(Re)mixed Eulerian numbers

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For any $n \geq 1$, we define and study and a family of polynomials in q, the remixed Eulerian numbers $A_{\mathbf{c}}(q)$ indexed by

$$W_n := \{\mathbf{c} = (c_1, \cdots, c_n) \mid c_i \in \mathbb{N}, \sum_{i=1}^{n} c_i = n\}.$$

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PLAN

1) Mixed Eulerian numbers $A_{\mathbf{c}} := A_{\mathbf{c}}(1)$.

2) Definition of $A_{\mathbf{c}}(q)$ and probabilistic interpretation.

3) Special subfamilies.

4) General properties.

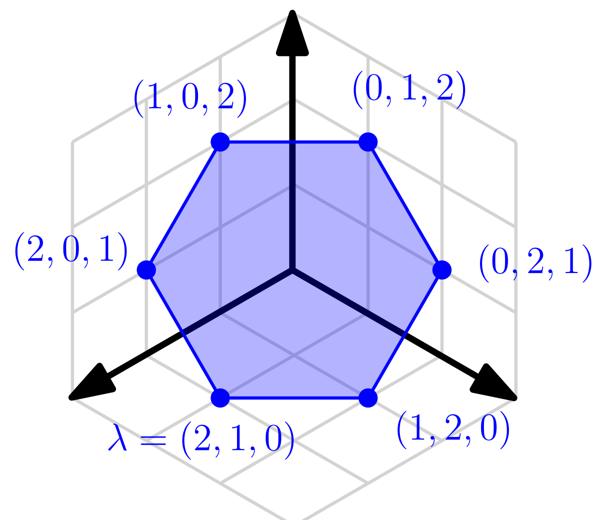
Postnikov's
Mixed Eulerian numbers

Permutahedron

Let $(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}$.

Definition The permutahedron $\mathrm{Perm}(\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ is the convex hull of the points $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n+1)})$ for $\sigma \in S_{n+1}$.

$\mathrm{Perm}(2, 1, 0)$



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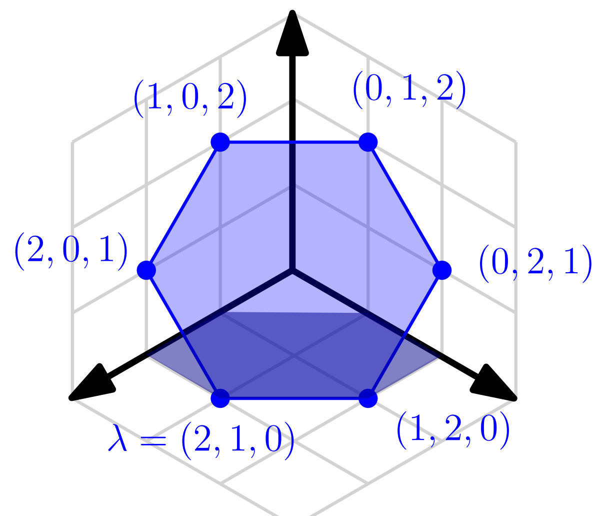

$V(2, 1, 0) = 3.$

The volume $V(\lambda_1, \lambda_2, \dots, \lambda_{n+1})$ is the volume of the permutahedron projected on $\{\lambda_{n+1} = 0\}$.

Mixed Eulerian numbers

The following results come from (Postnikov '09).

- $V(\lambda_1, \ldots, \lambda_{n+1})$ is a polynomial in the λ_i, homogeneous of degree n.

  Ex: $V(\lambda_1, \lambda_2, \lambda_3) = \frac{\lambda_1^2}{2} + \lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\frac{\lambda_2^2}{2} + \lambda_2\lambda_3 + \frac{\lambda_3^2}{2}$.

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- $V(\lambda_1, \cdots, \lambda_{n+1})$ only depends on the differences $\mu_i = \lambda_i - \lambda_{i+1}$.

  $\rightarrow \hat{V}(\mu_1, \ldots, \mu_n) := V(\lambda_1, \cdots, \lambda_{n+1})$.

  Ex: $\hat{V}(\mu_1, \mu_2) = \frac{\mu_1^2}{2} + 2\mu_1\mu_2 + \frac{\mu_2^2}{2}$

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Definition The mixed Eulerian numbers $A_{\mathbf{c}}$ are the normalized coefficients of $\hat{V}$

$$\hat{V}(\mu_1, \cdots, \mu_n) = \sum_{\mathbf{c} \in W_n} A_{\mathbf{c}} \frac{\mu_1^{c_1} \cdots \mu_n^{c_n}}{c_1! \cdots c_n!}$$

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with $\Delta_{k,n} = \mathrm{Perm}(1^k, 0^{n+1-k})$ the k^{th} hypersimplex.

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• By taking volumes in this decomposition, it expresses $A_{\mathbf{c}}$ as $n!$ times the **mixed volume** of hypersimplices, with $\Delta_{k,n+1}$ occurring c_k times.

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• One has $n! V(\Delta_{k,n+1}) = A_n^{k-1}$ (known already to Laplace). A_n^{k-1} is an **Eulerian number**: it counts permutations of S_n with $k-1$ descents.

It follows that $A_{\ldots,0,n,0\ldots} = A_n^{k-1}$

k^{th} position

Remixed Eulerian numbers

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Fix n. For any $\mathbf{c} \in W_n$, define $L_i(\mathbf{c}),\ R_i(\mathbf{c}) \in W_n$

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Definition-Theorem [N.-Tewari '21] There exists a unique family $A_{\mathbf{c}}(q)$ with $\mathbf{c} \in W_n$ that satisfies

$$(q+1)A_{\mathbf{c}}(q) = qA_{L_i(\mathbf{c})}(q) + A_{R_i(\mathbf{c})}(q) \quad \forall\, \mathbf{c}, i \text{ with } c_i \geq 2$$

with the normalization $A_{1,\ldots,1}(q) = [n]_q!$.

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One must show that this system of linear equations has indeed a unique solution (necessarily in $\mathbb{Q}(q)$).

$A_{\mathbf{c}}(q)$ for $n = 3$

$$A_{111}(q) = [3]_q! = 1 + 2q + 2q^2 + q^3$$

$$A_{210}(q) = 1 + q$$
$$A_{021}(q) = q + 2q^2 + q^3$$

$$A_{120}(q) = 1 + 2q + q^2$$
$$A_{012}(q) = q^2 + q^3$$

$$A_{300}(q) = 1$$
$$A_{030}(q) = 2q + 2q^2$$
$$A_{003}(q) = q^3$$

$$A_{102}(q) = q + q^2 + q^3$$
$$A_{201}(q) = 1 + q + q^2$$

(The sum in each group is $[3]_q!$; we will explain that later.)

First properties

Recall the definition

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There holds $A_{\mathbf{c}}(1) = A_{\mathbf{c}}$ in general.

The proof goes by finding an alternative, direct definition of $A_{\mathbf{c}}(q)$ that uses "q-divided symmetrization", which is a q-deformation of a linear form defined by Postnikov to give a formula for $V(\lambda_1, \cdots, \lambda_{n+1})$.

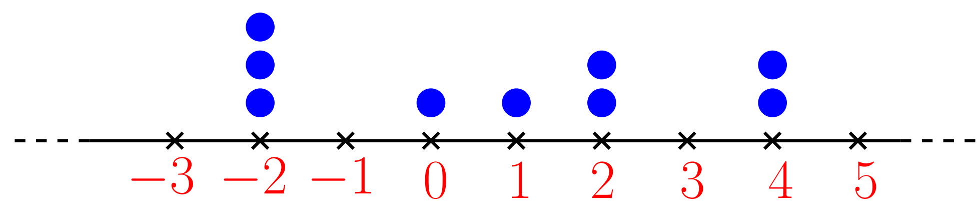
Remark: From that alternative definition follows moreover the existence of $A_{\mathbf{c}}(q)$, and the fact that $A_{\mathbf{c}}(q) \in \mathbb{Z}[q]$.

Probabilistic model for $A_{\mathbf{c}}(q)$ $(q \geq 0)$

States: Sequences $\mathbf{c} = (c_i)_{i \in \mathbb{Z}}$ with sum $\sum_i c_i = n$, seen as particle configurations

$$\mathbf{c} = (\ldots, 0, 3, 0, 1, 1, 2, 0, 2, 0, \ldots)$$

$\Leftrightarrow$



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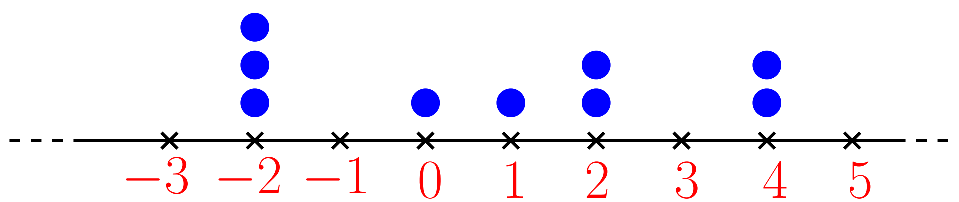
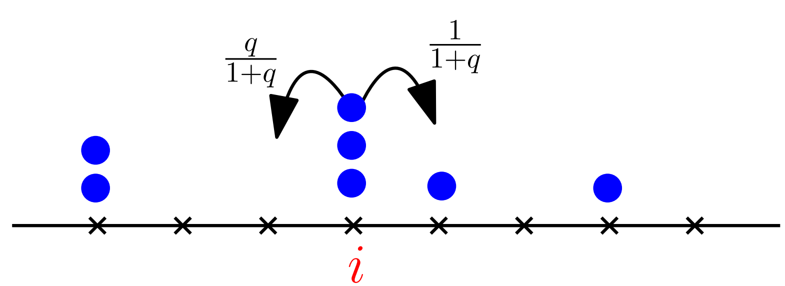


Transitions: If $c_i \geq 2$, particle at site i can jump:

- left with probability $\frac{q}{1+q}$ (reaches $L_i(\mathbf{c})$)
- right with probability $\frac{1}{1+q}$ (reaches $R_i(\mathbf{c})$)



Probabilistic model ($q \geq 0$)

Model: Start with an initial configuration $\mathbf{c}$. Then "let particles jump" until a stable configuration is reached.

(stable $=$ at most one particle per site, identified with $I \subset \mathbb{Z}$, $|I| < +\infty$)

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- (termination) The process stabilizes almost surely.
- (abelianness) The order in which particles jump does not change the probability of the final configuration.

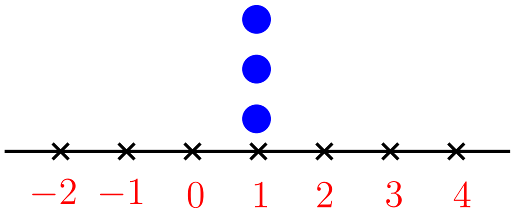
Definition Let $P(\mathbf{c} \to I)$ be the probability that, starting from $\mathbf{c}$, the final stable configuration is I.

Theorem (N.-Tewari '21)

If $\mathbf{c} \in W_n$, $P(\mathbf{c} \to \{1, \ldots, n\}) = \dfrac{A_{\mathbf{c}}(q)}{[n]_q!}$

Clearly $P(\mathbf{c} \to \{1, \ldots, n\}) = 0$ if $\mathbf{c} \notin W_n$.

Illustration $\mathbf{c} = (3, 0, 0)$



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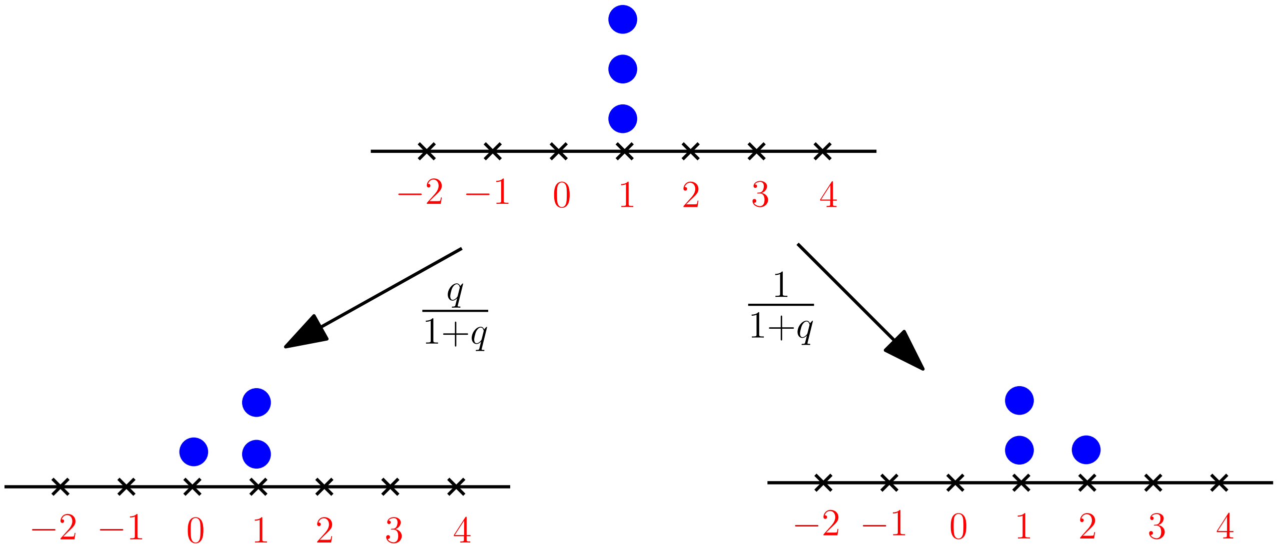

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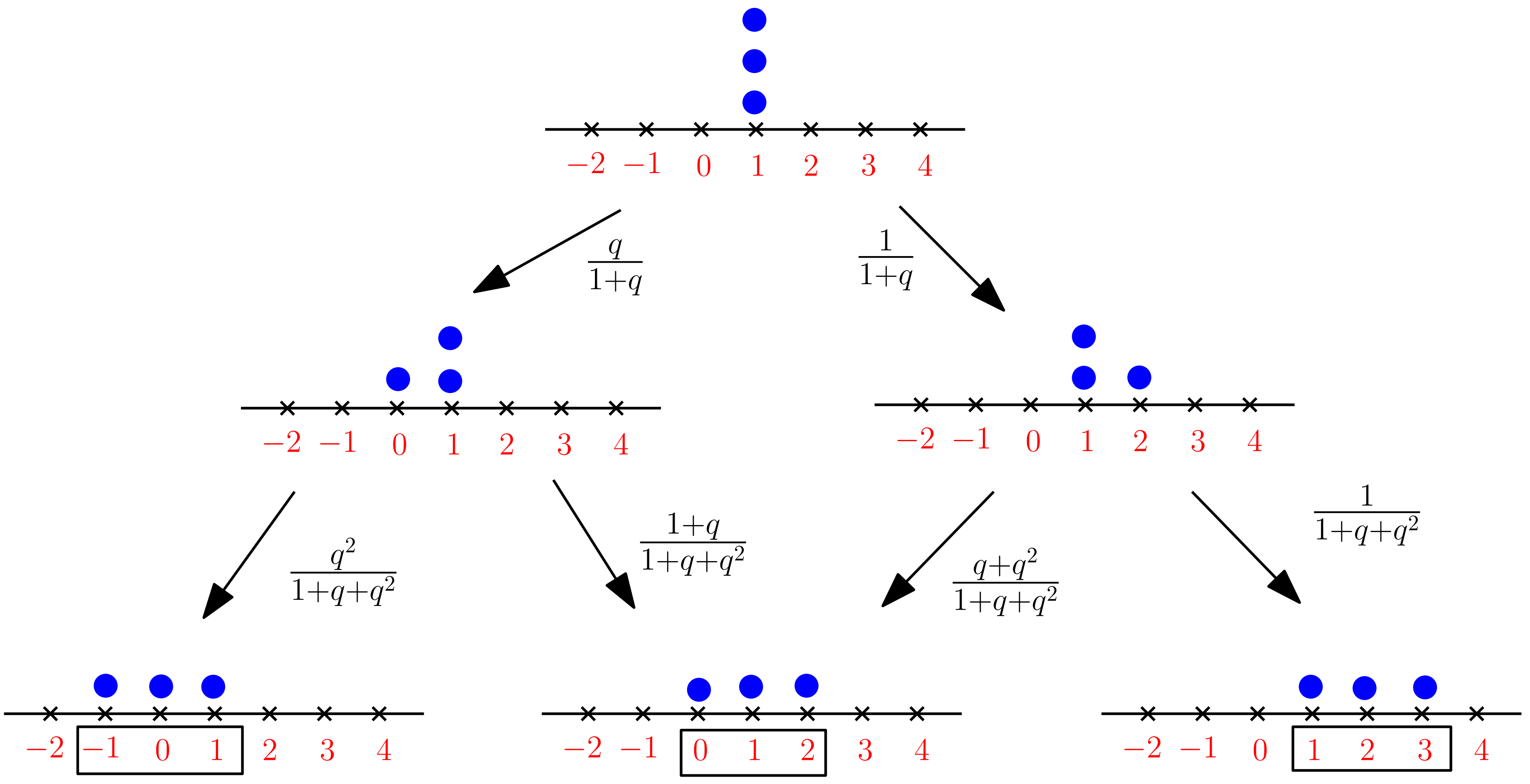

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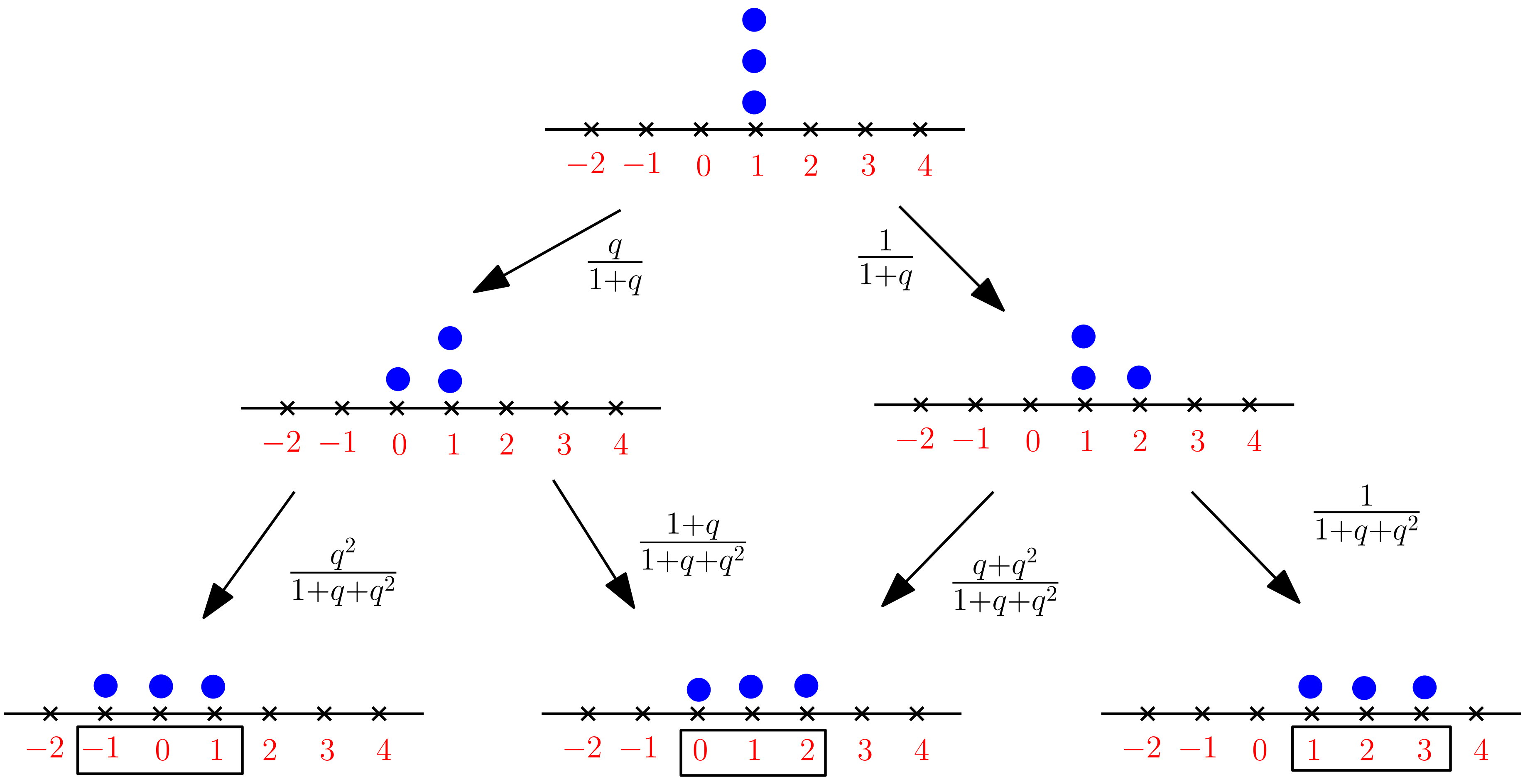

Prob $= \frac{q^3}{[3]_q!}$

Prob $= \frac{2q+2q^2}{[3]_q!}$

Prob $= \frac{1}{[3]_q!}$

Proof

Theorem (N.-Tewari '21) $P(\mathbf{c} \to \{1, \ldots, n\}) = \dfrac{A_{\mathbf{c}}(q)}{[n]_q!}$

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- If $c_i \geq 2$, then (abelian+transition)

$$P(\mathbf{c} \to I) = \tfrac{q}{1+q} P(L_i(\mathbf{c}) \to I) + \tfrac{1}{1+q} P(R_i(\mathbf{c}) \to I)$$

- If $\mathbf{c}$ stable, $P(c \to I) = 1$ if $\mathbf{c} = I$, 0 if $\mathbf{c} \neq I$.

$\longrightarrow$ $[n]_q! P(\mathbf{c} \mapsto \{1, \ldots, n\})$ satisfies the conditions of the definition of $A_{\mathbf{c}}(q)$. $\square$

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This is an example of Internal Diffusion Limited Aggregation process, introduced in (Diaconis-Fulton '93).

Special cases

Let $\mathbf{c} = (c_1, \ldots, c_n) \in W_n$.

- $\mathbf{c} = (\ldots, 0, n, 0, \ldots)$, n in kth position.

  $A_{\mathbf{c}}(q) =$ polynomial enumerating permutations in S_n with $k-1$ descents according to their inversion number.

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- $\mathbf{c} = (c_1, \cdots, c_n)$ with $\sum_{i \leq k} c_i \geq k$ for all k.

  $A_{\mathbf{c}}(q) =$ an explicit product of q-integers.

  Exercise

The interval case

We assume $\mathbf{c} = (\underbrace{c_1, c_2, \ldots, c_k}_{>0}, 0^{n-k})$.

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Theorem (N.-Tewari '20+)

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$q = 1$: (Berget-Spink-Tseng '20)

$k = 1$: special case of identity of MacMahon-Carlitz

In fact by comparison with work of (Garsia-Remmel '84), one recovers precisely the family of **hit polynomials** coming from rook theory.

A cyclic rule

- Write $\mathbf{c} \sim \mathbf{c}'$ if $(\mathbf{c}, 0)$ is a cyclic shift of $(\mathbf{c}', 0)$.

$$(3, 0, 1, 1, 0) \sim (0, 3, 0, 1, 1) \sim (1, 1, 0, 0, 3)$$

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Proof sketch: Consider the previous process on a discrete ring $\mathbb{Z}/(n+1)\mathbb{Z} = \{0, 1, \ldots, n\}$.

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For $q = 1$, conjectured by Stanley, proved by Postnikov using the Coxeter arrangement of affine type $\tilde{A}_n$.

Polynomial properties

$\mathbf{c} = (0, 3, 0, 0, 0, 1, 3) \in W_7$

$$A_{\mathbf{c}}(q) = 2q^{20} + 6q^{19} + 11q^{18} + 18q^{17} + 27q^{16} + 35q^{15} + 40q^{14} + 42q^{13} + 40q^{12} + 35q^{11} + 27q^{10} + 18q^{9} + 11q^{8} + 6q^{7} + 2q^{6}$$

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Proposition For any $\mathbf{c} \in W_n$, $A_{\mathbf{c}}(q)$ is palindromic.

This means $A_c(q) = q^{v_{\mathbf{c}}+d_{\mathbf{c}}} A_c(q^{-1})$, where $v_{\mathbf{c}}$ is the valuation of $A_{\mathbf{c}}(q)$ and $d_{\mathbf{c}}$ its degree.

In the example, $v_{\mathbf{c}} = 6$, $d_{\mathbf{c}} = 20$.

Polynomial properties

For $\mathbf{c} \in W_n$, define $h(\mathbf{c}) = (h_1, h_2, \ldots, h_n)$

by $h_i := (c_1 + c_2 + \cdots + c_i) - i$. for all i.

Proposition For any $\mathbf{c} \in W_n$, there holds

$$v_{\mathbf{c}} = \sum_{i, h_i < 0} |h_i| \quad \text{and} \quad d_{\mathbf{c}} = \binom{n}{2} - \sum_{i, h_i > 0} h_i$$

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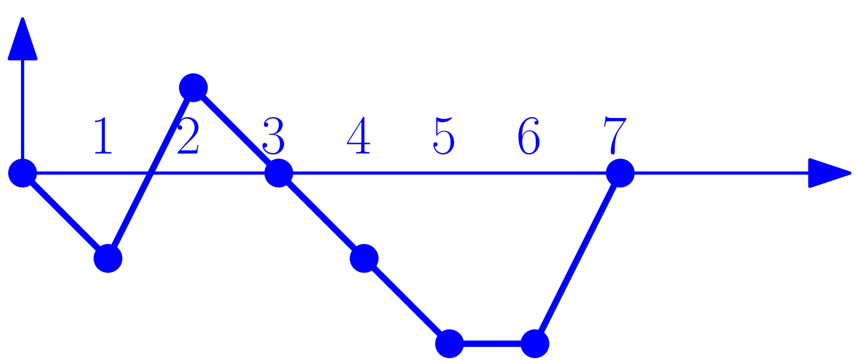
$$\mathbf{c} = (0, 3, 0, 0, 0, 1, 3)$$
$$h(\mathbf{c}) = (-1, 1, 0, -1, -2, -2, 0)$$



successive heights

Combinatorial interpretation

Given $\mathbf{c} = (c_1, \ldots, c_n)$ define $l_1 := 1, l_2, \ldots, l_{n+1}$ by

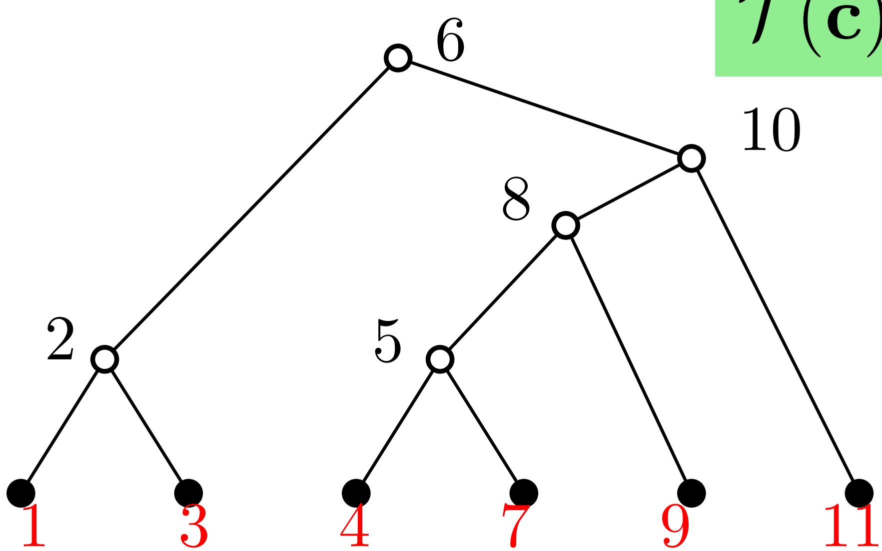
$$l_{i+1} - l_i = c_i + 1.$$

Consider complete, plane binary trees with $n+1$ leaves (thus n internal nodes) labeled with $\{1, 2, \ldots, 2n+1\}$:

(1) Leaves are labeled $l_1, \ldots, l_{n+1}$ from left to right.

(2) Each internal node has label larger than its left child and smaller than its right child.

$\mathcal{T}(\mathbf{c}) = \{$these trees$\}$



$\mathbf{c} = \quad 1 \quad 0 \quad 2 \quad 1 \quad 1$

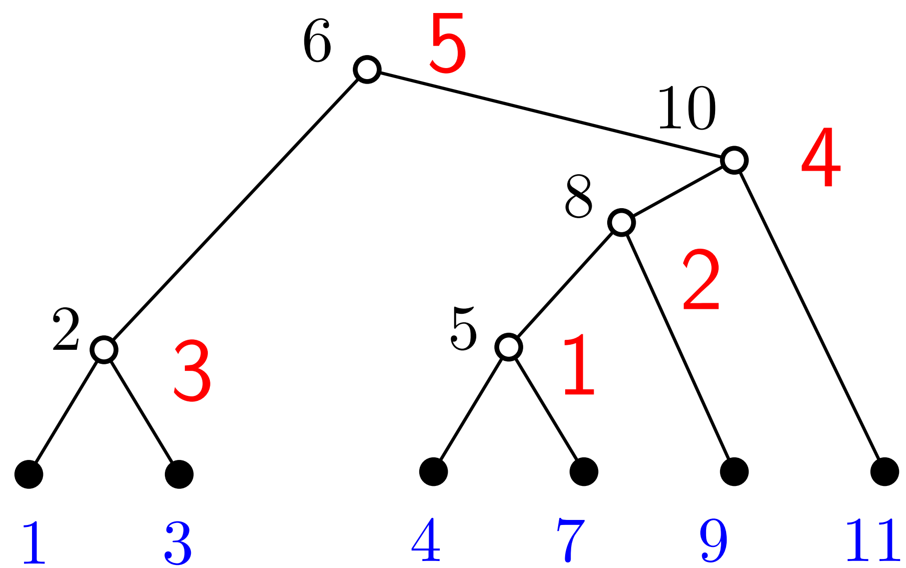
Combinatorial interpretation

Theorem (Liu '16 (q=1), N.-Tewari '20+)

$A_{\mathbf{c}}(1)$ is the number of pairs (T, ω) where :
(1) $T \in \mathcal{T}(\mathbf{c})$
(2) ω is a decreasing labeling on the nodes of $\mathbf{c}$.

Moreover, $A_{\mathbf{c}}(q)$ is obtained by counting each such (T, ω) with weight $q^{|\mathrm{Inv}(\omega)|}$.

ω viewed as permutation via projection.



$\omega \leftrightarrow 35124$

q^5

FIN

How we got into this

A one-page summary

Let $\mathcal{P}_n$ be the permutahedral variety over $\mathbb{C}$. It is a subvariety, of dimension n, inside the larger flag variety $\mathrm{Flags}(\mathbb{C}^{\mathrm{n+1}})$.

To get some information on it, we intersect it with some special subvarieties, the Schubert varieties X_w indexed by $w \in S_{n+1}$ with n inversions.

The intersection consists of a bunch of points: our leading question is how many ?

$$\longrightarrow \quad \boxed{a_w := \#(\mathcal{P}_n \cap X_w) \in \mathbb{N}}$$

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Using a rather long and winding road, these can be decomposed as follows:

(N.-Tewari '20)

$$a_w = \frac{1}{n!} \sum_{\mathbf{i} \in \mathrm{Red}(w)} A_{c(\mathbf{i})}(1).$$

From 1 to q

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... there's more:

$$a_w(q) := \frac{1}{[n]_q!} \sum_{\mathbf{i} \in \mathrm{Red}(w)} A_{c(\mathbf{i})}(q).$$

turns out to solve an analogous intersection problem in characteristic $p > 0$ when $q = p^f$.

($\mathcal{P}_n$ replaced with a Deligne-Lusztig variety).