Symmetric decompositions, triangulations and real-rootedness

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- Polynomials with nonnegative coefficients and only real roots arise frequently in combinatorics

- real rootedness of a polynomial $p(x) = c_n x^n + \cdots + c_1 x + c_0$ with $c_i \geq 0$ implies:

  - unimodality: $c_0 \leq c_1 \leq \cdots \leq c_k \geq c_{k+1} \geq \cdots \geq c_n$
  - log-concavity: $c_{i-1} c_{i+1} \leq c_i^2$
  - γ-positivity: if $p(x)$ is symmetric then $p(x) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k x^k (1+x)^{d-2k}$, with $\gamma_i \geq 0$

- Find nonnegative, real rooted polynomials in geometric combinatorics

  Brenti, Brändén, Borcea-Brändén, Jochemko, Mohammadi-Welker, Hlavacek-Solus,....

i.e. if P is a simplicial polytope, the h-polynomial $h(sd(P), x)$ of the barycentric subdivision of P is nonnegative and real rooted

- Every $p(x) \in \mathbb{R}[x]$ with $\deg(p(x)) \leq d$ can be uniquely decomposed as

$$p(x) = a(x) + x\, b(x)$$

➠ $\deg(a(x)) \leq d, \;\; a(x) = x^d a(\frac{1}{x})$ and $a(x) = \frac{p(x) - x^{d+1} p(1/x)}{1-x}$

➠ $\deg(b(x)) \leq d-1, \;\; b(x) = x^{d-1} b(\frac{1}{x})$ and $b(x) = \frac{x^d p(1/x) - p(x)}{1-x}$

➠ the decomposition depends on $p(x)$ and d

example: if $p(x) = 2x^3 - x^2 + x - 8$ then

for $d = 3$: $\quad p(x) = (-8x^3 - 9x^2 - 9x - 8) + x(10x^2 + 8x + 10)$

for $d = 4$: $\quad p(x) = (-8x^4 - 7x^3 - 10x^2 - 7x - 8) + x(8x^3 + 9x^2 + 9x + 8)$

- We say that $p(x) = a(x) + x\, b(x)$ has $\left.\begin{array}{l}\text{nonnegative}\\ \text{unimodal}\\ \text{real rooted}\\ \text{interlacing}\end{array}\right\}$ symmetric decomposition

  if both $a(x)$ and $b(x)$ have the corresponding property

- Find polynomials in geometric combinatorics which have nonnegative, real rooted symmetric decompositions

i.e. if Δ is a triangulation of a ball, then

$$h(\Delta, x) = h(\partial\Delta, x) + x\,\frac{(h(\Delta, x) - h(\partial\Delta, x))}{x}$$

is a symmetric decomposition

Under what conditions this symmetric decomposition has nice properties?

- Let $f(x), g(x) \in \mathbb{R}[x]$ be real rooted polynomials

  Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be the roots of f

  Let $b_1 \geq b_2 \geq \cdots \geq b_m$ be the roots of g $\quad (m = n$ or $n+1)$

- the polynomial f interlaces g $(f \prec g)$ if

$$b_{n+1} \leq a_n \leq b_n \leq \cdots \leq a_1 \leq b_1$$

- A sequence $(p_i(x))_{i=0}^{n}$ of real rooted polynomials in an interlacing sequence if

$$p_i(x) \prec p_j(x) \text{ for all } \quad 1 \leq i \leq j \leq n$$

[Wagner '92] If $(f_i)_{i=1}^n$ is an interlacing sequence and $\lambda_i \geq 0$, then

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n \quad \text{is real rooted}$$

[Brändén '15] If $(f_i)_{i=1}^n$ and $(g_i)_{i=1}^n$ are two interlacing sequences then

$$f_1 g_n + f_2 g_{n-1} + \cdots + f_n g_1 \quad \text{is real rooted}$$

[Savage,Visontai '15] If f, g be two polynomials with nonnegative coefficients.

Then $f \prec g$ iff $(\lambda x + \mu)f + g$ is real rooted for all $\lambda, \mu > 0$

Starting point: face enumeration of simplicial complexes

- Let Δ be a simplicial complex with $\dim(\Delta) = n - 1$

  f-vector: $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{n-1}(\Delta))$ where $f_i(\Delta) = \#$ of i-dim faces of Δ

$$f(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i$$

  h-vector: $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_n(\Delta))$

$$h(\Delta, x) := \sum_{i=0}^{n} h_i(\Delta) x^i = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1 - x)^{n-i}$$

- They are related by $f(\Delta, x) = (1 + x)\, h(\Delta, \frac{x}{1+x})$

Question: Let Δ be a simplicial complex. Find triangulations Δ' of Δ having nonnegative, real rooted $h(\Delta', x)$?

- [Brenti-Welker] Barycentric subdivisions
- [Jochemko] r-fold edgewise subdivisions
- [Anwar-Nazir,Mohammadi-Welker] Interval subdivisions (2-colored barycentric subdivisions)
- [Athanasiadis] r-colored barycentric subdivisions (for $r = 1$ they reduce to barycentric subdivisions)

Let Δ be a "nice" simplicial complex (i.e., sphere, ball, Cohen-Macualay,...)

Find broad classes of triangulations Δ' of Δ which have the property that $h(\Delta', x)$ is nonnegative and real rooted

[Athanasiadis '20] Uniform triangulations which have the strong interlacing property ✓

Find broad classes of triangulations Δ' of Δ which have the property that $h(\Delta', x)$ nonnegative real rooted symmetric decomposition.

[Athanasiadis,T.'21] Again, uniform triangulations which have the strong interlacing property lead to an affirmative answer under certain conditions

- we give conditions on $h(\Delta)$ under which the symmetric decomposition of $h(\Delta', x)$ is nonnegative and real rooted
- we give conditions on $h(\Delta)$ under which the above symmetric decomposition of $h(\Delta', x)$ is also interlacing

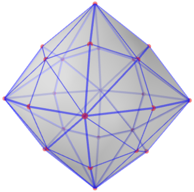
- Let Δ be a simplicial complex of $\dim(\Delta) \leq d$ and $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)_{-1\leq i \leq j \leq d-1}$

  A triangulation Δ' of Δ is $\mathcal{F}$-uniform if, for all $0 \leq i \leq j \leq d$

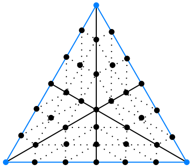
  each j-face of Δ contains the $\underbrace{\text{same number of } i\text{-faces}}_{f_{\mathcal{F}}(i+1,j+1)}$ of Δ'

- We call $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)_{0\leq i \leq j \leq d}$ an f-triangle of size d

- $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)$ is feasible if every simplex σ_j, $1 \leq j \leq d$ has an $\mathcal{F}$-uniform triangulation



1	7	12	6
1	3	2	
1	1		
1			



1	37	90	54
1	7	6	
1	1		
1			

Theorem [Athanasiadis '20] If $\mathcal{F}$ is a feasible f-triangle and Δ' an $\mathcal{F}$-uniform triangulation of Δ, then

$$h(\Delta', x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{\mathcal{F},n,k}(x)$$

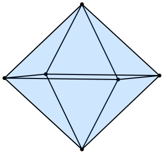
where $p_{\mathcal{F},n,k}(x)$ are polynomials with nonnegative coefficients, depending only on $\mathcal{F}$.

- $$h_{\mathcal{F}}(\sigma_n, x) = p_{\mathcal{F},n,0}(x) = \sum_{k=0}^{n} p_{\mathcal{F},n-1,k}(x)$$
- $x^n p_{\mathcal{F},n,k}(\frac{1}{x}) = p_{\mathcal{F},n,n-k}(x)$
- $$p_{\mathcal{F},n,k}(x) = p_{\mathcal{F},n,k-1}(x) + (x-1)p_{\mathcal{F},n-1,k-1}(x)$$
- $p_{\mathcal{F},n,k}$ is the h-polynomial of the relative simplicial complex obtained from the $\mathcal{F}$-triangulation of σ_n by removing all faces on k facets of $\partial\sigma_n$

Theorem [Brenti-Welker'08] For barycentric subdivisions we have $p_{\mathcal{F},n,k}(x) = \sum_{j=0}^{n} p_{\mathcal{F}}(n,k,j)x^j$, where

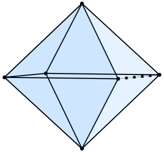
$$p_{\mathcal{F}}(n,k,j) = \#\text{ permutations of } \{1,\dots,n+1\} \text{ with } k \text{ descents and } \sigma(1) = j+1$$

$$\left.\begin{aligned} p_{3,0}(x) &= x^2 + 4x + 1 \\ p_{3,1}(x) &= 2x^2 + 4x \\ p_{3,2}(x) &= 4x^2 + 2x \\ p_{3,3}(x) &= x^3 + 4x^2 + x \end{aligned}\right\} \text{the polynomials } p_{\mathcal{F},3,i}(x) \text{ for the barycentric subdivision of the 2-dim simplex}$$



$h(\Delta) = (1,3,3,1)$

$$h(sd(\Delta)) = 1 \cdot p_{3,0}(x) + 3 \cdot p_{3,1}(x) + 3 \cdot p_{3,2}(x) + 1 \cdot p_{3,3}(x)$$
$$= x^3 + 23x^2 + 23x + 1$$



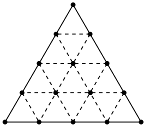
$h(\Delta) = (1,3,2,0)$

$$h(sd(\Delta)) = 1 \cdot p_{3,0}(x) + 3 \cdot p_{3,1}(x) + 2 \cdot p_{3,2}(x) + 0 \cdot p_{3,3}(x)$$
$$= 15x^2 + 20x + 1$$

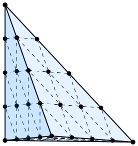
Properties of $p_{\mathcal{F},n,k}(x)$ [Athanasiadis '20]

- $h_{\mathcal{F}}(\sigma_n, x) = p_{\mathcal{F},n,0}(x) = \sum_{k=0}^{n} p_{\mathcal{F},n-1,k}(x)$
- $x^n p_{\mathcal{F},n,k}(\frac{1}{x}) = p_{\mathcal{F},n,n-k}(x)$
- $p_{\mathcal{F},n,k}(x) = p_{\mathcal{F},n,k-1}(x) + (x-1)p_{\mathcal{F},n-1,k-1}(x)$
- $p_{\mathcal{F},n,n+1}(x) := h_{\mathcal{F}}(\sigma_{n+1}) - h_{\mathcal{F}}(\partial\sigma_{n+1})$

$p_{2,0}(x) = 3x + 1$
$p_{2,1}(x) = 4x$
$p_{2,2}(x) = x^2 + 3x$
$p_{2,3}(x) = 2x^2 + 2x$



$p_{3,0}(x) = 3x^2 + 12x + 1$
$p_{3,1}(x) = 6x^2 + 10x$
$p_{3,2}(x) = 10x^2 + 6x$
$p_{3,3}(x) = x^3 + 12x^2 + 3x$
$p_{3,4}(x) = 0$



$p_{4,0}(x) = x^3 + 31x^2 + 31x + 1$
$p_{4,1}(x) = 4x^3 + 40x^2 + 20x$
$p_{4,2}(x) = 10x^3 + 44x^2 + 10x$
$p_{4,3}(x) = 20x^3 + 40x^2 + 4x$
$p_{4,4}(x) = x^4 + 31x^3 + 31x^2 + x$
$p_{4,5}(x) = 34x^3 + 124x^2 + 34$

- Let $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)_{0 \le i \le j \le n}$ be a feasible f-triangle.

  The triangle $\mathcal{F}$ has the strong interlacing property with respect to n if,

  ① $h_{\mathcal{F}}(\sigma_m, x)$ is real rooted for all $m = 2, \dots, n-1$

  ② $h_{\mathcal{F}}(\sigma_m, x) - h_{\mathcal{F}}(\partial\sigma_m, x)$ ➠ is identically zero or

  ➠ is real rooted of degree $m-1$ with nonnegative coefs interlaced by $h_{\mathcal{F}}(\sigma_{m-1}, x)$, for all $m = 2, \dots, n-1$

- useful property: If $\mathcal{F}$ has the strong interlacing property, then $\big(p_{\mathcal{F},n,0}(x), p_{\mathcal{F},n,1}(x), \dots, p_{\mathcal{F},n,n}(x)\big)$ is an interlacing sequence

Theorem [Athanasiadis'20]

If $h(\Delta) = (h_0(\Delta), \dots, h_n(\Delta))$ is nonnegative, then $h(\Delta', x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{\mathcal{F},n,k}(x)$ is real rooted

$h(\Delta', x) = \sum_{k=0}^{n} h_k(\Delta)\, p_{\mathcal{F},n,k}(x)$

Theorem [Athanasiadis'20]

Let $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)$ be a feasible triangle having the strong interlacing property w.r.t. n

Let $\mathcal{D}_{\mathcal{F},n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ with $\mathcal{D}_{\mathcal{F},n}(x^k) = p_{\mathcal{F},n,k}(x)$

If $p(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ with $c_i \geq 0$ then $\mathcal{D}_{\mathcal{F},n}(p(x))$ is nonnegative and real rooted

Theorem [Athanasiadis, T. '21]

Let $\mathcal{F} = \big(f_{\mathcal{F}}(i,j)\big)$ be a feasible triangle having the strong interlacing property w.r.t. n

Let $\mathcal{D}_{\mathcal{F},n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ with $\mathcal{D}_{\mathcal{F},n}(x^k) = p_{\mathcal{F},n,k}(x)$

and $p(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ with $c_i \geq 0$ for all i

① If $c_0 + c_1 + \cdots + c_i \leq c_n + c_{n-1} + \cdots + c_{n-i}$ for all $0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

then $\mathcal{D}_{\mathcal{F},n}(p(x))$ has a nonnegative, real-rooted symmetric decomposition w.r.t. n

② If, in addition, $c_i c_{n-i-1} \leq c_{i+1} c_{n-i}$ for all $0 \leq i \leq n-1$

then, the above decomposition is interlacing.

proof

Conditions of Theorem (1) in terms of h-vectors: $h_n \geq 0$

- Condition for $h(\Delta', x)$ having real rooted symmetric decomposition

① $h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta) \leq h_n(\Delta) + h_{n-1}(\Delta) + \cdots + h_{n-i}(\Delta)$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$

Question: Find families of simplicial complexes having h-vector satisfying ①

➠ Doubly Cohen-Macaulay simplicial complexes [Stanley]

- Condition for $h(\Delta', x)$ having interlacing symmetric decomposition

② $$\frac{h_0(\Delta)}{h_n(\Delta)} \leq \frac{h_1(\Delta)}{h_{n-1}(\Delta)} \leq \cdots \leq \frac{h_{n-1}(\Delta)}{h_1(\Delta)} \leq \frac{h_n(\Delta)}{h_0(\Delta)}$$

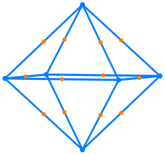
Question: Do all 2-Cohen-Macaulay simplicial compexes Δ satisfy ②?

Theorem [Athanasiadis,T.'21]

Let $\mathcal{F} = \left(f_{\mathcal{F}}(i,j)\right)$ be a feasible triangle having the strong interlacing property w.r.t. $n+1$

Let Γ be an n-simplicial complex with nonnegative h-vector and Δ its $(n-1)$-skeleton

- $h_{\mathcal{F}}(\Delta, x)$ has a nonnegative, real rooted and interlacing symmetric decomposition w.r.t. n



Example: If Δ is the 1-skeleton of the boundary of the cross-polytope:

$$h(sd(\Delta)) = 7x^2 + 16x + 1$$

$$= (x^2 + 10x + 1) + x(6x + 6)$$ real rooted and interlacing symmetric decomposition

Proof: We use the fact that $h_k(\Delta) = h_0(\Gamma) + h_1(\Gamma) + \cdots + h_k(\Gamma)$

Our main result (2)

Theorem [Athanasiadis, T. '21]

Let $\mathcal{D}_{\mathcal{F},n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ with $\mathcal{D}_{\mathcal{F},n}(x^k) = p_{\mathcal{F},n,k}(x)$

Let $\mathcal{F} = \left(f_{\mathcal{F}}(i,j)\right)$ be a feasible triangle having the strong interlacing property w.r.t. $n-1$

$p(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathbb{R}_n[x]$ with $c_i \geq 0$ for all i

① If $\quad c_0 + c_1 + \cdots + c_i \geq c_{n-1} + \cdots + c_{n-i}$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$

then $\mathcal{D}_{\mathcal{F},n}(p(x))$ has a nonnegative, real-rooted symmetric decomposition w.r.t. $n-1$

② If, in addition, $\quad c_i c_{n-i-1} \geq c_{i+1} c_{n-i}$ for all $1 \leq i \leq n-1$

then, the above decomposition is interlacing

proof: apply Theorem(1) to $x^n p(1/x)$

- Condition for $h(\Delta', x)$ having real rooted symmetric decomposition

① $h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta) \geq h_{n-1}(\Delta) + h_{n-2}(\Delta) + \cdots + h_{n-i}(\Delta)$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$

Question: Find families of simplicial complexes having h-vector satisfying ①

➠ every simplicial $(n-1)$-ball satisfies ① [Stanley '93]

- Condition for $h(\Delta', x)$ having interlacing symmetric decomposition

② $\frac{h_1(\Delta)}{h_{n-1}(\Delta)} \geq \frac{h_2(\Delta)}{h_{n-2}(\Delta)} \geq \cdots \geq \frac{h_{n-2}(\Delta)}{h_2(\Delta)} \geq \frac{h_{n-1}(\Delta)}{h_1(\Delta)}$

Question: Find families of simplicial complexes having h-vector satisfying ① and ②

➠ Is there a class of triangulations of simplicial balls satisfying ②?

- Is it true that for any polytope P the h-polynomial $h(sd(P), x)$ is nonnegative and real rooted?
- Is it true that for any "nice" polytopal complex Δ the h-polynomial $h(sd(\Delta), x)$ is nonnegative and real rooted?
- Is it true that for any "nice" polytopal complex Δ the h-polynomial $h(sd(\Delta), x)$ has a nonnegative real rooted symmetric decomposition?

Thank you for your attention!

ευχαριστώ για την προσοχή σας

proof (idea)

Let $p(x) = c_0 + c_1 x + \cdots + c_n x^n$ with $c_i \geq 0$ for all i

- We find the symmetric decomposition of $\mathcal{D}_{\mathcal{F},n}(p(x))$:

$$\mathcal{D}_{\mathcal{F},n}(p(x)) = \sum_{k=0}^{n} c_k \, p_{\mathcal{F},n,k}(x) = \sum_{k=0}^{n} c_k \Big(x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) + \sum_{i=k}^{n} p_{\mathcal{F},n-1,i}(x) \Big)$$

$$= \cdots =$$

$$= (c_0 + \cdots + c_n) p_{\mathcal{F},n-1,n}(x) + \sum_{i=0}^{n-1} \Big(c_0 + \cdots + c_i + (c_0 + \cdots + c_{n-1-i}) x \Big) p_{\mathcal{F},n-1,i}$$

$$+ x \sum_{i=0}^{n-1} (c_n + c_{n-1} + \cdots + c_{n-i} - c_0 - c_1 - \cdots - c_i) p_{\mathcal{F},n-1,i}(x)$$

$$\mathcal{D}_{\mathcal{F},n}(p(x)) = \sum_{k=0}^{n} c_k \, p_{\mathcal{F},n,k}(x) = \sum_{k=0}^{n} c_k \Big(x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) + \sum_{i=k}^{n} p_{\mathcal{F},n-1,i}(x) \Big)$$

- $a(x)$ and $b(x)$ are real rooted
  - $b(x)$ is a nonnegative linear combination of the polynomials $p_{\mathcal{F},n-1,i}(x)$
  - $a(x)$ is a sum of polynomials $\lambda_i(x) \cdot p_{\mathcal{F},n-1,i}$ with $\deg(\lambda_i(x)) \leq 1$ and nonnegative coefficients
  - $p_{\mathcal{F},_{n-1,i}}(x), \quad i = 0, \ldots, n$ is an interlacing sequence
- if $c_i c_{n-1-i} \leq c_{n+1} c_{n-i}$ then $a(x)$ and $b(x)$ are interlacing
  - $p_{\mathcal{F}}(x) := \mathcal{D}_{\mathcal{F},n}(p(x))$
  - It suffices to show that $p_{\mathcal{F}}(x)$ is interlaced by $x^n p_{\mathcal{F}}(\frac{1}{x})$
  - this is true if $\dfrac{c_i}{c_{i+1}} \leq \dfrac{c_{n-i}}{c_{n-i-1}}$