Quantum generalized Heisenberg algebras and their combinatorial trace

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From q-algebras to generalized Heisenberg algebras

- Throughout this presentation, $\mathbb{F}$ will be denoted an arbitrary field unless otherwise stated.
- Generalized Heisenberg Algebras were introduced in the physics literatures in two different routes (but with the same source). One has been raised under the view of the Russian scientists perspective of the physics and mathematics and the other one from the Brazilian scientists point of view.
- All has started from an algebra which is called the q-algebra or the algebra of q-deformed commutators (or the q-Oscillator algebra), which has been introduced by Ludvig Dmitrievich Faddeev, Petr Petrovich Kulish and Michio Jimbo independently in a series of articles related to the integrable models on quantum field theory and quantum spectral transform methods, between 1982 to 1989.

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- The q-algebra is generated by operators a, $a^\dagger$ and N subject to the relations $[a, a^\dagger]_q = q^{-N}$, $[N, a] = -a$ and $[N, a^\dagger] = a^\dagger$, where $[a, a^\dagger]_q = aa^\dagger - qa^\dagger a$ and $\dagger$ stands for the Hermitian conjugate (adjoint or transpose; we work on matrix operator space) and N will be considered as a self adjoint matrix operator (the diagonal matrix with $n_{ii} = i$ for $i = 0, 1, \ldots$), for q a fixed complex number.
- This algebra is related to quantum groups, whose properties have been intensively studied since their appearance in connection with a quantum mechanical problem in statistical mechanics: the quantum Yang-Baxter equation.

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- In 1997, Sergey Yurievich Vernov and Melita Nikolaevna Mnatsakanova, have considered algebras of a more general form than the q-algebra, namely the algebras in which condition $[a, a^\dagger]_q = q^{-N}$ is replaced by $a^\dagger a = \varphi(N)$, $aa^\dagger = \varphi_1(N)$, where $\varphi(N)$ and $\varphi_1(N)$ are functions.
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- The most interesting result which people have gotten out of this example (due to Kirkman and Small) is that by solving these equations we will encounter with a new system of relations which will give us the q-analogues of the universal enveloping algebra of the 3-dimensional Heisenberg algebra $\mathfrak{h}_1$ which is the quotient of the free algebra $\mathbb{F}\langle a, a^\dagger, L\rangle$ modulo the two sided ideal I generated by the elements $aa^\dagger - qa^\dagger a - L, La^\dagger - q^{-1}a^\dagger L$ and $La - qaL$.

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- Afterward, in 2000 and 2001, in a series of articles, Sergey and Melita, have considered a class of algebras which they called the "*generalized Heisenberg algebras*". These algebras are almost the same as the generalized q-algebras studied by the same authors in 1997, with the only difference that in the generalized Heisenberg algebras (generalized generalized q-algebras) (introduced by Sergey and Melita), they specified and fixed φ_1 in terms of φ i.e. they defined $\varphi_1(N) := \varphi(N+1)$.
- In this case, with a little care, we can see that, these algebras can be seen as the generalized Heisenberg algebras defined by Curado and Monteiro in 2001, if we set $\varphi(N) = N - 1$.
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- From the Brazilian side, in 1999, the first introduced generalization of the Heisenberg type (also known as Heisenberg-Weyl type) algebras (algebras of the Canonical Commutation Relations (CCR)), has been formulated in a preprint in physics literature “Logistic algebras”, where Rego-Monteiro, has tried to generalize the known algebraic structures appeared in the solution of the physical systems. He called them Logistic algebras, because of the logistic map used in his definition.
- In 2001, an alternative strategy to construct a solvable model working with selfadjoint Hamiltonians has been proposed by Evaldo Curado and Rego-Monteiro in a series of articles (as a generalization of logistic algebras) on what has been called generalized Heisenberg algebras (GHA), which are dependent on a general analytic function $f(h)$ of h. These kinds of algebras have been considered by $\mathcal{H}(f)$ for $f(h) \in \mathbb{C}[h]$.

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- We have to note that the original definition for the generalized Heisenberg algebras $\mathcal{H}(f)$ in the work of Curado and Monteiro was using any analytic function $f(h)$, but Lü and Zhao only studied generalized Heisenberg algebras for $f(h) \in \mathbb{C}[h]$ and in our work we will follow the definition which has been introduced by Lü and Zhao.

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Generalized Heisenberg algebras (GHA)

- But what are generalized Heisenberg algebras?
- **Definition 1:** Let $f \in \mathbb{C}[h]$ be a fixed polynomial over $\mathbb{C}$. The *generalized Heisenberg algebra* $\mathcal{H}(f)$ is the unital associative $\mathbb{C}$-algebra with generators x, y, h satisfying the relations:

$$hx = xf(h), \quad yh = f(h)y, \quad yx - xy = f(h) - h. \tag{1}$$

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- From the algebraic point of view, regarding the *GHA*s, the following results have been obtained by Lü, Zhao and Volodymyr Mazorchuk:
- Computation of the center $Z(\mathcal{H}(f))$ of $\mathcal{H}(f)$ and of a basis for $\mathcal{H}(f)$.
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- And the following results have been obtained by Samuel Lopes:
- $\mathcal{H}(f)$ is right (resp. left) Noetherian if and only if $\deg f = 1$.
- If $\deg f \leq 1$ then $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra.
- Determination of all locally nilpotent and all locally finite derivations of $\mathcal{H}(f)$, in case $\deg(f) > 1$.
- Computation of the automorphism group of $\mathcal{H}(f)$, in case $\deg(f) > 1$.
- $\mathcal{H}(f)$ is a prime ring for any $f \in \mathbb{F}[h]$.
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On the way to combinatorics through down-up algebras

- It starts from a poset (partially ordered set) P. More specifically from a differential poset.
- But what are differential posets?
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- Let $\mathbb{F}$ be any field of characteristic 0.
- Given any poset P, we may define an abstract vector space $\mathbb{F}P = \oplus_{x \in P} \mathbb{F}x$ of finite linear combinations of elements of P with coefficients in $\mathbb{F}$.
- If in addition P is locally finite and each element of P is a member of only finitely many cover relations, we may define two linear transformations d and u on $\mathbb{F}P$ as follows:

$$dx = \sum_{y \lessdot x} y \qquad \text{and} \qquad ux = \sum_{x \lessdot y} y \tag{2}$$

where $\lessdot$ stands for the cover relationship and we extend both to all of $\mathbb{F}P$ by linearity.
- The operators u and d in (2) will be called up and down operators which keep track of all possible steps "up" and "down" in the Hasse diagram from x.

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- Now let us study the behavior of combinations of d, u on an arbitrary $x \in P$.
- Applying the definitions of d and u directly will give the following:

$$dux = \sum_{\substack{y,z \\ z \lessdot y \text{ and } x \lessdot y}} z.$$

An element $z \in P$ appears in this sum exactly k times, where k is the number of elements of P which cover both x and z.

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- Thus, we see that $(du - ud)x = x$ if and only if x is covered by exactly one more element than it covers and for each $z \neq x \in P$, the number of elements covering both x and z is equal to the number of elements covered by both x and z.
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- **Definition 2:** We call a poset P differential if it satisfies the following three axioms:
- (D1) P is locally finite and graded with a unique minimal element (often denoted $\hat{0}$).
- (D2) If $x \neq y$ are two elements of P and there are k elements of P covered by both x and y, there are exactly k elements of P which cover both x and y.
- (D3) If $x \in P$ covers k elements of P then x is covered by exactly $k+1$ elements of P.
- And we have the following Propsition which can easily be proved by induction on n:
- **Proposition 3:** For a differential poset P, we have $du^n = nu^{n-1} + u^n d$ for all $n \geq 1$.

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- **Definition 2:** We call a poset P differential if it satisfies the following three axioms:
- (D1) P is locally finite and graded with a unique minimal element (often denoted $\hat{0}$).
- (D2) If $x \neq y$ are two elements of P and there are k elements of P covered by both x and y, there are exactly k elements of P which cover both x and y.
- (D3) If $x \in P$ covers k elements of P then x is covered by exactly $k + 1$ elements of P.
- And we have the following Propsition which can easily be proved by induction on n:
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- Thus, the action of d on u has a distinct resemblance to that of a differential operator.
- The resemblance is even more clear if we apply the operator to our minimal element $\hat{0}$. since $d\hat{0} = 0$ so $du^n\hat{0} = nu^{n-1}\hat{0}$.
- It is from a generalization of this result that the name "differential poset" arises (Stanley- Differential Posets).
- Young's lattice of all partitions of all nonnegative integers provides an important example of such a poset.
- Stanley found that many of the interesting enumerative and structural properties of Young's lattice could be deduced from the relation $du - ud = I$, for I the identity transformation on $\mathbb{F}P$.
- Stanley also considered more general posets which satisfy the relation $du - ud = rI$ for some fixed positive integer r and he referred to these kind of posets as "r-differential".

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- In 1994, Sergey Vladimirovich Fomin independently defined essentially the same class of posets for $r = 1$ calling them "Y-graphs", the terminology inspired by Young's lattice.
- In his study of uniform posets, Paul Terwilliger considered finite ranked posets P whose down and up operators satisfy the following relation

$$d_i d_{i+1} u_i = \alpha_i d_i u_{i-1} d_i + \beta_i u_{i-2} d_{i-1} d_i + \gamma_i d_i, \quad (3)$$

where d_i and u_i denote the restriction of d and u to the elements of rank i.

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$$d_{i+1} u_i u_{i-1} = \alpha_i u_{i-1} d_i u_{i-1} + \beta_i u_{i-1} u_{i-2} d_{i-1} + \gamma_i u_{i-1}, \quad (4)$$

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- In many classical cases the constants in the above relations (relations (3) and (4)) do not depend on the rank of the poset. A particular instance of this provides a q-analogue of the notion of differential poset.
- **Definition 4:** A partially ordered set whose down and up operators satisfy

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\begin{aligned}
d^2u &= q(q+1)dud - q^3ud^2 + rd \\
du^2 &= q(q+1)udu - q^3u^2d + ru,
\end{aligned}
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where q and r are fixed complex numbers is said to be "(q, r)-differential poset".

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- In 1998, Georgia Benkart and Tom Roby have studied certain infinite-dimensional associative algebras whose generators satisfy relations more general than relations (5) and they called these class of algebras, down-up algebras:
- **Definition 5:** We say a unital associative algebra $A = A(\alpha, \beta, \gamma)$ over the complex numbers $\mathbb{C}$ with generators u, d and defining relations

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- Generalized down-up algebras (GDUA, for short) were introduced by Thomas Cassidy and Brad Shelton in 2004 as a generalization of the down-up algebras $A(\alpha, \beta, \gamma)$ of Benkart and Roby and are defined as follows:
- **Definition 6:** Let $f \in \mathbb{F}[h]$. Define $L := L(f, r, s, \gamma)$ to be the unital associative $\mathbb{F}$-algebra generated by x, y and h with relations

$$yh - rhy = \gamma y, \quad hx - rxh = \gamma x, \quad yx - sxy + f(h) = 0;$$

Then L is called a generalized down-up algebra $(GDUA)$.
- Over an algebraically closed field, generalized down-up algebras include all down-up algebras.

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A deep dive into quantum generalized Heisenberg algebras

- Our work aims to generalize the class of GHAs to a larger class of algebras which we call quantum generalized Heisenberg algebras as they can be seen simultaneously as multiparameter deformations and as generalizations of the GHAs.
- These depend on an arbitrary base field $\mathbb{F}$, a *quantum* parameter $q \in \mathbb{F}$ and two polynomials f and g.
- In spite of the name, the class of GHAs does not include the enveloping algebra of the Heisenberg Lie algebra nor its quantum deformations, nor the enveloping algebra of $\mathfrak{sl}_2$!
- These and many other like algebras are now included in the class of *qGHA*s.

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A deep dive into quantum generalized Heisenberg algebras

- Our main motivation for introducing a generalization of this class, besides providing a broader framework for the investigation of the underlying physical systems, comes from the observation that the classes of generalized Heisenberg algebras and of (generalized) down-up algebras intersect, although neither one contains the other.
- **Proposition 7:** The GHA $\mathcal{H}(f)$ is isomorphic to a generalized down-up algebra (GDUA) if and only if $\deg f \leq 1$.
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Quantum generalized Heisenberg algebras (qGHA)

- But what are quantum generalized Heisenberg algebras?
- **Definition 8:** Let $\mathbb{F}$ be an arbitrary field. Then for any fixed $f, g \in \mathbb{F}[h]$ and $q \in \mathbb{F}$, the quantum generalized Heisenberg algebra (*qGHA*, for short) $\mathcal{H}_q(f, g)$ is the unital associative algebra over $\mathbb{F}$ generated by x, y, and h subject to the defining relations

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- Working over an arbitrary field $\mathbb{F}$, we can thus view qGHAs as a generalization of the GHAs, by deforming and generalizing the relation $yx - xy = f(h) - h$, turning it into a skew-commutation relation and allowing the skew-commutator to equal a generic polynomial, independent from f.
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Quantum generalized Heisenberg algebras (qGHA)

- The first natural example of quantum generalized Heisenberg algebras is the class of GHAs, $\mathcal{H}(f) = \mathcal{H}_1(f, f - h)$.
- The universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_1$ is the quotient of the free algebra $\mathbb{F}\langle X, Y, H\rangle$ modulo the two sided ideal I generated by elements $XY - YX - H$, $XH - HX$ and $YH - HY$. It is easy to see that $U(\mathfrak{h}_1)$ is isomorphic to the quantum generalized Heisenberg algebra$\mathcal{H}_1(h, -h)$ if we consider the correspondence $X \leftrightarrow x$, $Y \leftrightarrow y$ and $H \leftrightarrow h$.
- Consider the 3-dimensional Lie algebra $\mathfrak{sl}_2$, with basis elements x, y, h and Lie bracket given by $[x, h] = 2x$, $[h, y] = 2y$ and $[y, x] = h$. We can view its enveloping algebra as $qGHA$ $\mathcal{H}_1(h - 2, h)$.

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Quantum generalized Heisenberg algebras (qGHA)

- Fix $g \in \mathbb{C}[x]$. Define $S = \mathbb{C}[A, B, H]$ subject to the relations

$$[H, A] = A, \qquad [H, B] = -B, \qquad AB - BA = g(H).$$

  Then S will be called the Smith algebra due to Paul Smith, and it is easy to see that S is isomorphic to the quantum generalized Heisenberg algebra $\mathcal{H}_1(h-1, g)$ if we consider the correspondence $B \leftrightarrow x$, $A \leftrightarrow y$ and $H \leftrightarrow h$.

- Fix $g \in \mathbb{C}[x]$ and $ç \in \mathbb{C}, ç \neq 0$. Define $R = \mathbb{C}[A, B, H]$ subject to the relations

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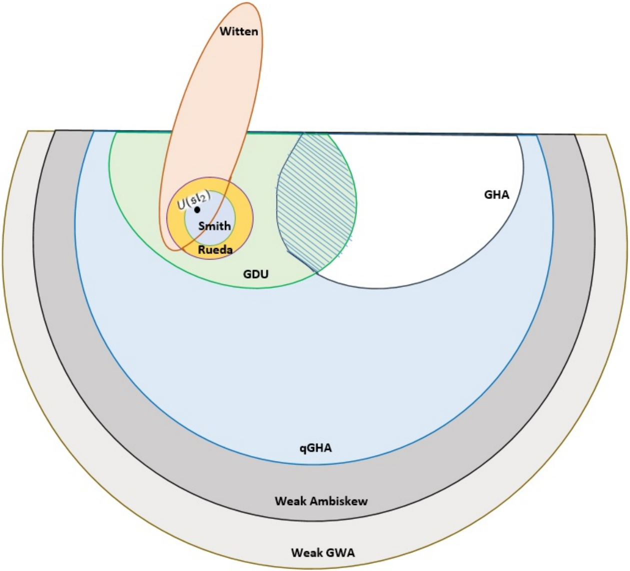

Quantum generalized Heisenberg algebras (qGHA)

- In our papers, with Arxiv identification numbers 2009.05270 and 2004.09301, we classified all simple finite dimensional $\mathcal{H}_q(f,g)$-modules, we determined automorphism groups of the $\mathcal{H}_q(f,g)$ when $\deg f > 1$ and we also solved the isomorphism problem for this class of algebras and we have specified when a quantum generalized Heisenberg algebra is Noetherian. We also have studied their ring-theoretical properties like Gelfand-Kirillov dimension and being domain and the primality of qGHAs. And in another direction, we have investigated and still are investigating Hopf quantum generalized Heisenberg algebras for some general class of qGHAs and their simple module theory.

Future works

- One of the longest-standing open conjectures in algebra is the *Dixmier conjecture* which asks if every endomorphism of the Weyl algebra $A_1(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0, or more generally of the n-th Weyl algebra $A_n(\mathbb{F}) = A_1(\mathbb{F})^{\otimes n}$ is an automorphism. This problem remains open even for $A_1(\mathbb{F})$ and in general Tsuchimoto and, independently, Kanel-Belov and Kontsevich, proved that the Dixmier conjecture is stably equivalent to the Jacobian conjecture from the field of algebraic geometry (of Keller).
- It is natural to ask similar questions regarding other algebras. Note that in characteristic 0 the Weyl algebras $A_n(\mathbb{F})$ are simple, so any endomorphism of the Weyl algebra is necessarily injective. Thus, a natural generalization of this question for other families of non-simple algebras is to ask whether all monomorphisms are automorphisms.

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Future works

- This type of question has been answered for several classes of algebras, some of which are *GWA* and some not, but to our knowledge it hasn't been investigated for GHA. Given the relevance of the Dixmier conjecture, this could be an interesting problem to study within the class of *GHA*.

Future works

- Representation theory is of great importance in almost all areas of pure and applied mathematics, and in theoretical physics. In spite of this, it is generally considered a hopeless problem to classify all simple representations for a given infinite-dimensional algebra, except under suitably special conditions.
- As has been indicated above, all finite-dimensional simple $\mathcal{H}(f)$-modules have been classified by Lü, Zhao and Mazorchuk. In their paper on simple weight modules over weak generalized Weyl algebras, they have considered the problem of classifying the simple weight modules over weak generalized Weyl algebras over a polynomial ring in just one variable. Although the latter does not directly cover *GHA*.

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- Study particular classes of representations of $\mathcal{H}(f)$, starting with an attempt to arrive at a good definition of Whittaker modules, and of torsion free module structures on suitable commutative subalgebras of $\mathcal{H}(f)$. The latter will involve studying the methods employed by Jonathan Nilsson in his paper on the simple $\mathfrak{sl}_{n+1}$-module structures on $U(\mathfrak{h})$ (for $\mathfrak{h}$ the standard Cartan subalgebra of $\mathfrak{sl}_{n+1}$) and also the references to other types of representations included in Jonathan's paper.

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- Computing the global and Krull dimension of $\mathcal{H}_q(f,g)$.
- Determining those quantum generalized Heisenberg algebras all of whose finite-dimensional representations are completely reducible.
- Studying simple weight modules for $\mathcal{H}_q(f,g)$.
- Determining the primitive ideals of $\mathcal{H}_q(f,g)$.
- Investigating Hochschild (co)homology of $\mathcal{H}_q(f,g)$.
- Going back to the Physics literature on generalized Heisenberg algebras, where these appeared to be defined over rings more general than the polynomial ring $\mathbb{F}[h]$, it would also be of interest to consider quantum generalized Heisenberg algebras defined over a Laurent polynomial ring $\mathbb{F}[h^{\pm 1}]$, a power series ring $\mathbb{F}[[h]]$ or the rational function field $\mathbb{F}(h)$.

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