Séminaire Lotharingien de Combinatoire **87** (2023), Article B87b

HAGLUND'S POSITIVITY CONJECTURE FOR MULTIPLICITY ONE PAIRS

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Abstract. Haglund's conjecture states that $\frac{\langle J_\lambda(q,q^k), s_\mu\rangle}{(1-q)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[q]$ for all partitions λ, μ and all non-negative integers k, where J_λ is the integral form Macdonald symmetric function and s_μ is the Schur function. This paper proves Haglund's conjecture in the cases when the pair (λ, μ) satisfies $K_{\lambda,\mu} = 1$ or $K_{\mu',\lambda'} = 1$ where K denotes the Kostka number. We also obtain some general results about the transition matrix between Macdonald symmetric functions and Schur functions.

1. Introduction

The Macdonald symmetric functions $P_\lambda(q,t)$, $\lambda \in Par$, are a remarkable family of symmetric functions depending on two parameters q and t, indexed by the set of partitions Par. They simultaneously generalize many known bases of symmetric functions.

We denote by $K^{(1)}_{\lambda,\mu}(q,t)$ the coefficient of the monomial symmetric function m_μ in the monomial expansion of $P_\lambda(q,t)$. This is a rational function in q and t which has the Kostka number $K_{\lambda,\mu}$ as its limit when q and t tend to 1. There are various expressions for $K^{(1)}_{\lambda,\mu}(q,t)$ in the literature: in [Mac95] a tableaux formula is given, in [HHL04], [HHL06] a formula in terms of nonattacking fillings is found, and in [RY08] a formula in terms of alcove walks is given. A nice survey of some of the monomial expansions can be found in [GR21] and [GR21supplement].

In contrast, very little is known about the Schur expansion of $P_\lambda(q,t)$. Some particular Schur coefficients of the integral form Macdonald polynomials $J_\lambda(q,t)$ were found in [Yoo12] and [Yoo15]. The integral form Macdonald polynomial $J_\lambda(q,t)$ is a certain normalization of $P_\lambda(q,t)$, whose monomial coefficients are in $\mathbb{Z}[q,t]$.

In [HHL04] it was shown that the coefficient $\langle J_\lambda(q,t), h_\mu\rangle$ of m_μ in $J_\lambda(q,t)$ has the following positivity property:

$$\frac{\langle J_\lambda(q,q^k), h_\mu\rangle}{(1-q)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[q] \quad \text{for all } k \in \mathbb{Z}_{\geq 0} \text{ and } \lambda, \mu \in Par. \tag{1}$$

Haglund [Hag10] conjectured that the above equation holds true if h_μ is replaced by s_μ.

Conjecture A (Haglund)**.** *For partitions λ and μ,*

$$\frac{\langle J_\lambda(q,q^k), s_\mu\rangle}{(1-q)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[q] \quad \textit{for all } k \in \mathbb{Z}_{\geq 0}. \tag{Hag(λ,μ)}$$

Yoo showed this is true in some special cases in [Yoo12] and [Yoo15] by obtaining explicit formulas for the coefficients. Among them, some particular cases are when λ

2020 *Mathematics Subject Classification.* 05E05.

has only one row, when λ is of hook shape with $\ell(\lambda) \geq \lambda_1 - 2$, when λ has at most two columns, when μ is of hook shape, or when λ and μ both have length at most 2.

This paper extends Yoo's results in a different direction, showing Haglund's conjecture to be true for all 'multiplicity one pairs', i.e., for (λ, μ) such that $K_{\lambda,\mu} = 1$ or $K_{\mu',\lambda'} = 1$.

The following is the main result of this paper.

Theorem 1.1. $\mathrm{Hag}(\lambda, \mu)$ *holds true for all pairs* λ, μ *such that either* $K_{\lambda,\mu} = 1$ *or* $K_{\mu',\lambda'} = 1$.

In fact, what we establish is a dual version of Haglund's conjecture which can be easily seen to be equivalent to the original. Namely, let $k_{\lambda,\mu}(q,t) = \langle s_\lambda, J_\mu(q,t)\rangle_{q,t}$ where $\langle \; , \; \rangle_{q,t}$ is the (q,t)-deformation of the Hall scalar product (cf. [Mac95]). Then we have the following.

Conjecture B (Dual Haglund's Conjecture)**.** *For partitions* λ, μ*, we have*

$$\frac{k_{\lambda,\mu}(t^k, t)}{(1-t)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[t] \quad \textit{for all } k \in \mathbb{Z}_{\geq 0}. \qquad (\mathrm{Hag}'(\lambda,\mu))$$

We also define another q,t-analogue $K^{(2)}_{\lambda,\mu}(q,t)$ of the Kostka numbers by

$$s_\lambda = \sum_{\mu \in Par} K^{(2)}_{\lambda,\mu}(q,t) P_\mu(q,t) \text{for all } \lambda \in Par. \qquad (2)$$

$K^{(2)}_{\lambda,\mu}(q,t)$ coincides with $k_{\lambda,\mu}(q,t)$ up to a constant. Analogous to $K^{(1)}_{\lambda,\mu}(q,t)$, this is a rational function in q,t, and $K^{(2)}_{\lambda,\mu}(0,t) = K_{\lambda,\mu}(t)$, the Kostka–Foulkes polynomial. The matrix $(K^{(2)}_{\lambda,\mu})_{\lambda,\mu \vdash n}$ has the interesting property that (cf. [Mac95, Chapter VI, (5.1″)])

$$K^{(2)}_{\lambda,\mu}(q,t) = (K^{(2)})^{-1}_{\mu',\lambda'}(t,q). \qquad (3)$$

Since the entries of $(K^{(2)})^{-1}$ are the Schur coefficients of the $P_\lambda(q,t)$, finding $K^{(2)}$ is roughly equivalent to finding these.

We use Macdonald's description of $K^{(1)}_{\lambda,\mu}(q,t)$ and basic properties of $P_\lambda(q,t)$ to find some general reduction principles. In particular, we show that (see Lemma 4.2), if for a pair of partitions (λ, μ) we have $\lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r$ for some r, then, for $i = 1, 2$, we have

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^1,\mu^1} \cdot (K^{(i)})^{\pm 1}_{\lambda^2,\mu^2} \;,$$

where $\lambda^1 = (\lambda_1, \ldots, \lambda_r)$, $\mu^1 = (\mu_1, \ldots, \mu_r)$, $\lambda^2 = (\lambda_{r+1}, \ldots)$, $\mu^2 = (\mu_{r+1}, \ldots)$.

A dual version of this is given in Lemma 4.3, where instead of breaking the Young diagrams across a row, we break across a column.

Let $\lambda, \mu \in Par$ such that $\lambda \geq \mu$. We can always decompose $\lambda = (\lambda^1, \ldots, \lambda^r)$ and $\mu = (\mu^1, \ldots, \mu^r)$ where $\lambda^1, \ldots \lambda^r, \mu^1, \ldots, \mu^r$ are partitions such that, for $i \in \{1, \ldots r\}$, $\lambda^i \geq \mu^i$ and, for $i \in \{1, \ldots r-1\}$, $\ell(\lambda^i) = \ell(\mu^i)$. Then, for $i = 1, 2$, we have

$$(K^{(i)})^{\pm 1}_{\lambda,\mu}(q,t) = \prod_{j=1}^{r} (K^{(i)})^{\pm 1}_{\lambda^j,\mu^j}(q,t). \qquad (4)$$

For $j \in \{1, \ldots, r-1\}$ each pair (λ^j, μ^j) contains a common rectangle of row length at least λ^{j+1}_1. Proving the dual Haglund conjecture is equivalent to proving it for each pair

(λ^j, μ^j) with these rectangles removed (see Corollary 4.1). We can repeat this process for each pair until no further decomposition is possible.

This allows us to only look at pairs of partitions (λ, μ) which are *irreducible*, i.e., which satisfy $\lambda \geq \mu$ and $\lambda_1 + \cdots + \lambda_i > \mu_1 + \cdots + \mu_i$ for $1 \leq i \leq \ell(\lambda)$. Berenshteĭn and Zelevinskiĭ in [BZ90] gave a criterion for a pair of partitions (λ, μ) to satisfy $K_{\lambda,\mu} = 1$. This can be restated as follows.

Theorem 1.2. *Let λ, μ be an irreducible pair. Then $K_{\lambda,\mu} = 1$ if and only if either*

(1) $\lambda = (m^n)$ *for some* $m \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 0}$ *and* $\ell(\mu) = n + 1$,
or
(2) $\lambda = (n)$ *and* $\mu \vdash n$, $\mu \neq \lambda$ *for some* $n \in \mathbb{Z}_{\geq 1}$.

We find (see Corollary 5.1) that, if λ, μ are partitions contained in a rectangle (m^{n+1}), then, for $i = 1, 2$, we have

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^c,\mu^c} \ , \tag{5}$$

where λ^c, μ^c denote the complements of the partitions λ, μ inside the rectangle (m^{n+1}). Since the complement of the partition (m^n) is (m), this reduces all computation of $K^{(2)}_{\lambda,\mu}$ when $K_{\lambda,\mu} = 1$ to the case where λ is a single row.

In this case we calculate $K^{(2)}_{\lambda,\mu}$ by the use of the Cauchy identity and Macdonald's principal specialization formula (see Proposition 3.2.)

Finally, in Section 6 we finish the proof of the dual version of Haglund's conjecture when $K_{\lambda,\mu} = 1$ by analyzing the normalization factors for the integral form Macdonald polynomials. By exploiting duality of Macdonald polynomials (see Lemma 2.1) we get a proof when $K_{\mu',\lambda'} = 1$.

Our reduction principles may also be applied to instances where the final reduction is to pairs (λ, μ) of the form studied by Yoo. Therefore explicit formulas for $k_{\lambda,\mu}(q,t)$ can be obtained in a wider set of cases.

An interesting feature of our formulas and many of Yoo's formulas is that $K^{(2)}_{\lambda,\mu}(q,t)$ can be written as a sum of $K_{\lambda,\mu}$ many terms, where each term is a product with factors of the form $(t^a - q^b t^c)$ in the numerator and $(1 - q^d t^e)$ in the denominator with $a, b, c, d, e \in \mathbb{Z}_{\geq 0}$. This is especially striking if $K_{\mu',\lambda'} = 1$, where $K_{\lambda,\mu}$ could be larger than 1.

Acknowledgments. We would like to thank R. Venkatesh for organizing the Workshop on Macdonald Polynomials 2021. We are very grateful to A. Ram for his inspiring lectures and discussions. Finally, a lot of thanks to S. Viswanath for numerous discussions, going through the whole paper and giving helpful suggestions.

2. Definitions

We first review some basic definitions in the theory of symmetric functions. The main reference for this section is [Mac95].

2.1. **Partitions.** By a partition we mean a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers which is weakly decreasing and has a finite sum. By Par we denote the set of all partitions. A partition λ may be realized as a set of boxes upper left justified and arranged in rows so that there are λ_i boxes in row i. This is called the Young diagram of λ. For a partition λ the conjugate partition λ' is the partition obtained by interchanging the rows and columns in the diagram of λ. The length of a partition λ is $\ell(\lambda) = \lambda'_1$, and the size is $|\lambda| = \sum \lambda_i$

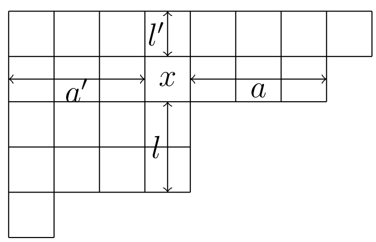
The set of partitions is partially ordered by the dominance order defined by

$$\lambda \geq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } i \geq 1, \text{ and } |\lambda| = |\mu|. \tag{6}$$

For a box $x = (r, c)$ in the diagram of λ, where r and c are the row and column numbers of x with row numbering starting from 1 in the top row and column numbering starting from 1 in the leftmost column, we write $x \in \lambda$ and define the arm length $a_\lambda(x)$, coarm length $a'_\lambda(x)$, leg length $l_\lambda(x)$ and the coleg length $l'_\lambda(x)$ of x, by

$$a_\lambda(x) = \lambda_r - c, \ a'_\lambda(x) = c - 1 \ , \tag{7}$$

$$l_\lambda(x) = \lambda'_c - r, \ l'_\lambda(x) = r - 1 \ . \tag{8}$$



$$a_\lambda(x) = 3, \ a'_\lambda(x) = 3$$
$$l_\lambda(x) = 2, \ l'_\lambda(x) = 1$$

FIGURE 1. Young diagram for $\lambda = (8, 7, 4, 4, 1)$ and arm, coarm, leg, coleg lengths of $x = (2, 4)$

When the partition is understood, we drop λ from the notation and simply write $a(x), a'(x), l(x), l'(x)$.

The content of a box $x \in \lambda$ is $c(x) = a'(x) - l'(x)$. The hook length of the box x is $h(x) = a(x) + 1 + l(x)$. For a partition λ, define $n(\lambda) = \sum_{x \in \lambda} l'(x)$.

2.2. **Symmetric Functions.** Let q and t be complex variables. We work in the ring Λ of symmetric functions in infinitely many variables $x_1, x_2, \ldots$ with coefficients from $\mathbb{C}(q, t)$.

The space of symmetric functions has a few known bases indexed by $\lambda \in Par$, such as the monomial basis (m_λ), the elementary basis (e_λ), the homogeneous basis (h_λ), the power sum basis (p_λ), and the Schur basis (s_λ). Their definitions can be found in [Mac95].

For a partition λ and for $i \geq 1$, let $m_i(\lambda)$ be the number of parts of λ equal to i, and let $z_\lambda = \prod_{i=1}^{\ell(\lambda)} i^{m_i(\lambda)} m_i(\lambda)!$. The Hall scalar product on Λ, denoted by $\langle \ , \ \rangle$, is defined by $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$. We also have $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ and $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$.

2.3. **Plethysm.** The power sums $p_k(X) = \sum_{i \geq 0} x_i^k$, $k \geq 1$, form an algebraically independent generating set for Λ over $\mathbb{C}(q, t)$. For a formal series of rational functions $E = E(z_1, z_2, \ldots)$ we denote by $p_k[E]$ the plethystic substitution of E into p_k, defined to be $E(z_1^k, z_2^k, \ldots)$. In other words, $p_k[E]$ is the result of replacing each variable in E by its k-th power, and the constant is kept as it is. Since any symmetric function $f \in \Lambda$ is a polynomial in the p_k, we can define $f[E]$ as the unique $\mathbb{C}$-algebra homomorphism extending $p_k \mapsto p_k[E]$. Note that we are not extending as $\mathbb{C}(q, t)$-algebra homomorphism, i.e., q, t are also treated as variables. When using plethysm, we write alphabets

$X = \{x_1, x_2, \dots\}$, $Y = \{y_1, y_2, \dots\}$ as $X = \sum x_i$, $Y = \sum y_i$. So $p_k[X] = \sum x_i^k = p_k(X)$ for all k, and by extension, $f[X] = f(X)$ for any symmetric function f.

In particular,

$$p_k\left[X\frac{1-q}{1-t}\right] = p_k(X)\frac{1-q^k}{1-t^k}\ , \tag{9}$$

$$p_k\left[\frac{1-q}{1-t}\right] = \frac{1-q^k}{1-t^k}\ . \tag{10}$$

For more on plethysm see [Hai99] and [Hag06].

2.4. **Tableaux.** A semistandard Young tableau or a tableau T of shape λ is a filling of the diagram λ with entries from $1, 2, \dots$ such that the entries are weakly increasing left to right along the rows and strictly increasing top to bottom along columns. If for each $i \geq 1$ there are μ_i many i's in T then T is said to have content $\mu = (\mu_1, \mu_2, \dots)$. We denote the set of tableaux of shape λ, content μ by $SSYT(\lambda, \mu)$. The cardinality of this set is the Kostka number $K_{\lambda,\mu}$.

We have

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu\ . \tag{11}$$

By a matrix indexed by partitions we always mean that the rows and columns are indexed by partitions, and the indexing is compatible with dominance ordering, i.e., the partitions are listed in a sequence such that if $\lambda \geq \mu$ in dominance order then λ precedes μ in the sequence, and if $|\lambda| < |\mu|$ then λ precedes μ. These are infinite matrices. For a matrix M indexed by partitions, we call it upper-unitriangular if $M_{\lambda,\mu} = 0$ unless $\mu \leq \lambda$, and if $M_{\lambda,\lambda} = 1$. The set of upper unitriangular matrices forms a group.

We denote by K the matrix indexed by partitions whose (λ, μ)-th entry is $K_{\lambda,\mu}$. This matrix K is upper-unitriangular.

2.5. **Macdonald Symmetric Functions.** The scalar product $\langle\ ,\ \rangle_{q,t}$ on Λ is defined by

$$\langle f\ , g\rangle_{q,t} = \left\langle f[X], g\left[X\frac{1-q}{1-t}\right]\right\rangle, \tag{12}$$

where $\langle\ ,\ \rangle$ on the right-hand side is the Hall scalar product.

The Macdonald P-functions $(P_\lambda[X; q, t])_{\lambda \in Par}$ are characterized by the following two properties:

$$P_\lambda[X; q, t] - m_\lambda[X] \in \sum_{\mu<\lambda} \mathbb{C}(q, t) m_\mu[X]\ , \tag{13}$$

$$\langle P_\lambda[X; q, t], P_\mu[X; q, t]\rangle_{q,t} = 0 \text{ if } \lambda \neq \mu\ , \tag{14}$$

where $\leq$ is the dominance order on partitions.

When the variables are understood, we may drop X from the notation and just write $P_\lambda(q, t)$.

Let $(Q_\lambda(q,t))$ denote the dual basis to $(P_\lambda(q,t))$ under the q,t scalar product $\langle\ ,\ \rangle_{q,t}$. For a partition μ, let

$$c_\mu(q,t) = \prod_{x\in\mu}(1-q^{a(x)}\ t^{l(x)+1})\ , \tag{15}$$

$$c'_\mu(q,t) = \prod_{x\in\mu}(1-q^{a(x)+1}\ t^{l(x)})\ , \tag{16}$$

$$b_\mu(q,t) = \frac{c_\mu(q,t)}{c'_\mu(q,t)}\ . \tag{17}$$

Then we have the following relation.

Theorem 2.1 ([Mac95, Chapter VI, (6.19)])**.**

$$Q_\mu(q,t) = b_\mu(q,t)P_\mu(q,t)\ .$$

The *integral form Macdonald polynomials* are

$$J_\mu(q,t) = c_\mu(q,t)P_\mu(q,t) = c'_\mu(q,t)Q_\mu(q,t)\ . \tag{18}$$

The monomial coefficients of $J_\mu(q,t)$ are in $\mathbb{Z}[q,t]$.

2.6. q,t**-Kostka Functions.** We define two q,t-analogues of the Kostka numbers.

Definition 2.1. *For $\lambda,\mu\in Par$, let $K^{(1)}_{\lambda,\mu}(q,t)$ and $K^{(2)}_{\lambda,\mu}(q,t)$ be defined as the following change of basis coefficients:*

$$P_\lambda(q,t) = \sum_\mu K^{(1)}_{\lambda,\mu}(q,t)\ m_\mu\ , \tag{19}$$

$$s_\lambda = \sum_\mu K^{(2)}_{\lambda,\mu}(q,t)\ P_\mu(q,t)\ . \tag{20}$$

In other words, $K^{(1)}_{\lambda,\mu}(q,t) = \langle P_\lambda(q,t), h_\mu\rangle$ and $K^{(2)}_{\lambda,\mu}(q,t) = \langle s_\lambda, Q_\mu\rangle_{q,t}$. Note that both the change of basis matrices are upper- unitriangular, i.e., for $i\in\{1,2\}$, we have $K^{(i)}_{\lambda,\mu}(q,t)=0$ unless $\mu\le\lambda$ and, for all $\lambda\in Par$, we have $K^{(i)}_{\lambda,\lambda}(q,t)=1$. For $K^{(1)}$ this follows from the definition of Macdonald polynomials, and for $K^{(2)}$ this is because

$$K = K^{(2)}K^{(1)}\ , \tag{21}$$

where K is the matrix of Kostka numbers, which is known to be upper-unitriangular.

2.7. **Macdonald's Formula for** $K^{(1)}_{\lambda,\mu}(q,t)$**.** Macdonald gave a formula for $K^{(1)}_{\lambda,\mu}(q,t)$ in [Mac95], which we now describe. For two partitions λ,μ with $\mu_i\le\lambda_i$ for all i, λ/μ is called a horizontal strip if $\lambda'_j-\mu'_j\in\{0,1\}$ for all $j\in\mathbb{Z}_{>0}$. If λ/μ is a horizontal strip, then define

$$\psi_{\lambda/\mu}(q,t) = \prod_{\substack{x=(r,c)\in\mu\\ \lambda_r\ne\mu_r\\ \lambda'_c=\mu'_c}} \frac{(1-q^{a_\mu(x)}\ t^{\,l_\mu(x)+1})}{(1-q^{a_\mu(x)+1}\ t^{\,l_\mu(x)})}\frac{(1-q^{a_\lambda(x)+1}\ t^{\,l_\lambda(x)})}{(1-q^{a_\lambda(x)}\ t^{\,l_\lambda(x)+1})}\ . \tag{22}$$

Given a tableau T, let $T_{\leq i}$ denote the shape obtained by the boxes with content $1, \dots, i$. In a tableau T, $T_{\leq i}/T_{\leq(i-1)}$ is by definition a horizontal strip for each i. For a tableau T, define

$$\psi_T = \prod_{i \geq 1} \psi_{T_{\leq i}/T_{\leq(i-1)}} \ . \tag{23}$$

Then we have the following formula.

Theorem 2.2 ([Mac95, Chapter VI, (7.13′)]). *For partitions λ, μ,*

$$K^{(1)}_{\lambda,\mu}(q,t) = \sum_{T \in SSYT(\lambda,\mu)} \psi_T(q,t) \ . \tag{24}$$

2.8. **Specializations.**

(1) At $q = 0$, $K^{(2)}_{\lambda,\mu}(0,t)$ is the Kostka–Foulkes polynomial $K_{\lambda,\mu}(t)$. It is a polynomial in t with non-negative integer coefficients. A combinatorial formula for the Kostka–Foulkes polynomial was given by Lascoux and Schützenberger. There is a function $ch : SSYT(\lambda,\mu) \to \mathbb{Z}_{\geq 0}$ called charge, such that

$$K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{ch(T)} \ . \tag{25}$$

(2) At $t = 1$, $K^{(2)}_{\lambda,\mu}(q,1) = K_{\lambda,\mu}$ for arbitrary q.

(3) At $q = 1$, the matrix $K^{(2)}(1,t) = J(K^{-1})^{tr} J$, where J is the matrix indexed by partitions with $J_{\lambda,\mu} = \delta_{\lambda,\mu'}$. So the matrix $K^{(2)}$ contains information on both the matrix K and its inverse. A general version is Lemma 2.1 below.

2.9. **Duality.** By $(K^{(i)})^{-1}_{\lambda,\mu}$ we mean taking the (λ,μ)-entry of the inverse matrix.

The following lemma says that expanding the Macdonald functions in the Schur basis is roughly equivalent to its inverse problem, namely expanding the Schur functions in the Macdonald basis.

Lemma 2.1 ([Mac95, Chapter VI, (5.1″)]).

$$K^{(2)}_{\lambda,\mu}(q,t) = (K^{(2)})^{-1}_{\mu',\lambda'}(t,q) \ . \tag{26}$$

Since $c_\lambda(q,t) = c'_{\lambda'}(t,q)$ (cf. [Mac95, Chapter VI, (8.2)]) from (18),(26) and Definition 2.1 we get

$$\langle J_\lambda(t,q), s_\mu \rangle = \langle J_{\lambda'}(q,t), s_{\mu'} \rangle_{q,t} \ . \tag{27}$$

2.10. **Haglund's Conjecture.** In [HHL06] it was shown that the monomial coefficients $\langle J_\lambda(q,t), h_\mu \rangle$ of $J_\lambda(q,t)$ have the following positivity property:

$$\frac{\langle J_\lambda(q,q^k), h_\mu \rangle}{(1-q)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[q] \quad \text{for all } k \in \mathbb{Z}_{\geq 0} \ . \tag{28}$$

Haglund made the following conjecture in [Hag10] about the Schur coefficients $\langle J_\lambda(q,t), s_\mu \rangle$ of $J_\lambda(q,t)$.

Conjecture 1 (Haglund). *For partitions λ and μ,*

$$\frac{\langle J_\lambda(q,q^k), s_\mu \rangle}{(1-q)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[q] \quad \textit{for all } k \in \mathbb{Z}_{\geq 0} \ . \tag{Hag(λ,μ)}$$

Yoo [Yoo12, Yoo15] showed that this is true in some special cases.

Let

$$k_{\lambda,\mu}(q,t) = \langle s_\lambda, J_\mu(q,t)\rangle_{q,t} \ . \tag{29}$$

Then

$$k_{\lambda,\mu}(q,t) = K^{(2)}_{\lambda,\mu}(q,t)c'_\mu(q,t) = (K^{(2)})^{-1}_{\mu',\lambda'}(t,q)c'_\mu(q,t) \ . \tag{30}$$

Note that $c'_\mu(0,t) = 1$, so

$$k_{\lambda,\mu}(0,t) = K_{\lambda,\mu}(t) \ , \tag{31}$$

where the right-hand side is the Kostka–Foulkes polynomial as in Subsection 2.8. By (27), we have the following dual version of Haglund's conjecture.

Conjecture 2. *For partitions λ, μ,*

$$\frac{k_{\lambda,\mu}(t^k,t)}{(1-t)^{|\lambda|}} \in \mathbb{Z}_{\geq 0}[t] \quad \textit{for all } k \in \mathbb{Z}_{\geq 0} \ . \tag{$\mathrm{Hag}'(\lambda,\mu)$}$$

3. The Case Where λ Is a Single Row or μ Is a Single Column

In this section we derive formulas for $K^{(2)}_{(n),\mu}(q,t)$ and $K^{(2)}_{\lambda,1^n}(q,t)$, and use them to show $\mathrm{Hag}'(\lambda,\mu)$ is true in these cases.

We begin by reviewing Cauchy formulas and principal specializations.

3.1. **Cauchy Identity.** For a series of rational functions E we let $\Omega[E] = \sum_{n\geq 0} h_n[E]$. Let $(u_\lambda)_{\lambda\in\mathrm{Par}}$, $(v_\lambda)_{\lambda\in\mathrm{Par}}$ be Hall-dual bases. Then the Cauchy identity says that

$$\Omega[XY] = \sum_{\lambda\in\mathrm{Par}} u_\lambda[X]v_\lambda[Y] \ . \tag{32}$$

If $(u_\lambda)_{\lambda\in\mathrm{Par}}$, $(v_\lambda)_{\lambda\in\mathrm{Par}}$ are q,t-dual bases, then by definition of $\langle,\rangle_{q,t}$, the bases $\left(u_\lambda\left[X\frac{1-q}{1-t}\right]\right)_{\lambda\in Par}$ and $(v_\lambda[X])_{\lambda\in Par}$ are Hall-dual, hence,

$$\Omega[XY] = \sum_{\lambda\in Par} u_\lambda\left[X\frac{1-q}{1-t}\right]v_\lambda[Y] \ . \tag{33}$$

Equivalently, by replacing X by $X\dfrac{1-t}{1-q}$,

we get the (q,t)-Cauchy identity

$$\Omega\left[XY\frac{1-t}{1-q}\right] = \sum_{\lambda\in Par} u_\lambda[X]v_\lambda[Y] \ . \tag{34}$$

Since $(h_\lambda)_{\lambda\in Par}$ and $(m_\lambda)_{\lambda\in Par}$ are Hall-dual, $\left(h_\lambda\left[X\frac{1-t}{1-q}\right]\right)_{\lambda\in Par}$ is q,t-dual to $(m_\lambda[X])_{\lambda\in Par}$.

We give a quick proof of the following well-known result (cf. [Mac95, Chapter VI, (5.5)]) to illustrate the power of plethystic methods.

Proposition 3.1. *We have*

$$Q_{(n)}[X;q,t] = h_n\left[X\frac{1-t}{1-q}\right] . \tag{35}$$

Proof. The Cauchy identity gives

$$\sum_{\lambda\in Par} h_\lambda\left[X\frac{1-t}{1-q}\right] m_\lambda[Y] = \sum_{\lambda\in Par} Q_\lambda[X;q,t]P_\lambda[Y;q,t] , \tag{36}$$

Substitute $Y = y$, where y is a variable. Since

$$m_\lambda[y] = m_\lambda(y,0,0,\dots) = \begin{cases} y^n, & \text{if } \lambda = (n) \text{ for some } n \geq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

and by unitriangularity for P_λ, we get

$$P_\lambda[y] = \begin{cases} y^n, & \text{if } \lambda = (n) \text{ for some } n \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

Hence, by (36) above,

$$\sum_{n\geq 0} h_{(n)}\left[X\frac{1-t}{1-q}\right] y^n = \sum_{n\geq 0} Q_{(n)}[X;q,t]y^n . \tag{39}$$

Comparing coefficient of y^n we get the result. □

3.2. **Principal Specializations.** We can use the Cauchy identity and plethystic substitution to calculate the first row of various transition matrices. The next lemma and the corollaries illustrate that. We will need Macdonald's evaluation identity stated below.

Theorem 3.1 ([Mac95, Chapter VI, (6.17)])**.** *Let z be a complex variable. Then*

$$P_\lambda\left[\frac{1-z}{1-t};q,t\right] = \prod_{s\in\lambda} \frac{t^{l'(s)} - q^{a'(s)}z}{1-q^{a(s)}t^{l(s)+1}} , \tag{40}$$

$$Q_\lambda\left[\frac{1-z}{1-t};q,t\right] = \prod_{s\in\lambda} \frac{t^{l'(s)} - q^{a'(s)}z}{1-q^{a(s)+1}t^{l(s)}} . \tag{41}$$

3.3. **The Case Where λ Is a Row or μ Is a Column.** We now calculate $k_{\lambda,\mu}(q,t)$ in these special cases. Replacing Y by $\dfrac{1-z}{1-t}$ in the Cauchy identity (34) and using Theorem 3.1 we get the following lemma.

Lemma 3.1. *Let z be a complex variable. For a pair $(u_\lambda)_{\lambda\in Par}$ and $(v_\lambda)_{\lambda\in Par}$ of q,t-dual bases of Λ we have*

$$h_n\left[X\frac{1-z}{1-q}\right] = \sum_{\lambda\vdash n} v_\lambda\left[\frac{1-z}{1-t}\right] u_\lambda[X] . \tag{42}$$

In particular,

$$h_n\left[X\frac{1-z}{1-q}\right] = \sum_{\lambda\vdash n}\prod_{x\in\lambda} \frac{t^{l'(x)} - q^{a'(x)}z}{1-q^{a(x)+1}t^{l(x)}} P_\lambda[X;q,t] . \tag{43}$$

At $t = q$,

$$h_n\left[X\frac{1-z}{1-q}\right] = \sum_{\lambda\vdash n} q^{n(\lambda)} \prod_{x\in\lambda} \frac{1-q^{c(x)}z}{1-q^{h(x)}} s_\lambda[X] \ . \tag{44}$$

Proposition 3.2. *We have*

$$k_{(n),\mu}(q,t) = t^{n(\mu)} \prod_{x\in\mu}(1-q^{a'(x)+1}t^{-l'(x)}) \ . \tag{45}$$

Proof. We substitute $z = q$ in (43), use (18), (29) and the fact that $s_{(n)} = h_n$. □

We can use Lemma 3.1 to get another proof of the following result of [Yoo12] in the dual formalism.

Proposition 3.3. *We have*

$$k_{\lambda,1^n}(q,t) = \frac{t^{n(\lambda')}[n]_t!}{\prod_{x\in\lambda}[h(x)]_t} \prod_{x\in\lambda}(1-t^{-c(x)}q) \tag{46}$$

$$= K_{\lambda,1^n}(t) \prod_{x\in\lambda}(1-t^{-c(x)}q) \ , \tag{47}$$

where $[j]_t = \dfrac{1-t^j}{1-t}$ *for* $j \in \mathbb{Z}_{\geq 1}$ *and* $[n]_t! = [n]_t[n-1]_t \dots [1]_t$.

Proof. Substituting $z = t$ in (44), we get

$$h_n\left[X\frac{1-t}{1-q}\right] = \sum_{\lambda\vdash n} \prod_{x\in\lambda} \frac{q^{l'(x)}-q^{a'(x)}t}{1-q^{h(x)}} s_\lambda[X] \ . \tag{48}$$

By Proposition 3.1, the left-hand side is $Q_n(q,t)$, and so the product in front of $s_\lambda[X]$ on the right-hand side is $\langle Q_n(q,t), s_\lambda\rangle$. Observe that $c'_{(n)}(q,t) = (q;q)_n$, where by $(a;x)_n$ we mean the product $\prod_{i=0}^{n-1}(1-ax^i)$. Since $c'_{(n)}Q_n = J_n$, Equation (27) gives

$$k_{\lambda',1^n}(q,t) = (t;t)_n \prod_{x\in\lambda} \frac{t^{l'(x)}-t^{a'(x)}q}{1-t^{h(x)}} \ . \tag{49}$$

Since conjugation interchanges coleg lengths and coarm lengths, and the set of hook lengths remains unchanged, we have

$$k_{\lambda,1^n}(q,t) = (t;t)_n \prod_{x\in\lambda} \frac{t^{a'(x)}-t^{l'(x)}q}{1-t^{h(x)}} \tag{50}$$

$$= \frac{t^{n(\lambda')}(t;t)_n}{\prod_{x\in\lambda}(1-t^{h(x)})} \prod_{x\in\lambda}(1-t^{-c(x)}q) \ . \tag{51}$$

Since $k_{\lambda,\mu}(0,t) = K_{\lambda,\mu}(t)$, the result follows. □

Corollary 3.1. $\mathrm{Hag}'(\lambda,\mu)$ *holds when* λ *is a row or when* μ *is a column.*

Proof. When $\lambda = (n)$ and μ is arbitrary, from (45) we have

$$k_{(n),\mu}(t^k,t) = t^{n(\mu)} \prod_{x\in\mu}(1-t^{k(a'(x)+1)-l'(x)}) \quad \text{for all } k \in \mathbb{Z}_{\geq 0} \ . \tag{52}$$

Since the quantity $k(a'(x)+1)-l'(x)$ decreases by 1 down a column and it is non-negative in the top row, if it is negative for some $x \in \mu$ then it must also attain the value 0 above x in the same column. Hence

$$\frac{k_{(n),\mu}(t^k,t)}{(1-t)^n} = \begin{cases} t^{n(\mu)} \prod_{x\in\mu}[k(a'(x)+1)-l'(x)]_t, & \text{if } \ell(\mu) \le k, \\ 0, & \text{otherwise,} \end{cases} \tag{53}$$

where $[m]_t = 1+t+\cdots+t^{m-1}$. In both cases $\dfrac{k_{(n),\mu}(t^k,t)}{(1-t)^n} \in \mathbb{Z}_{\ge 0}[t]$.

Next, when $\mu = 1^n$ and λ is arbitrary, we get

$$k_{\lambda,1^n}(t^k,t) = K_{\lambda,1^n}(t) \prod_{x\in\lambda}(1-t^{k-c(x)}) \ . \tag{54}$$

Along a row the quantity $k-c(x)$ decreases by 1 as one moves from left to right, and for the first box in the row (in the first column) $k-c(x) \ge 0$. So if $k-c(x)<0$ for some $x \in \lambda$ then in that row there is a box y such that $k-c(y)=0$, implying $k_{\lambda,1^n}(t^k,t)=0$. Otherwise $k-c(x) \ge 0$ for all $x \in \lambda$, and hence

$$\frac{k_{\lambda,1^n}(t^k,t)}{(1-t)^n} = \begin{cases} K_{\lambda,1^n}(t) \prod_{x\in\lambda}[k-c(x)]_t, & \text{if } \lambda_1 \le k, \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

Since $K_{\lambda,\mu}(t) \in \mathbb{Z}_{\ge 0}[t]$, we obtain $\dfrac{k_{\lambda,1^n}(t^k,t)}{(1-t)^n} \in \mathbb{Z}_{\ge 0}[t]$. □

4. Multiplication

We state a general lemma on posets that will be useful in this section.

Lemma 4.1. *Let M be an upper unitriangular matrix indexed by a partially ordered set P, i.e., for all $a \in P$, $M_{a,a}=1$ and $M_{a,b}=0$ unless $b \le a$. For an interval I in P let $M|_I = (M_{a,b})_{a,b\in I}$ denote the matrix restricted to the interval I. Then, for any interval J containing a,b, we have $(M^{-1})_{a,b} = (M|_J)^{-1}_{a,b}$.*

Proof. Let $N = M^{-1}$. The lemma follows since one can simply write down the solution to the equations

$$\delta_{a,b} = \sum_{a\le c\le b} N_{a,c}M_{c,b} \tag{56}$$

recursively, starting from $b=a$. □

Let λ^1, λ^2 be two partitions such that the least non-zero part of λ^1 is at least the largest part of λ^2. In this case by (λ^1,λ^2) we mean the partition $(\lambda^1_1,\ldots,\lambda^1_{\ell(\lambda^1)},\lambda^2_1,\ldots)$.

Lemma 4.2. *Let $\lambda = (\lambda^1,\lambda^2)$ and $\mu = (\mu^1,\mu^2)$ where $\lambda^j \ge \mu^j$ (in particular, $|\lambda^j| = |\mu^j|$) for $j=1,2$ and $\ell(\lambda^1) = \ell(\mu^1)$.*

Then for $i=1,2$,

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^1,\mu^1}(K^{(i)})^{\pm 1}_{\lambda^2,\mu^2} \ . \tag{57}$$

Proof. Let $T \in SSYT(\lambda, \mu)$. Since $\ell(\lambda^1) = \ell(\mu^1)$ and $|\lambda^1| = |\mu^1|$, in the first $\ell(\lambda^1)$ rows of T the content must be μ^1. We can break the tableau T into two pieces; let T^1 comprise the first $\ell(\lambda^1)$ rows, and T^2 be the tableau formed by the remaining rows of T in which we subtract $\ell(\lambda^1)$ from each entry. It is clear that we get a bijection

$$\begin{aligned} SSYT(\lambda, \mu) &\to SSYT(\lambda^1, \mu^1) \times SSYT(\lambda^2, \mu^2), \\ T &\mapsto (T^1, T^2). \end{aligned}$$

In particular, $K_{\lambda,\mu} = K_{\lambda^1,\mu^1} K_{\lambda^2,\mu^2}$.

For $1 \leq i \leq \ell(\lambda^1)$ the horizontal strips $T_{\leq i}/T_{\leq i-1}$ and $T^1_{\leq i}/T^1_{\leq i-1}$ are the same, so $\psi_{T_{\leq i}/T_{\leq i-1}} = \psi_{T^1_{\leq i}/T^1_{\leq i-1}}$.

For $i > \ell(\lambda^1)$ the horizontal strips $T_{\leq i}/T_{\leq i-1}$ differ from $T^2_{\leq i-\ell(\lambda^1)}/T^2_{\leq i-\ell(\lambda^1)-1}$ in the first $\ell(\lambda^1)$ rows, but none of those boxes contribute to $\psi_{T_{\leq i}/T_{\leq i-1}}$, and the contribution from the remaining boxes is the same since the arms and legs are the same. So $\psi_{T_{\leq i}/T_{\leq i-1}} = \psi_{T^2_{\leq i-\ell(\lambda^1)}/T^2_{\leq i-\ell(\lambda^1)-1}}$.

Therefore the above bijection preserves ψ, that is,

$$\psi_T = \psi_{T^1}\psi_{T^2} \ . \tag{58}$$

By (24), we get

$$K^{(1)}_{\lambda,\mu} = K^{(1)}_{\lambda^1,\mu^1} K^{(1)}_{\lambda^2,\mu^2} \ . \tag{59}$$

Let $\lambda \geq \nu \geq \mu$. Setting $\nu^1 = (\nu_1, \ldots, \nu_{\ell(\lambda^1)})$ and $\nu^2 = (\nu_{\ell(\lambda^1)+1}, \ldots)$, we have $\lambda^i \geq \nu^i \geq \mu^i$ for $i = 1, 2$ and $\ell(\lambda^1) = \ell(\nu^1) = \ell(\mu^1)$. So the interval $[\lambda, \mu]$ in the dominance order becomes $[\lambda, \mu] = [\lambda^1, \mu^1] \times [\lambda^2, \mu^2]$. The matrix $K^{(1)}$ restricted to $[\lambda, \mu]$ is then the tensor product

$$K^{(1)}|_{[\lambda,\mu]} = K^{(1)}|_{[\lambda^1,\mu^1]} \otimes K^{(1)}|_{[\lambda^2,\mu^2]} \ . \tag{60}$$

In particular, $K|_{[\lambda,\mu]} = K|_{[\lambda^1,\mu^1]} \otimes K|_{[\lambda^2,\mu^2]}$.

Since the inverse of the tensor product of two matrices is the tensor product of their inverses, we get

$$(K^{(1)})^{-1}|_{[\lambda,\mu]} = (K^{(1)})^{-1}|_{[\lambda^1,\mu^1]} \otimes (K^{(1)})^{-1}|_{[\lambda^2,\mu^2]} \ . \tag{61}$$

By (21), we have

$$K^{(2)}|_{[\lambda,\mu]} = K^{(2)}|_{[\lambda^1,\mu^1]} \otimes K^{(2)}|_{[\lambda^2,\mu^2]} \ . \tag{62}$$

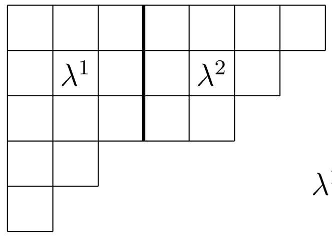
Once again, taking inverses we get

$$(K^{(2)})^{-1}|_{[\lambda,\mu]} = (K^{(2)})^{-1}|_{[\lambda^1,\mu^1]} \otimes (K^{(2)})^{-1}|_{[\lambda^2,\mu^2]} \ . \tag{63}$$

By Lemma 4.1, we conclude that, for $i = 1, 2$,

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^1,\mu^1} (K^{(i)})^{\pm 1}_{\lambda^2,\mu^2} \ . \qquad \square$$

Let λ^1, λ^2 be two partitions such that the last non-zero column length of λ^1 is greater than or equal to the first column length of λ^2. Then by $\lambda^1 + \lambda^2$ we denote the partition $(\lambda^1_1 + \lambda^2_1, \lambda^1_2 + \lambda^2_2, \ldots)$. Note that, if $\lambda = \lambda^1 + \lambda^2$, then $\lambda' = (\lambda^{1'}, \lambda^{2'})$ in the notation of Lemma 4.2. By $\lambda - \lambda^1$ we mean the partition λ^2.



$$\lambda^1 = (3,3,3,2,1)$$
$$\lambda^2 = (4,3,2)$$
$$\lambda^1 + \lambda^2 = (7,6,5,2,1)$$

FIGURE 2. Example of $\lambda^1 + \lambda^2$

Lemma 4.3. *Let $\lambda, \mu \in Par$ such that $\lambda = \lambda^1 + \lambda^2$ and $\mu = \mu^1 + \mu^2$ with $\lambda^1 \geq \mu^1$, $\lambda^2 \geq \mu^2$, and $\lambda^1_1 = \mu^1_1$. Then, for $i = 1, 2$, we have*

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^1,\mu^1} (K^{(i)})^{\pm 1}_{\lambda^2,\mu^2} \ . \tag{64}$$

Proof. Note that $\lambda' = (\lambda^{1'}, \lambda^{2'})$ and $\mu' = (\mu^{1'}, \mu^{2'})$, $\mu^{1'} \geq \lambda^{1'}$ and $\ell(\mu^{1'}) = \mu^1_1 = \lambda^1_1 = \ell(\lambda^{1'})$. Therefore we can apply Lemma 4.2 to μ', λ'. By Lemma 2.1,

$$K^{(2)}_{\lambda,\mu}(q,t) = (K^{(2)})^{-1}_{\mu',\lambda'}(t,q) \tag{65}$$
$$= (K^{(2)})^{-1}_{\mu^{1'},\lambda^{1'}}(t,q)(K^{(2)})^{-1}_{\mu^{2'},\lambda^{2'}}(t,q) \tag{66}$$
$$= K^{(2)}_{\lambda^1,\mu^1}(q,t) K^{(2)}_{\lambda^2,\mu^2}(q,t) \ . \tag{67}$$

In particular,

$$K_{\lambda,\mu} = K_{\lambda^1,\mu^1} K_{\lambda^2,\mu^2} \ . \tag{68}$$

If a partition γ satisfies $\lambda \geq \gamma \geq \mu$ then $\mu' \geq \gamma' \geq \lambda'$. By our earlier argument, $[\mu', \lambda'] = [\mu^{1'}, \lambda^{1'}] \times [\mu^{2'}, \lambda^{2'}]$. Then $[\lambda, \mu] = [\lambda^1, \mu^1] + [\lambda^2, \mu^2]$, and any $\gamma^1 \in [\lambda^1, \mu^1]$ satisfies $\lambda^1_1 = \gamma^1_1 = \mu^1_1$. Thus,

$$K^{(2)}|_{[\lambda,\mu]} = K^{(2)}|_{[\lambda^1,\mu^1]} \otimes K^{(2)}|_{[\lambda^2,\mu^2]} \ . \tag{69}$$

As before, we take inverses and use (21) to prove the statement. □

Let $\lambda^1, \lambda^2, \mu^1, \mu^2$ be as in Lemma 4.2. Note that, since $\ell(\lambda^1) = \ell(\mu^1)$ and $\lambda^1 \geq \mu^1$, we have $\mu^1_{\ell(\lambda^1)} \geq \lambda^1_{\ell(\lambda^1)}$. Thus both λ^1 and μ^1 contain the rectangle R of row length $\lambda^1_{\ell(\lambda^1)}$ and column length $\ell(\lambda^1)$.

Corollary 4.1. *Let $\lambda^1, \lambda^2, \mu^1, \mu^2$ be as in Lemma 4.2, and let R be the rectangular partition of row length $\lambda^1_{\ell(\lambda^1)}$ and column length $\ell(\lambda^1)$. If* $\mathrm{Hag}'(\lambda, \mu)$ *is true for $\lambda^1 - R, \mu^1 - R$ and for λ^2, μ^2 then it is true for λ, μ.*

Proof. By applying Lemma 4.3 to $\lambda^1 = R + (\lambda^1 - R)$ and $\mu^1 = R + (\mu^1 - R)$, for $i = 1, 2$ we get

$$K^{(i)}_{\lambda^1,\mu^1} = K^{(i)}_{R,R} K^{(i)}_{\lambda^1 - R,\mu^1 - R} = K^{(i)}_{\lambda^1 - R,\mu^1 - R} \ . \tag{70}$$

So, by Lemma 4.2,

$$K^{(i)}_{\lambda,\mu} = K^{(i)}_{\lambda^1,\mu^1} K^{(i)}_{\lambda^2,\mu^2} = K^{(i)}_{\lambda^1 - R,\mu^1 - R} K^{(i)}_{\lambda^2,\mu^2} \ . \tag{71}$$

Since the arms and legs of boxes in $\mu^1 - R$ have no intersection with those of μ^2, it follows that

$$\frac{c'_\mu}{c'_{\mu^1-R}c'_{\mu^2}} = \prod_{x\in R}(1-q^{a_\mu(x)+1}t^{l_\mu(x)}) \ . \tag{72}$$

From (30), we get

$$k_{\lambda,\mu}(q,t) = k_{\lambda^1-R,\mu^1-R}(q,t)\ k_{\lambda^2,\mu^2}(q,t)\ \prod_{x\in R}(1-q^{a_\mu(x)+1}t^{l_\mu(x)}) \ . \tag{73}$$

So,

$$\frac{k_{\lambda,\mu}(t^k,t)}{(1-t)^{|\lambda|}} = \frac{k_{\lambda^1-R,\mu^1-R}(t^k,t)}{(1-t)^{|\lambda^1|-|R|}}\ \frac{k_{\lambda^2,\mu^2}(t^k,t)}{(1-t)^{|\lambda^2|}}\ \frac{\prod_{x\in R}(1-t^{k(a_\mu(x)+1)+l_\mu(x)})}{(1-t)^{|R|}} \ . \tag{74}$$

The last term on the right-hand side is a product of t-numbers and hence in $\mathbb{Z}_{\geq 0}[t]$. □

As particular cases we get the next corollary.

Corollary 4.2. *If* $\mathrm{Hag}'(\lambda,\mu)$ *is true for* λ,μ *then it is true for* (R,λ), (R,μ), *and for* $(S+\lambda),(S+\mu)$, *where* R,S *are rectangular partitions, with row-length of* R *at least as big as* λ_1 *and column length of* S *at least as big as* $\ell(\mu)$.

Remark 4.1. We deduce [Yoo12, Theorem 3.1.1] in the dual setup by (73) and using Proposition 3.2. In particular this proves that *Hag*$'(\lambda,\mu)$ holds for $\lambda=(a+k,b-k)$ and $\mu=(a,b)$. Similarly, we obtain [Yoo15, Proposition 3.4] in the dual setup by applying (73) and using Proposition 3.3.

Example 4.1. We demonstrate the two principles of Lemma 4.2 and Lemma 4.3 in the following example (see (71)):

$$K^{(2)}_{533,44111}(q,t) = K^{(2)}_{53,44}(q,t)K^{(2)}_{3,111}(q,t) = K^{(2)}_{2,11}(q,t)K^{(2)}_{3,111}(q,t) \ .$$

5. Complementation

In this section we switch to type GL_n Macdonald polynomials and then come back to symmetric functions. The reference for this section is [Mac95, Chapter VI, Section 9].

Let $n\in\mathbb{Z}_{>0}$ be fixed, and let q and t be two complex numbers with $t=q^k$ for some fixed $k\in\mathbb{Z}_{\geq 0}$.

Let $\mathcal{P}=\mathbb{Z}^n$, and $\mathcal{P}^+=\{\alpha\in\mathcal{P}:\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n\}$. For $\alpha=(\alpha,\ldots,\alpha_n)\in\mathbb{Z}^n$ let $x^\alpha=(x_1^{\alpha_1},\ldots,x_n^{\alpha_n})$. Let $W=S_n$. By $\mathbb{C}[\mathcal{P}]$ we denote the ring of Laurent polynomials $\mathbb{C}[x_1^{\pm 1},x_2^{\pm 1},\ldots,x_n^{\pm 1}]$. There is a ring homomorphism $\Lambda\to\mathbb{C}[\mathcal{P}]^W$ defined by sending $x_{n+i}\mapsto 0$ for $i\geq 1$. We denote the image of a symmetric function f by $f(X_n)$. For any $\lambda\in\mathcal{P}^+$ (not necessarily with non-negative coordinates) we can define $P_\lambda(x;q,t)\in\mathbb{C}[\mathcal{P}]^W$ by the relations

$$P_\lambda(x;q,t) = P_\lambda(X_n;q,t), \quad \text{if } \ \lambda_n\geq 0, \tag{75}$$

$$P_{\lambda+(1^n)}(x;q,t) = (x_1\cdots x_n)P_\lambda(x;q,t), \quad \text{for all } \lambda\in\mathcal{P}^+. \tag{76}$$

We also define, for all $\lambda\in\mathcal{P}^+$, $m_\lambda(x)=P_\lambda(x;q,1)$ and $s_\lambda(x)=P_\lambda(x;q,q)$.

Furthermore, $\mathcal{P}^+$ carries the dominance partial order $\geq$ defined by

$$\lambda\geq\mu \text{ if and only if } \lambda-\mu\in\mathbb{Z}_{\geq 0}\{e_i-e_j: 1\leq i<j\leq n\}, \tag{77}$$

where e_i denotes the standard basis vector $(0, \dots, 1, \dots, 0)$ with 1 in the i-th position and 0 everywhere else.

Let $^- : \mathbb{C}[\mathcal{P}] \to \mathbb{C}[\mathcal{P}]$ be the $\mathbb{C}$-algebra involution defined by $x_i \mapsto x_i^{-1}$ for $i \in \{1, \dots, n\}$. Then there is a scalar product on $\mathbb{C}[\mathcal{P}]$ defined by

$$\langle f, g\rangle' = \frac{1}{n!}\mathrm{ct}(f\overline{g}\Delta) \ , \tag{78}$$

where ct denotes the constant term map on the Laurent polynomial ring, i.e., $\mathrm{ct}(f) =$ coefficient of x^0 in f, and

$$\Delta = \prod_{i \neq j}\prod_{r=0}^{k-1}(1 - q^r\frac{x_i}{x_j}) \ . \tag{79}$$

The next theorem characterizes the $P_\lambda(x; q, t) : \lambda \in \mathcal{P}^+$.

Theorem 5.1. *The set $\{P_\lambda(x; q, t) : \lambda \in \mathcal{P}^+\}$ is the unique family of elements in $\mathbb{C}[\mathcal{P}]^W$ satisfying*

$$P_\lambda(x; q, t) = m_\lambda(x; q, t) + \sum_{\mu<\lambda} a_{\lambda,\mu}(q, t)m_\mu(x) \ , \tag{80}$$

$$\langle P_\lambda(x; q, t), P_\mu(x; q, t)\rangle' = 0 \ for \ \ \lambda \neq \mu \ , \tag{81}$$

for some $a_{\lambda,\mu}(q, t) \in \mathbb{C}(q, t) \subset \mathbb{C}$.

Proof. If $\lambda \in Par$ with $\ell(\lambda) \leq n$, then $P_\lambda(x; q, t)$ is a homogeneous polynomial of degree $|\lambda|$. By (75), $P_\lambda(X_n; q, t)$ is a homogeneous Laurent polynomial with degree $\lambda_1+\cdots+\lambda_n$ for all $\lambda \in \mathcal{P}^+$. By comparing degrees, it is clear that, if $\lambda_1 + \cdots + \lambda_n \neq \mu_1 + \cdots + \mu_n$, then $\langle P_\lambda(x; q, t), P_\mu(x; q, t)\rangle' = 0$. By (75) and the definition of $\langle \ , \ \rangle'$ it is enough to check orthogonality of the set $\{P_\lambda(X_n; q, t) : \lambda \in \mathcal{P}^+ \text{ with } \lambda_n \geq 0\}$. This is shown in [Mac95, Chapter VI, (9.5)]. Since $\{m_\lambda(x) : \lambda \in \mathcal{P}^+\}$ is a basis of $\mathbb{C}[\mathcal{P}]^W$, so is $\{P_\lambda(x; q, t) : \lambda \in \mathcal{P}^+\}$.

Any two bases of $\mathbb{C}[\mathcal{P}]^W$ satisfying the above conditions will be related by a triangular orthogonal matrix and any such matrix is necessarily diagonal, so the uniqueness follows. □

Let $w_0 \in S_n$ be the permutation that sends i to $n + 1 - i$ for $i \in \{1, \dots, n\}$. Let $\phi : \mathbb{C}[\mathcal{P}] \to \mathbb{C}[\mathcal{P}]$ be the $\mathbb{C}$-algebra homomorphism defined by $\phi(x^\alpha) = x^{-w_0\alpha}$. So $\phi(f)(x) = w_0\overline{f}(x)$.

Proposition 5.1. *For $\lambda \in \mathcal{P}^+$, we have $\phi(P_\lambda(x; q, t)) = P_{-w_0\lambda}(x; q, t)$. In particular, $\phi(s_\lambda(x)) = s_{-w_0\lambda}(x)$.*

Proof. Since $\phi(m_\mu(x)) = m_{-w_0\mu}(x)$ for all $\mu \in \mathcal{P}^+$, we have

$$\phi(P_\lambda(x; q, t)) = m_{-w_0\lambda}(x) + \sum_{\mu<\lambda} a_{\lambda,\mu}(q, t)m_{-w_0\mu}(x) \ . \tag{82}$$

Note that $\mu \leq \lambda$ if and only if $-w_0\mu \leq -w_0\lambda$, so $\phi(P_\lambda)$ satisfies the unitriangularity property. To prove $\phi(P_\lambda(x; q, t)) = P_{-w_0\lambda}(x; q, t)$, we only need to prove that, for $\lambda \neq \mu$, we have $\langle\phi(P_\lambda(x; q, t)), \phi(P_\mu(x; q, t))\rangle' = 0$. This follows from the fact that the constant term of a Laurent polynomial is S_n-invariant, w_0 is a homomorphism, and that Δ is S_n-invariant. □

Corollary 5.1. *Let $\lambda, \mu \subset (m^n)$ be partitions. Let $\lambda^c = (m-\lambda_n, m-\lambda_{n-1}, \ldots, m-\lambda_1)$ and $\mu^c = (m-\mu_n, m-\mu_{n-1}, \ldots, m-\mu_1)$. Then, for $i = 1, 2$, we have*

$$(K^{(i)})^{\pm 1}_{\lambda,\mu} = (K^{(i)})^{\pm 1}_{\lambda^c,\mu^c} \ . \tag{83}$$

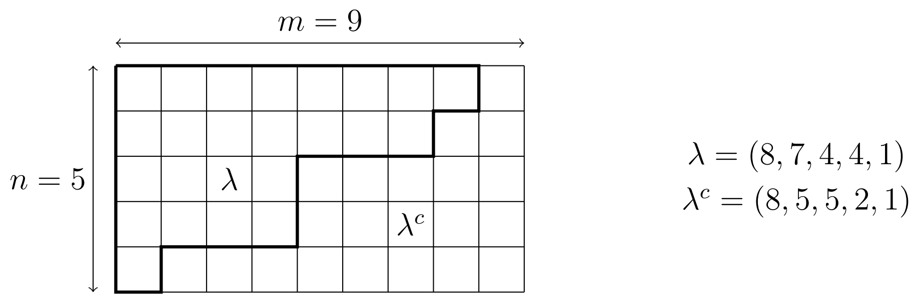


FIGURE 3. Example of λ, λ^c when $m = 9, n = 5$

Proof. Let $w_0 \in S_n$ be as before. First note that, if $\lambda \subset (m^n)$ is a partition, then $\lambda^c = (m-\lambda_n, \ldots, m-\lambda_1)$ is also a partition. So it makes sense to talk about $K^{(i)}_{\lambda^c,\mu^c}(q,t)$ in this situation.

In the ring Λ, we have

$$P_\lambda(q,t) = \sum_{\nu \le \lambda} K^{(1)}_{\lambda,\nu}(q,t)\ m_\nu \ . \tag{84}$$

Specializing to n variables, we have

$$P_\lambda(X_n;q,t) = \sum_{\nu \le \lambda} K^{(1)}_{\lambda,\nu}(q,t)\ m_\nu(X_n) \ . \tag{85}$$

On the right-hand side, if $\ell(\nu) > n$, then the corresponding term vanishes. Now applying the map ϕ to this equation, we get

$$P_{-w_0\lambda}(X_n;q,t) = \sum_{\nu \le \lambda} K^{(1)}_{\lambda,\nu}(q,t) m_{-w_0\nu}(X_n) \ . \tag{86}$$

Multiplying by $(x_1 \cdots x_n)^m$ and using (75), we get

$$P_{(m^n)-w_0\lambda}(X_n;q,t) = \sum_{\nu \le \lambda} K^{(1)}_{\lambda,\nu}(q,t) m_{(m^n)-w_0\nu}(X_n) \ . \tag{87}$$

Again, consider the equation in Λ

$$P_{\lambda^c}(q,t) = \sum_{\gamma \le \lambda^c} K^{(1)}_{\lambda^c,\gamma}(q,t) m_\gamma \ , \tag{88}$$

and specialize to n variables. We see that the coefficient of $m_{\mu^c}(X_n)$ is $K^{(1)}_{\lambda^c,\mu^c}(q,t)$. Here we have used the fact that $m_{\mu^c}(X_n) \neq 0$ since $\mu \subset (m^n)$. Comparing with (87), we get the desired statement for $K^{(1)}$. By considering

$$m_\lambda = \sum_{\nu \le \lambda} (K^{(1)})^{-1}_{\lambda,\nu}(q,t) P_\nu(q,t) \ , \tag{89}$$

we get the inverse statement for $K^{(1)}$. The proof for $(K^{(2)})^{\pm 1}$ is obtained by replacing the m_μ's in the above argument by s_μ's everywhere. $\square$

6. Multiplicity-one pairs

In this section we finish the proof that $\mathrm{Hag}'(\lambda,\mu)$ holds for all pairs (λ,μ) such that either $K_{\lambda,\mu}=1$ or $K_{\mu',\lambda'}=1$. We recall the reduction principle given in the introduction.

Let $\lambda,\mu\in Par$ such that $\lambda\geq\mu$. We can always decompose $\lambda=(\lambda^1,\dots,\lambda^r)$ and $\mu=(\mu^1,\dots,\mu^r)$ where $\lambda^1,\dots\lambda^r,\mu^1,\dots,\mu^r$ are partitions such that, for $i\in\{1,\dots r\}$, $\lambda^i\geq\mu^i$ and, for $i\in\{1,\dots r-1\}$, $\ell(\lambda^i)=\ell(\mu^i)$. Then, for $i=1,2$, by Lemma 4.2 we have

$$(K^{(i)})^{\pm1}_{\lambda,\mu}(q,t)=\prod_{j=1}^{r}(K^{(i)})^{\pm1}_{\lambda^j,\mu^j}(q,t)\ . \tag{90}$$

For $j\in\{1,\dots,r-1\}$, each pair (λ^j,μ^j) contains a common rectangle of row length at least λ_1^{j+1}. By Corollary 4.1, proving the dual Haglund conjecture is equivalent to proving it for each pair (λ^j,μ^j) with these rectangles removed. We can repeat this process for each pair until no further decomposition is possible.

Therefore, to prove the dual Haglund's conjecture we only need to prove $\mathrm{Hag}'(\lambda,\mu)$ for pairs of partitions (λ,μ) which are *irreducible*, i.e., which satisfy $\lambda\geq\mu$ and $\lambda_1+\cdots+\lambda_i>\mu_1+\cdots+\mu_i$ for $1\leq i\leq\ell(\lambda)$.

6.1. The case where $K_{\lambda,\mu}=1$.

Recall the characterization of irreducible pairs (λ,μ) with $K_{\lambda,\mu}=1$ given in Theorem 1.2. Case 2 of Theorem 1.2 is already done by Corollary 3.1. Now consider Case 1 of Theorem 1.2. Let $\lambda=(m^n)\geq\mu$ and $\ell(\mu)=n+1$. Since λ and μ are both contained in (m^{n+1}), by Corollary 5.1 we have

$$(K^{(i)})^{\pm1}_{\lambda,\mu}(q,t)=(K^{(i)})^{\pm1}_{(m),\mu^c}(q,t),\quad \text{for } i=1,2. \tag{91}$$

Here $\mu^c=(m-\mu_{n+1},\cdots-\mu_1)$ is the complement of μ in the rectangle (m^{n+1}), and (m) is the complement of λ.

In particular, $K^{(2)}_{\lambda,\mu}=K^{(2)}_{(m),\mu^c}$ implies

$$k_{\lambda,\mu}(q,t)=k_{(m),\mu^c}(q,t)\frac{c'_\mu(q,t)}{c'_{\mu^c}(q,t)}\ . \tag{92}$$

Note that c'_μ is a product of $|\mu|=mn$ many terms, c'_{μ^c} is a product of $m(n+1)-|\mu|=m$ many terms. If all terms in the denominator cancel off with terms in c'_μ then $\dfrac{c'_\mu(q,t)}{c'_{\mu^c}(q,t)}$ will be a product of $mn-m$ terms each of the form $(1-q^\alpha t^\beta)$ for $\alpha,\beta\in\mathbb{Z}_{\geq0}$. Substituting $q=t^k$ and dividing by $(1-t)$, we get $\dfrac{1-t^{k\alpha+\beta}}{1-t}\in\mathbb{Z}_{\geq0}[t]$. Since we know $\dfrac{k_{(m),\mu^c}(t^k,t)}{(1-t)^m}\in\mathbb{Z}_{\geq0}$ by Corollary 3.1, this would imply that $\dfrac{k_{\lambda,\mu}(t^k,t)}{(1-t)^{mn}}\in\mathbb{Z}_{\geq0}[t]$. To prove that $\dfrac{c'_\mu(q,t)}{c'_{\mu^c}(q,t)}$ is a product of $mn-m$ terms, each of the form $(1-q^\alpha t^\beta)$ for $\alpha,\beta\in\mathbb{Z}_{\geq0}$, we consider $f_\mu(q,t)=\sum_{x\in\mu}q^{a(x)}t^{l(x)}$ and show that $f_\mu(q,t)-f_{\mu^c}(q,t)\in\mathbb{Z}_{\geq0}[q,t]$. Since each monomial $q^\alpha t^\beta$ (counted with multiplicity) that occurs in $f_\mu(q,t)-f_{\mu^c}(q,t)$ corresponds to a term $(1-q^{\alpha+1}t^\beta)$ in $\dfrac{c'_\mu}{c'_{\mu^c}}$, this is clearly sufficient.

Lemma 6.1. *Let $\mu \in Par$. Then*

$$f_\mu(q,t) = \sum_{j=0}^{\ell(\mu)-1} t^j \left(\sum_{i=1}^{\ell(\mu)-j} q^{\mu_i - \mu_{i+j}} [\mu_{i+j} - \mu_{i+j+1}]_q \right). \tag{93}$$

Proof. For $i \in \{1, \dots \ell(\mu)\}$, in the i-th row of μ the rightmost $\mu_i - \mu_{i+1}$ many boxes have leg length 0 and arm lengths $0, \dots, \mu_i - \mu_{i+1} - 1$. The next $\mu_{i+1} - \mu_{i+2}$ many boxes from the right have leg length 1 and arm lengths $\mu_i - \mu_{i+1}, \dots, \mu_i - \mu_{i+2} - 1$ and so on. Hence the contribution of the i-th row to $f_\mu(q,t)$ is

$$\sum_{j=0}^{\ell(\mu)-i} t^j q^{\mu_i - \mu_{i+j}} [\mu_{i+j} - \mu_{i+j+1}]_q \,. \tag{94}$$

So,

$$\begin{aligned} f_\mu(q,t) &= \sum_{i=1}^{\ell(\mu)} \sum_{j=0}^{\ell(\mu)-i} t^j q^{\mu_i - \mu_{i+j}} [\mu_{i+j} - \mu_{i+j+1}]_q && (95) \\ &= \sum_{j=0}^{\ell(\mu)-1} t^j \left(\sum_{i=1}^{\ell(\mu)-j} q^{\mu_i - \mu_{i+j}} [\mu_{i+j} - \mu_{i+j+1}]_q \right). \end{aligned}$$

□

Corollary 6.1. *Let $\mu \subset (m^{n+1})$, with $\mu_1 < m$ and $\ell(\mu) = n+1$. Let μ^c denote the complement of μ in (m^{n+1}). Then*

$$f_\mu(q,t) - f_{\mu^c}(q,t) = \frac{1}{1-q} \sum_{j=0}^{n} t^j (q^{\mu^c_{n+1-j}} - q^{\mu_{n+1-j}}) \,. \tag{96}$$

Proof. We have

$$f_\mu(q,t) = \sum_{j=0}^{n} t^j \left(\sum_{i=1}^{n-j} q^{\mu_i - \mu_{i+j}} [\mu_{i+j} - \mu_{i+j+1}]_q + q^{\mu_{n+1-j} - \mu_{n+1}} [\mu_{n+1}]_q \right). \tag{97}$$

Since $(\mu^c)_i = m - \mu_{n+2-i}$, we have $\ell(\mu^c) = n+1$ since $\mu_1 < m$. Thus

$$\begin{aligned} f_{\mu^c}(q,t) &= \sum_{j=0}^{n+1} t^j \Bigg(\sum_{i=1}^{n-j} q^{\mu_{n+2-i-j} - \mu_{n+2-i}} [\mu_{n+1-i-j} - \mu_{n+2-i-j}]_q \\ &\qquad + q^{\mu_1 - \mu_{n+2-n-1+j}} [m - \mu_1]_q \Bigg) \\ &= \sum_{j=0}^{n+1} t^j \left(\sum_{i'=1}^{n-j} q^{\mu_{i'+1} - \mu_{i'+j+1}} [\mu_{i'} - \mu_{i'+1}]_q + q^{\mu_1 - \mu_{j+1}} [m - \mu_1]_q \right). \end{aligned}$$

Therefore the coefficient of t^j in $f_\mu(q,t) - f_{\mu^c}(q,t)$ is

$$\frac{1}{1-q}\Bigg(\sum_{i=1}^{n-j}(q^{\mu_i-\mu_{i+j}} - q^{\mu_i-\mu_{i+j+1}}) - \sum_{i'=1}^{n-j}(q^{\mu_{i'+1}-\mu_{i'+j+1}} - q^{\mu_{i'}-\mu_{i'+j+1}})$$
$$\ker n3cm + q^{\mu_{n+1-j}-\mu_{n+1}} - q^{\mu_{n+1-j}} - q^{\mu_1-\mu_{j+1}} + q^{m-\mu_{j+1}}\Bigg)$$
$$= \frac{1}{1-q}\left(q^{\mu_1-\mu_{1+j}} - q^{\mu_{n+1-j}-\mu_{n+1}} + q^{\mu_{n+1-j}-\mu_{n+1}} - q^{\mu_{n+1-j}} - q^{\mu_1-\mu_{j+1}} + q^{m-\mu_{j+1}}\right)$$
$$= \frac{1}{1-q}(q^{m-\mu_{j+1}} - q^{\mu_{n+1-j}}) = \frac{1}{1-q}(q^{\mu^c_{n+1-j}} - q^{\mu_{n+1-j}}). \qquad \square$$

Lemma 6.2. *Suppose $\mu \vdash mn$, $\ell(\mu) = n+1$, and $\mu \subset (m^{n+1})$. Then, for all $i \in \{1,\dots,n+1\}$, we have $\mu_i^c \le \mu_i$.*

Proof. Suppose $\mu_i^c = m - \mu_{n+2-i} > \mu_i$ for some $i \in \{1,\dots,n+1\}$.

If $i \neq n+2-i$, then

$$(n-1)\mu_1 \ge \sum_{j\notin\{i,n+2-i\}} \mu_j = |\mu| - (\mu_i + \mu_{n+2-i}) > (n-1)m, \tag{98}$$

which implies $\mu_1 > m$, a contradiction.

If $i = n+2-i$, then $m - \mu_{n+2-i} > \mu_i$ implies $2\mu_i < m$. Consequently,

$$2(n-1)\mu_1 \ge 2\sum_{j\neq i}\mu_j = 2|\mu| - 2\mu_i > m(2n-1). \tag{99}$$

Since $m \ge \mu_1$, we get $2(n-1) > 2n-1$, a contradiction. $\square$

With the assumptions of Corollary 6.1, by Lemma 6.2 we have

$$f_\mu(q,t) - f_{\mu^c}(q,t) \in \mathbb{Z}_{\ge 0}[q,t]. \tag{100}$$

Corollary 6.2. $\mathrm{Hag}'(\lambda,\mu)$ *is true whenever* $K_{\lambda,\mu} = 1$.

Proof. We showed that, if λ,μ is an irreducible pair with $K_{\lambda,\mu} = 1$, then $\mathrm{Hag}'(\lambda,\mu)$ is true. Corollary 4.1 shows that $\mathrm{Hag}'(\lambda,\mu)$ remains true when patching together irreducible pairs. $\square$

6.2. **The case where $K_{\mu',\lambda'} = 1$.**

Proposition 6.1. $\mathrm{Hag}'(\lambda,\mu)$ *is true whenever* $K_{\mu',\lambda'} = 1$.

Proof. Suppose $\lambda,\mu \in Par$ such that $K_{\mu',\lambda'} = 1$. To compute $k_{\lambda,\mu}(q,t)$, in the earlier case we used $K^{(2)}_{\lambda,\mu}(q,t)$. Here we use $(K^{(2)})^{-1}_{\mu',\lambda'}(t,q)$. This amounts to applying the equality between the far left-hand side and the far right-hand side in (30). As before, by use of Lemma 4.2 and Lemma 4.3, we reduce the calculation of $(K^{(2)})^{-1}_{\mu',\lambda'}(t,q)$ to the case where μ',λ' is an irreducible pair. The case where $\mu' = (m)$ for some m is done by Corollary 3.1. Let $\mu' = (m^n)$ and $\ell(\lambda') = n+1$. By Corollary 5.1, we have

$$(K^{(2)})^{-1}_{\mu',\lambda'}(t,q) = (K^{(2)})^{-1}_{(m),\lambda'^c}(t,q). \tag{101}$$

Hence, by (30),

$$k_{\lambda,\mu}(q,t) = k_{\lambda^c,(1^m)}(q,t)\frac{c_{(m^n)}(q,t)}{c_{(m)}(q,t)}\,. \tag{102}$$

In $\frac{c_{(m^n)}(q,t)}{c_{(m)}(q,t)}$, the contribution from the last row of (m^n) cancels with the contribution from (m), so it is a product of $mn-1$ terms each of the form $1-q^\alpha t^\beta$ for some $\alpha,\beta \in \mathbb{Z}_{\geq 0}$. Since, by Corollary 3.1, $\frac{k_{\lambda^c,1^m}(t^k,t)}{(1-t)^m} \in \mathbb{Z}_{\geq 0}[t]$, this implies $\mathrm{Hag}'(\lambda,\mu)$ for λ,μ, and hence for all pairs λ,μ with $K_{\mu',\lambda'}=1$ when μ',λ' is irreducible. Applying Corollary 4.1, we conclude the statement for general λ,μ with $K_{\mu',\lambda'}=1$. □

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