$

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The first few pentagonal numbers are

$$0 = p_1(0) = p_2(0) < 1 = p_1(1) < 2 = p_2(1) < 5 = p_1(2) < \dots$$
$$\dots < p_1(n) < p_2(n) < p_1(n+1) < \dots$$

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Taking differences we see that

$$p_2(n) - p_1(n) = n \quad \text{and} \quad p_1(n+1) - p_2(n) = 2n+1.$$

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Lemma

For any $M > 0$ the gap between successive pentagonal numbers is

$$p_2(n) - p_1(n) > M \quad \textit{and} \quad p_1(n+1) - p_2(n) > M,$$

for all $n > M$.

We have

$$(q;q)_\infty = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} \dots .$$

Then

$$(q^2;q)_\infty = \frac{(q;q)_\infty}{1-q},$$

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Then

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&= 1 - q^2 - q^3 - q^4 + q^7 + q^8 + q^9 + q^{10} \dots
\end{aligned}$$

Heights of q-Rising Factorials and Related Series

Theorem

The power series of

$$(q^2;q)_\infty := \sum_{j\geq 0} a_j q^j$$

is of Bloch–Pólya type. Furthermore, for any $j \in \mathbb{Z}_{\geq 0}$ *there exists unique* $n \in \mathbb{Z}_{\geq 0}$ *such that*

$$p_2(2n) \leq j < p_2(2n+2)$$

and

$$a_j = \begin{cases} 1, & \text{if } p_2(2n) \leq j < p_1(2n+1), \\ -1, & \text{if } p_2(2n+1) \leq j < p_1(2n+2), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

Let $(q^3;q)_\infty := \sum_{i\geq 0} b_i q^i$, *there exists a unique integer* $n \geq 0$ *such that*

$$p_1(2n) - 2 \leq i \leq p_1(2n+2) - 3.$$

Then,

$$b_i = \begin{cases} -n, & \text{if } p_1(2n)-2 \leq i \leq p_2(2n)-1, \\ 1-n+\lfloor \frac{i-p_2(2n)}{2} \rfloor, & \text{if } p_2(2n) \leq i \leq p_1(2n+1)-2, \\ 1+n, & \text{if } p_1(2n+1)-1 \leq i \leq p_2(2n+1)-2 \text{ and } i \equiv p_2(2n) \\ n, & \text{if } p_1(2n+1)-1 \leq i \leq p_2(2n+1)-2 \text{ and } i \not\equiv p_2(2n) \\ n - \lceil \frac{i-p_2(2n+1)}{2} \rceil, & \text{if } p_2(2n+1)-1 \leq i \leq p_1(2n+2)-3, \end{cases}$$

where $\lfloor\cdot\rfloor$, *and* $\lceil\cdot\rceil$ *are floor and ceiling functions, respectively.*

This formula not only says that the coefficients are unbounded but also tells that any integer appears as a coefficient of $(q^3;q)_\infty$ infinitely many times.

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Some first appearances of coefficient sizes are as follows:

$$\begin{aligned}(q^3;q)_\infty = 1 + \dots + 2q^{11} + \dots + 3q^{34} \dots + \dots\\ + 4q^{69} + \dots + 5q^{116} + \dots + 6q^{175} + \dots + 7q^{246} + \dots\end{aligned}$$

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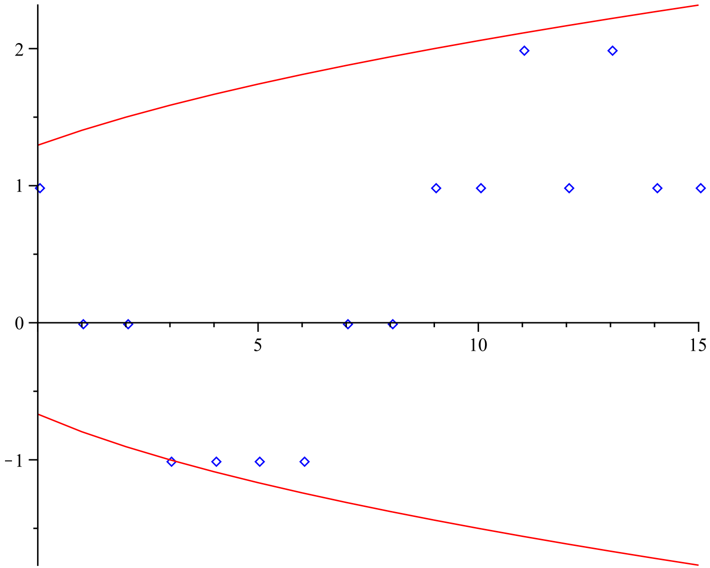
Example: if $i = 10^{100}$, then $n = 408248290463863016366214012450981898660991246776 11$.

Moreover, for this particular i the second case of the formula above applies. Hence, after the addition of three numbers we get, $b_i = -1988809025139063991081835693862813068960274101837 9$.


2
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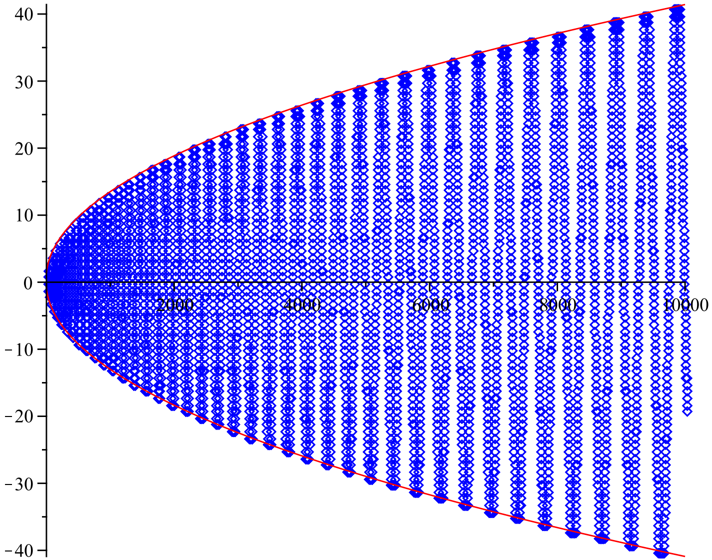
40
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Heights of q-Rising Factorials and Related Series

Theorem (q-Binomial Theorem)

$$\sum_{n\geq 0} \frac{(-a;q)_n}{(q;q)_n} t^n = \frac{(-ta;q)_\infty}{(t;q)_\infty}.$$

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Let $a \mapsto -a$ and $(a,q,t) = (0,q,q^m)$ in the above theorem and multiply both sides with $(q;q)_\infty$, we have:

$$\begin{aligned}(q;q)_{m-1} &= \sum_{i\geq 0} q^{mi}(q^{i+1};q)_\infty \\ &= (q;q)_\infty + q^m(q^2;q)_\infty + q^{2m}(q^3;q)_\infty \dots .\end{aligned}$$

We call a polynomial (or series) with height ≤ 1 (the coefficients from the set $\{-1, 0, 1\}$) a polynomial (or series, resp.) of Bloch–Pólya type.

Theorem

For $m \in \mathbb{Z}_{\geq 0}$, $(q;q)_m$ is of Bloch–Pólya type iff $m = 0,\ 1,\ 2,\ 3$ or 5.

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For any $2m \leq n < 3m$ the contribution for the coefficient of q^n of $(q;q)_{m-1}$ only comes from the first three terms.

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Remember:

$$(q^3;q)_\infty = 1 + \cdots + 2q^{11} + \cdots + 3q^{34} + \cdots + 4q^{69} + \dots .$$

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Remember:

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Therefore, for any $m > 69$ one can deduce that

$$2 \leq [\![q^{2m+69}]\!](q;q)_{m-1} \leq 6.$$

$$
\begin{aligned}
\llbracket q^{7} \rrbracket (q;q)_{6} &= 2,\\
\llbracket q^{12} \rrbracket (q;q)_{m-1} &= -2, \ \text{for} \ \ 8 \le m \le 10,\\
\llbracket q^{15} \rrbracket (q;q)_{10} &= -2,\\
-3 \le \llbracket q^{2m+20} \rrbracket (q;q)_{m-1} &\le -2, \ \text{for} \ \ 12 \le m \le 21,
\end{aligned}
$$

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$$\llbracket q^{12} \rrbracket (q;q)_{m-1} = -2, \text{ for } 8 \leq m \leq 10,$$
$$\llbracket q^{15} \rrbracket (q;q)_{10} = -2,$$
$$-3 \leq \llbracket q^{2m+20} \rrbracket (q;q)_{m-1} \leq -2, \text{ for } 12 \leq m \leq 21,$$

Moreover,

$$2 \leq \llbracket q^{2m+69} \rrbracket (q;q)_{m-1} \leq 12 \text{ for } 22 \leq m \leq 69 \text{ and } m \neq 42.$$

and

$$\llbracket q^{51} \rrbracket (q;q)_{41} = 2.$$

Theorem

For $m \in \mathbb{Z}_{\geq 0}$, $(q;q)_m$ *is of Bloch–Pólya type iff* $m = 0, 1, 2, 3$ *or* 5.

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The same argument can be used to prove other height arguments for $(q;q)_m$ as well.

Table: List of height sets S_h for $h = 1 \ldots 30$ with the cut-off values of m.

h	S_h	Cut-off	h	S_h	Cut-off	h	S_h	Cut-off
1	$\{0,1,2,3,5\}$	69	11	$\{23\}$	1079	21	$\{27\}$	3289
2	$\{4,6,7,8,9,11\}$	116	12	$\emptyset$	1246	22	$\emptyset$	3576
3	$\{10,13,14\}$	175	13	$\emptyset$	1425	23	$\emptyset$	3875
4	$\{12,15\}$	246	14	$\emptyset$	1616	24	$\emptyset$	4186
5	$\{17\}$	329	15	$\emptyset$	1819	25	$\emptyset$	4509
6	$\{16,18\}$	424	16	$\{24,25\}$	2034	26	$\emptyset$	4844
7	$\{19\}$	531	17	$\emptyset$	2261	27	$\emptyset$	5191
8	$\{20,21\}$	650	18	$\emptyset$	2500	28	$\{28\}$	5550
9	$\emptyset$	781	19	$\{26\}$	2751	29	$\{29\}$	5921
10	$\{22\}$	924	20	$\emptyset$	3014	30	$\emptyset$	6304

Conjecture (For Wadim: “Expectation:")

$$\textit{Either } S_h = \emptyset, \textit{ or } S_h = \{i(h)\},$$

for $h > 16$, *where* $i(h)$ *is a positive integer, and*

$$i(h_1) > i(h_2) \textit{ when } h_1 > h_2 > 16.$$

Moreover, for $h > 5$, *the set*

$$S_1 \cup S_2 \cup \cdots \cup S_h = \{0, 1, 2, \ldots, M(h)\}$$

consists of all consecutive integers from 0 up to some positive $M(h)$.

Let

$$F_{k,M}(q) := \sum_{j=0}^{M} q^{kj}(q;q)_j,$$
$$F_k(q) := \lim_{M\to\infty} F_{k,M}(q) = \sum_{j\geq 0} q^{kj}(q;q)_j.$$

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Lemma

For $k \geq 1$

$$q^{k+1}F_{k+1,M}(q) = 1 + (q^k - 1)F_{k,M}(q) - q^{k(M+1)}(q;q)_{M+1}.$$

Let

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Lemma

$$qF_{1,M}(q) = \sum_{j=0}^{M} q^{j+1}(q;q)_j = 1 - (q;q)_{M+1}.$$

Let

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Lemma

$$q^{k(k+1)/2}F_k(q) = \sum_{i=0}^{k-1}(-1)^i(q^{k-i};q)_i q^{(k-1-i)(k-i)/2} + (-1)^k(q;q)_{k-1}(q;q)_\infty$$

Theorem

i. $F_1(q)$ *and* $F_2(q)$ *are of Bloch–Pólya type,*

ii. $F_3(q) - q^9$ *and* $F_4(q) - q^{16} + q^{18} + q^{30} - q^{31}$ *are both of Bloch–Pólya type,*

iii. $F_5(q)$ *is not Bloch–Pólya type series and there is no polynomial* $f(q)$ *such that* $F_5(q) + f(q)$ *is one,*

iv. $F_6(q) - f_6(q)$ *is Bloch–Pólya type series, where*

$$f_6(q) := q^{29} - q^{32} + q^{36} - q^{38} + q^{43} - q^{45} \dots$$
$$\dots - q^{110} - q^{239} + q^{241} + q^{280} - q^{281},$$

v. *and for* $k \geq 7$, *there is no polynomial* $f(q)$ *such that* $F_k(q) + f(q)$ *is of Bloch-Pólya type.*

Lemma

$$q^{k(k+1)/2}F_k(q) = \sum_{i=0}^{k-1}(-1)^i(q^{k-i};q)_i q^{(k-1-i)(k-i)/2} + (-1)^k(q;q)_{k-1}(q;q)_\infty$$

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Lemma

For any $M > 0$ *the gap between successive pentagonal numbers is*

$$p_2(n) - p_1(n) > M \quad \text{and} \quad p_1(n+1) - p_2(n) > M,$$

for all $n > M$.

Example Proof: $F_4(q) - q^{16} + q^{18} + q^{30} - q^{31}$ is of Bloch polya type:

$$q^{10}F_4(q) = -1 + 2q + q^2 - 2q^3 - 2q^4 - q^5 + 4q^6 + (1 - q - q^2 + q^4 + q^5 - q^6)(q;q)_\infty.$$

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$$\begin{aligned} q^{10} F_4(q) = & q^{10} + q^{14} - q^{15} + q^{18} - q^{19} - q^{20} + q^{21} + q^{22} - q^{23} \\ & - q^{24} + 2q^{26} - 2q^{28} + q^{30} + q^{31} - q^{32} - q^{35} + q^{36} \\ & + q^{37} - q^{39} - 2q^{40} + 2q^{41} + q^{42} - q^{44} - q^{45} + q^{46} + q^{51} \\ & - q^{52} - q^{53} + q^{55} + q^{56} - q^{58} - q^{59} + q^{61} + q^{62} - q^{63} \\ & - (q;q)_3 (q^{70} + q^{77} - q^{92} - q^{100} + q^{117} \dots). \end{aligned}$$

Table: List of height sets $\hat{S}_h$ of $F_k(q)$ for $h = 1 \dots 40$.

h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$	h	$\hat{S}_h$
1	$\{1, 2\}$	11	$\{16\}$	21	$\emptyset$	31	$\emptyset$
2	$\{3, 4, 6\}$	12	$\{17\}$	22	$\emptyset$	32	$\emptyset$
3	$\{5, 8\}$	13	$\emptyset$	23	$\emptyset$	33	$\emptyset$
4	$\{7, 9\}$	14	$\{18\}$	24	$\{22\}$	34	$\emptyset$
5	$\emptyset$	15	$\{19\}$	25	$\emptyset$	35	$\emptyset$
6	$\{10, 12\}$	16	$\emptyset$	26	$\emptyset$	36	$\emptyset$
7	$\{11, 14\}$	17	$\{20\}$	27	$\emptyset$	37	$\{25\}$
8	$\{13, 15\}$	18	$\{21\}$	28	$\emptyset$	38	$\emptyset$
9	$\emptyset$	19	$\emptyset$	29	$\{23\}$	39	$\emptyset$
10	$\emptyset$	20	$\emptyset$	30	$\{24\}$	40	$\emptyset$

Theorem (Euler's Pentagonal Number Theorem, 1750s)

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

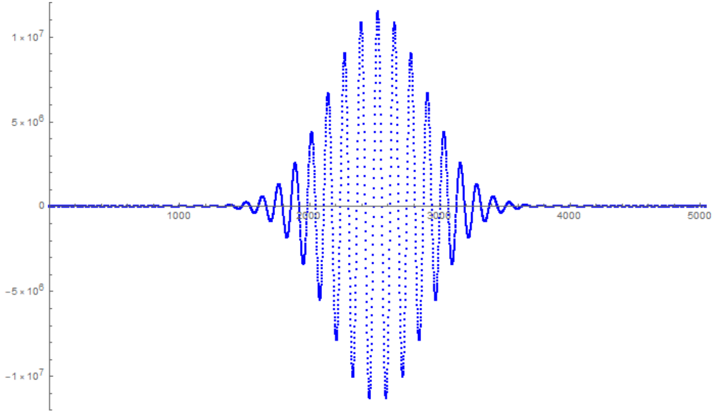
Let N be a positive integer,

$$(q;q)_N := \sum_{i=0}^{N(N+1)/2} a_{i,N}\ q^i.$$

What about the $a_{i,N}$?

$$(q;q)_4 = 1 - q - q^2 + 2q^5 - q^8 - q^9 + q^{10}.$$

Plot of $(q;q)_{100}$



Maximum absolute coefficient is $a_{2525,100} = 11,493,312.$

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A160089 The maximum of the absolute value of the coefficients of Pn = (1-x)(1-x^2)(1-x^3)...(1-x^n). 4

1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 4, 3, 3, 4, 6, 5, 6, 7, 8, 8, 10, 11, 16, 16, 19, 21, 28, 29, 34, 41, 50, 56, 68, 80, 100, 114, 135, 158, 196, 225, 269, 320, 388, 455, 544, 644, 786, 921, 1111, 1321, 1600, 1891, 2274, 2711, 3280, 3895, 4694, 5591, 6780, 8051, 9729, 11624 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,5

COMMENTS If n is even then a(n) is the absolute value of the coefficient of z^(n(n+1)/4). If n is odd, it is an open question as to which coefficient is a(n).

Where is the maximum abs. coefficient of $(q;q)_n$?

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$$= \sum_{i=0}^{N(N+1)/2} a_{i,N}\, q^i.$$

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$$= \sum_{i=0}^{N(N+1)/2} a_{i,N}\, q^i.$$

Therefore,

$$a_{i,N} = a_{i,N-1} - a_{i-N,N-1}.$$

Necessary resources for the calculation

That's where a big computer like MACH2[1] becomes a necessity!

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and $L(N)$ be the smallest integer such that $a_{L(N),N} = \mathcal{M}_N$.

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$$D_N = \frac{N(N+1)}{4} - L(N) \text{ and } E_N = D_N - D_{N-4}.$$

What did we calculate?

- $L(N)$ location of the maximum absolute coefficient,
- D_N distance of the maximum absolute coefficient from the midpoint,
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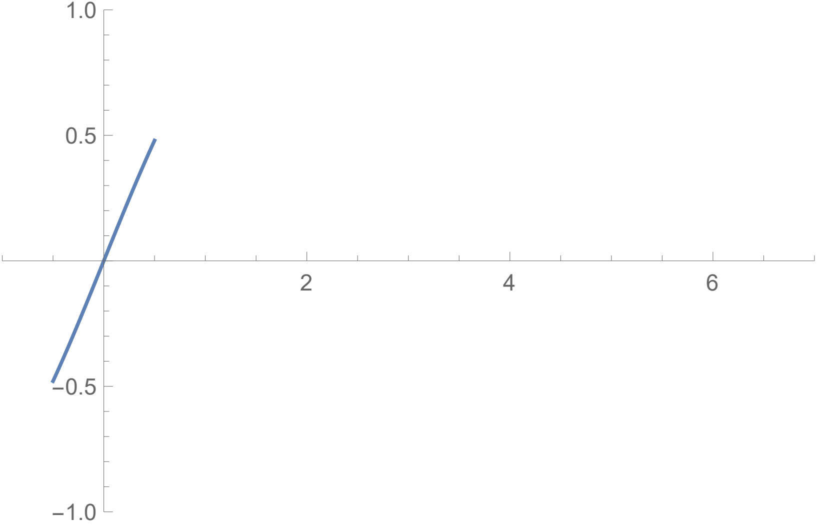
$$L(N) = \frac{N(N+1)}{4} - \left(D_m + \sum_{i=1}^{(N-m)/4} E_{m+4i} \right).$$

Interlude: Periodicity!?

Would you believe the periodicity of this graph?



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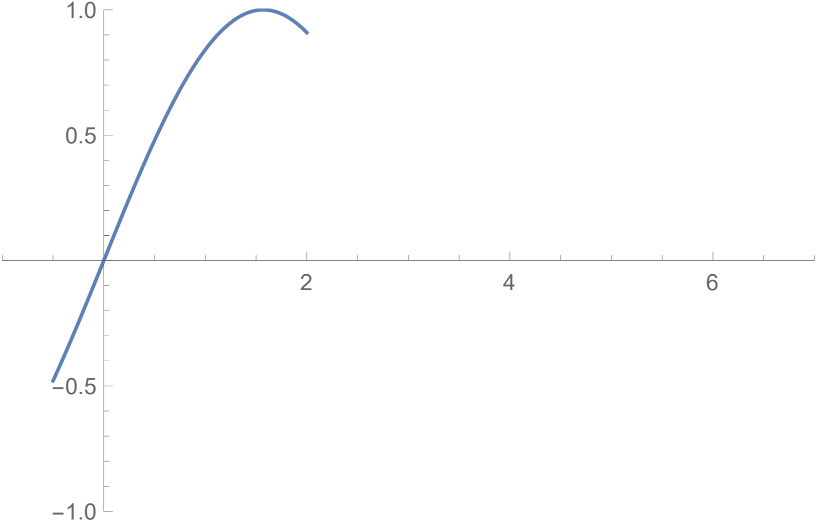

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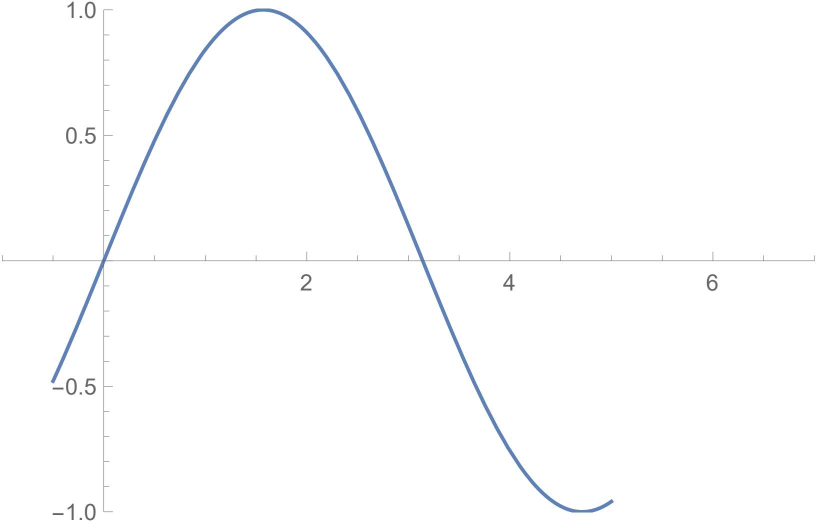

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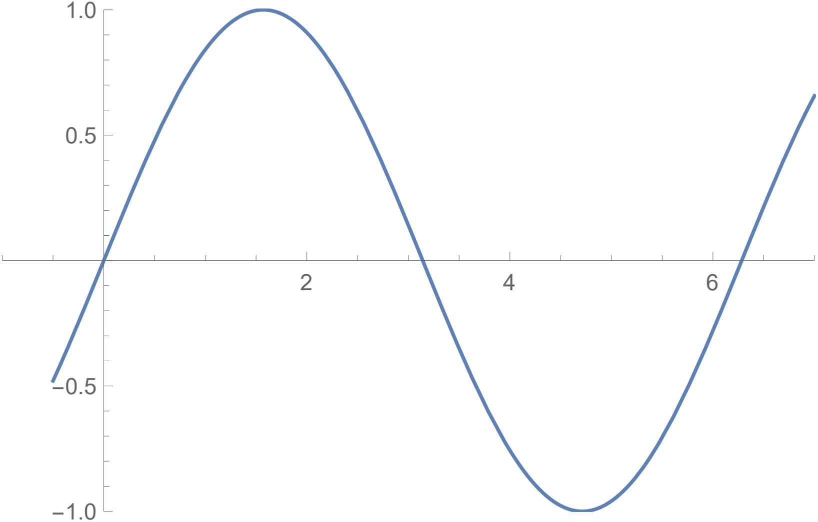

End of the interlude.

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Conjecture

$E_N \in \{1, 2\}$ *for all* $N \geq 61$.

$$L(N) = \frac{N(N+1)}{4} - \left(D_m + \sum_{i=1}^{(N-m)/4} E_{m+4i} \right).$$

E_N's are almost periodic with period 19

Starting from $N = 53$, the E_N's for 1 mod 4 parts look as follows:

$$2112111211121112111$$
$$2112111211121112111$$
$$2112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121112112$$
$$1112111211121121112$$
$$1112111211121121112$$
$$1112111211121121112$$

$$\cdots\cdots$$

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1112111211121112112
1112111211121112112
1112111211121121112
1112111211121121112
1112111211121121112

......

The alphabet

$$
\begin{aligned}
a &= \overbrace{2112111211121112112}^{19}, & k &= \overbrace{1211121112111211121}^{19},\\
b &= 1112111211121112112, & l &= 1211121112111211211,\\
c &= 1112111211121121112, & m &= 1211121112112111211,\\
d &= 1112111211211121112, & n &= 1211121121112111211,\\
e &= 1112112111211121112, & o &= 1211211121112111211,\\
f &= 1121112111211121112, & p &= 2111211121112111211,\\
g &= 1121112111211121121, & q &= 2111211121112112111,\\
h &= 1121112111211211121, & r &= 2111211121121112111,\\
i &= 1121112112111211121, & s &= 2111211211121112111,\\
j &= 1121121112111211121, & t &= 2112111211121112111.
\end{aligned}
$$

The words using this alphabet

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$$\underbrace{a^1 b^4 c^4 d^4 e^4 f^3 g^4 h^5 i^4 j^4 k^3 l^4 m^4 n^4 o^5 p^3 q^4 r^4 s^4 t^3}_{\text{word 1}} \underbrace{a^1 b^4 c^4 d^4 e^4 f^3 g^4 h^4 i^5 j^4 k^3 \dots}_{\text{word 2, etc.}}.$$

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$a^1\ b^4\ c^4\ d^4\ e^4\ f^3\ g^4\ h^5\ i^4\ j^4\ k^3\ l^4\ m^4\ n^4\ o^5\ p^3\ q^4\ r^4\ s^4\ t^3$
$a^1\ b^4\ c^4\ d^4\ e^4\ f^3\ g^4\ h^4\ i^5\ j^4\ k^3\ l^4\ m^4\ n^4\ o^5\ p^3\ q^4\ r^4\ s^4\ t^3$
$a^1\ b^3\ c^5\ d^4\ e^4\ f^3\ g^4\ h^4\ i^5\ j^4\ k^3\ l^4\ m^4\ n^4\ o^4\ p^4\ q^4\ r^4\ s^4\ t^3$
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$a^1\ b^3\ c^4\ d^5\ e^4\ f^3\ g^4\ h^4\ i^4\ j^5\ k^3\ l^4\ m^4\ n^4\ o^4\ p^3\ q^5\ r^4\ s^4\ t^3$
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$a^1\ b^3\ c^4\ d^4\ e^5\ f^3\ g^4\ h^4\ i^4\ j^4\ k^3\ l^5\ m^4\ n^4\ o^4\ p^3\ q^4\ r^5\ s^4\ t^3$
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$a^1\ b^3\ c^4\ d^4\ e^4\ f^4\ g^4\ h^4\ i^4\ j^4\ k^3\ l^4\ m^5\ n^4\ o^4\ p^3\ q^4\ r^4\ s^5\ t^3$
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$a^1\ b^3\ c^4\ d^4\ e^4\ f^3\ g^5\ h^4\ i^4\ j^4\ k^3\ l^4\ m^4\ n^5\ o^4\ p^3\ q^4\ r^4\ s^4\ t^4$
$a^1\ b^3\ c^4\ d^4\ e^4\ f^3\ g^4\ h^5\ i^4\ j^4\ k^3\ l^4\ m^4\ n^5\ o^4\ p^3\ q^4\ r^4\ s^4\ t^3$
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$a^1\ b^3\ c^5 \ldots$

The words using this alphabet

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$$\begin{array}{c}
a^1\ b^4\ c^4\ d^4\ e^4\ f^3\ g^4\ h^5\ i^4\ j^4\ k^3\ l^4\ m^4\ n^4\ o^5\ p^3\ q^4\ r^4\ s^4\ t^3 \\
\hline
11\left\{\begin{array}{l}
a^1\ b^4\ c^4\ d^4\ e^4\ f^3\ g^4\ h^4\ i^5\ j^4\ k^3\ l^4\ m^4\ n^4\ o^5\ p^3\ q^4\ r^4\ s^4\ t^3 \\
a^1\ b^3\ c^5\ d^4\ e^4\ f^3\ g^4\ h^4\ i^5\ j^4\ k^3\ l^4\ m^4\ n^4\ o^4\ p^4\ q^4\ r^4\ s^4\ t^3 \\
a^1\ b^3\ c^5\ d^4\ e^4\ f^3\ g^4\ h^4\ i^4\ j^5\ k^3\ l^4\ m^4\ n^4\ o^4\ p^4\ q^4\ r^4\ s^4\ t^3 \\
a^1\ b^3\ c^4\ d^5\ e^4\ f^3\ g^4\ h^4\ i^4\ j^5\ k^3\ l^4\ m^4\ n^4\ o^4\ p^3\ q^5\ r^4\ s^4\ t^3 \\
a^1\ b^3\ c^4\ d^5\ e^4\ f^3\ g^4\ h^4\ i^4\ j^4\ k^4\ l^4\ m^4\ n^4\ o^4\ p^3\ q^4\ r^5\ s^4\ t^3 \\
a^1\ b^3\ c^4\ d^4\ e^5\ f^3\ g^4\ h^4\ i^4\ j^4\ k^3\ l^5\ m^4\ n^4\ o^4\ p^3\ q^4\ r^5\ s^4\ t^3 \\
a^1\ b^3\ c^4\ d^4\ e^4\ f^4\ g^4\ h^4\ i^4\ j^4\ k^3\ l^5\ m^4\ n^4\ o^4\ p^3\ q^4\ r^4\ s^5\ t^3 \\
a^1\ b^3\ c^4\ d^4\ e^4\ f^4\ g^4\ h^4\ i^4\ j^4\ k^3\ l^4\ m^5\ n^4\ o^4\ p^3\ q^4\ r^4\ s^5\ t^3 \\
a^1\ b^3\ c^4\ d^4\ e^4\ f^3\ g^5\ h^4\ i^4\ j^4\ k^3\ l^4\ m^5\ n^4\ o^4\ p^3\ q^4\ r^4\ s^4\ t^4 \\
a^1\ b^3\ c^4\ d^4\ e^4\ f^3\ g^5\ h^4\ i^4\ j^4\ k^3\ l^4\ m^4\ n^5\ o^4\ p^3\ q^4\ r^4\ s^4\ t^4 \\
a^1\ b^3\ c^4\ d^4\ e^4\ f^3\ g^4\ h^5\ i^4\ j^4\ k^3\ l^4\ m^4\ n^5\ o^4\ p^3\ q^4\ r^4\ s^4\ t^3
\end{array}\right. \\
\hline
a^1\ b^4\ c^4\ d^4\ e^4\ f^3\ g^4\ h^4\ i^5\ j^4\ k^3\ l^4\ m^4\ n^4\ o^5\ p^3\ q^4\ r^4\ s^4\ t^3 \\
a^1\ b^3\ c^5 \ldots
\end{array}$$

Conjecture (Berkovich-U. 2019[3])

For any $k \geq 0$, $r = 0, 1, 2, \ldots 10$, *let*

$$A(k,r) = 5700r + 62624k - 76\,\delta_{r,11},$$

where $\delta_{i,j}$ *is the Kronecker delta, then,*

$$L(5909 + A(k,r)) = \frac{(5909 + A(k,r))(5910 + A(k,r))}{4} - D_{5909+A(0,r)} - 19787k,$$

where the full list of needed seed values of D_n*'s are*

$$\begin{array}{lll} D_{5909} = 1867.5, & D_{11609} = 3668.5, & D_{17309} = 5469.5, \\ D_{23009} = 7270.5, & D_{28709} = 9071.5, & D_{34409} = 9675.5, \\ D_{40109} = 12673.5, & D_{45809} = 14474.5, & D_{51509} = 16275.5, \\ D_{57209} = 18076.5, & D_{62833} = 19853.5. & \end{array}$$

[3] Where do the maximum absolute q-series coefficients of $(1-q)(1-q^2)(1-q^3)\ldots(1-q^{n-1})(1-q^n)$ occur?, with A. Berkovich.

Thank you for your time

Where do the maximum absolute q-series coefficients of $(1-q)(1-q^2)(1-q^3)\ldots(1-q^{n-1})(1-q^n)$ occur?

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