Minuscule Exceptional Schubert Varieties

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BACKGROUND

Generic rings and finite free resolutions

R - Noetherian ring. A finite free resolution of R-modules is an acyclic complex $\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_1 \xrightarrow{d_1} F_0$

with $F_i = R^{f_i}$, $\operatorname{rank}(d_i) = r_i$, $f_i = r_i + r_{i+1}$ for $1 \le i \le n$.

The format of $\mathbb{F}_\bullet$ is the sequence $(f_0, f_1, \dots, f_n) =: \underline{f}$.

Def. A pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ is a generic resolution with format $\underline{f}$

commutative ring

free complex over R_{gen}

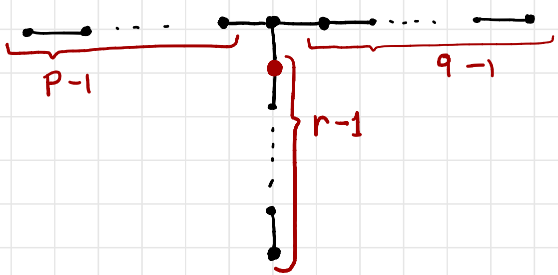
iff:

① $\mathbb{F}_\bullet^{gen}$ is acyclic over R_{gen}.

② For every free acyclic complex $\mathbb{G}_\bullet$ of the same format over a Noetherian ring S there exists a ring homomorphism $\phi : R_{gen} \longrightarrow S$ such that $\mathbb{G}_\bullet = F_{gen} \otimes_{R_{gen}} S$

Theorem (Bruns, 1984) A generic resolution always exists for each format $f = (f_0, \dots, f_n)$. It is however, not unique, and in general not Noetherian.

Theorem (Weyman, 2016) For $n=3$, there exists a specific generic ring R_{gen} which is Noetherian iff and only if f comes from a Dynkin diagram.

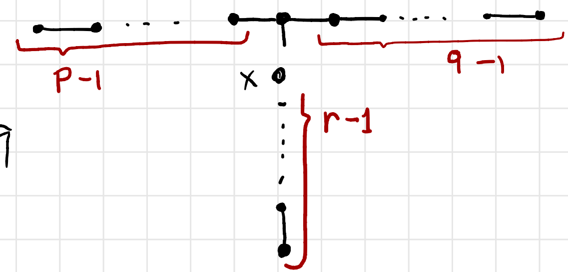


$$0 \to R_1 \xrightarrow{d_1} R_2 \xrightarrow{d_2} R_3 \xrightarrow{d_3} R$$

$$\begin{aligned} \operatorname{rank}(d_1) + 1 &= p \\ \operatorname{rank}(d_2) - 1 &= q \\ \operatorname{rank}(d_3) + 1 &= r \end{aligned}$$

"Resolution of format $T_{p,q,r}$".

Motivation



- $T_{p,q,r}$ is of **Dynkin type** $\iff$ R_{gen} is Noetherian (Weyman)

Format $T_{p,q,r}$

$$0 \to R_1 \xrightarrow{d_1} R_2 \xrightarrow{d_2} R_3 \xrightarrow{d_3} R$$

$\text{rank}(d_1) + 1 = p$
$\text{rank}(d_2) - 1 = q$
$\text{rank}(d_3) + 1 = r$

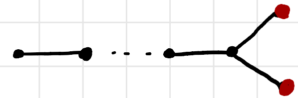
- For all Dynkin types, there exists an opposite Schubert variety of codimension 3 in G/P_x, whose intersection with the big open cell has coordinate ring whose resolution has format $T_{p,q,r}$. (Sam-Weyman)

Theorem (S.A.F. - J.W. - T.) $/\mathbb{C}$

Let G be a reductive group of exceptional type and $P \subseteq G$ a standard maximal parabolic subgroup stabilising a minuscule fundamental weight. Let $U \subseteq G/P$ be the "big open cell".
Let $X \subseteq G(E_6)/P_1$ or $X \subseteq G(E_7)/P_7$ (+ some constraints) be an opposite Schubert variety. Then $X \cap U$ is:

- a complete intersection (c.i.) or a c.i. in:
- the variety of pure spinors
- the variety of complexes
- the Huneke-Ulrich ideal of deviation 2
- 2×2 minors of a 2×3 generic matrix
- 4×4 Pfaffians of a 6×6 skew-symmetric matrix

The variety of pure spinors



Let x be one of the red vertices. Fix $Q : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \longrightarrow \mathbb{C}$ a non-degenerate, symmetric bilinear form. A subspace $V \subseteq \mathbb{C}^{2n}$ is isotropic iff $Q(v,w)=0$ forall $v,w \in V$. The isotropic Grassmannian is

$$IG(n,2n) = \{ V \in Gr(n,2n) \mid V \text{ is isotropic} \}.$$

The homogeneous space $SO(2n,\mathbb{C})/P_x$ is one of the two connected components of $IG(n,2n)$.

- Each of these connected components is called a variety of (even, odd, resp.) pure spinors.

$$\rightsquigarrow SO(2n,\mathbb{C})/P_x \hookrightarrow \mathbb{P}(V(S^+)) \text{ or } \mathbb{P}(V(S^-))$$

$g, f \in \mathbb{Z}_{>0}$

The variety of complexes Let X, Y be matrices of indeterminates, of sizes $1 \times g$ and $g \times f$, respectively. Define the ideal $J \subseteq k[X, Y]$:

$J = \text{minors}(1, XY) + \text{minors}(\min\{f, g\}, Y)$. In affine space $\mathbb{A}^{fg+g}$, the variety V of all complexes $0 \to k^f \xrightarrow{\theta_2} k^g \xrightarrow{\theta_1} k$.

with $\text{rank}(\theta_2) < \min\{f, g\}$ is called the variety of complexes.

Theorem (De Concini – Strickland, 1981)

The ideal J is the defining ideal of the variety of complexes V.

(Moreover, it is a perfect ideal of grade $\max\{f, g\}$.

Recall: perfect $\overset{\text{def}}{\iff}$ proj. dim $=$ length of maximal regular sequence)

The deviation two Gorenstein rings of Huneke and Ulrich.

Theorem (Huneke-Ulrich, 1985)

(R, m, k) commutative, noetherian local ring and X, Y matrices with entries in m, of sizes $1 \times 2n$ and $2n \times 2n$, respectively. Assume Y is skew-symmetric. Then the ideal

$$J = \text{minors}(1, XY) + \text{Pf}(X)$$

is a perfect Gorenstein ideal of deviation 2.

the minimal number of generators of J is 2 more than the grade of J.

Weights and patterns

- Integral weights $\xleftrightarrow{1:1}$ Labellings of the Dynkin diagram with integers $\sigma_i \in \mathbb{Z}$.

$\sigma = (\sigma_1, \dots, \sigma_n)$

- The Weyl group $W = \langle s_1, \dots, s_n \rangle$ acts on the weight σ by $\sigma_i \mapsto -\sigma_i$ for each s_i and adding σ_i to all σ_j such that j is a node adjacent to i.

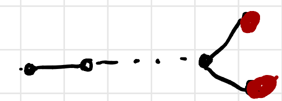
- The fundamental weight ω_i is defined by $\sigma_j = \delta_{ij}$.

<u>Example</u> (E_6) $\swarrow \omega_2$

$$\begin{array}{ccccc} {}_1 0 & 0_3 & 0_4 & 0_5 & 0_6 \\ & & 1_2 & & \end{array} \xrightarrow[s_2]{} \begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ & & -1 & & \end{array} \xrightarrow[s_4]{} \begin{array}{ccccc} 0 & 1 & -1 & 1 & 0 \\ & & 0 & & \end{array}$$

Fundamental Representations

- Nodes in Dynkin diagram $\overset{1:1}{\longleftrightarrow}$ Fundamental weights
- Fundamental weight ω_i $\rightsquigarrow$ Fundamental representation $V(\omega_i)$
- Type A_{n-1} : $V(\omega_i) = \Lambda^i \mathbb{C}^n$ for $i = 1, \dots, n-1$
- Type D_n : $V(\omega_i) = \Lambda^i \mathbb{C}^{2n}$ for $i = 1, \dots, n-2$ + two half-spin representations $\underset{\text{"even spinors"}}{V(S^+)}, \underset{\text{"odd spinors"}}{V(S^-)}$



Exceptional types: later.

Minuscule Representations

Def. A fundamental representation is minuscule if the Weyl group acts transitively on its set of weights.

- For type A_n, all fundamental representations are minuscule.
- For type D_n, the minuscule representations are the two half-spin representations and the natural representation.

Example of non-minuscule representation

$G = SL(3,\mathbb{C})$, V = adjoint representation

T = diagonal matrices

$$\begin{tikzcd}
& -1\ 2 \arrow[r, "2"] & 1\ -2 \arrow[dr, "1"] & \\
1\ 1 \arrow[ur, "1"] \arrow[dr, "2"'] & & & -1\ -1 \\
& 2\ -1 \arrow[r, "1"] & -2\ 1 \arrow[ur, "2"'] &
\end{tikzcd}$$

- For type E_6, there are two (dual) minuscule representations of dimension 27. They are determined by the well-known configuration of 27 lines on a cubic surface.

- For type E_7, there is one minuscule rep. of dimension 56 (28 pairs of bitangents to a plane quartic surface).

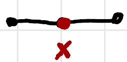
- There are no minuscule representations in type E_8.

Parabolic subgroups and quotients

Node x in Dynkin diagram $\rightsquigarrow$ maximal parabolic subgroup $P_x \subseteq G$

$$\underbrace{G/P_x}_{\substack{\text{projective}\\\text{variety,}\\\text{smooth}}} \hookrightarrow \mathbb{P}(V(\omega_x))$$

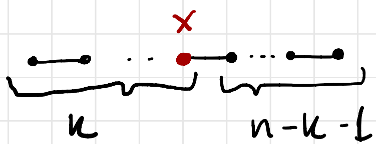
Example $n = 4$



$$P_x = \left\{ \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{pmatrix} \right\}$$

$$GL(4,\mathbb{C})/P_x \cong Gr(2,4)$$

In general:



$\rightsquigarrow$

$$P_x = \left\{ \begin{array}{c} \\ \} k \\ \\ \end{array} \begin{pmatrix} \multicolumn{2}{c}{\overbrace{\hspace{3em}}^{n}} \\ * & * \\ 0 & * \\ & \underbrace{\hspace{2em}}_{n-k} \end{pmatrix} \right\}$$

$$G/P_x \cong Gr(k,n) \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n)$$

Plücker embedding

Plücker coordinates Let $\mathbb{C}^n$ with canonical basis $e_1, \dots, e_n$.

Let $W \in Gr(k,n)$ be spanned by $w_1, \dots, w_k \in \mathbb{C}^n$.

The assignment

$$Gr(k,n) \xrightarrow{\varphi} \mathbb{P}(\Lambda^k \mathbb{C}^n)$$
$$W \longmapsto [w_1 \wedge \cdots \wedge w_k]$$

$i_1 < \cdots < i_k$.

is a well-defined map called the **Plücker embedding**. For indices $i_1, \dots, i_k$ we denote by $P_{i_1 \cdots i_k}(W)$ the projection of $\varphi(W)$ to the coordinate $[e_{i_1} \wedge \cdots \wedge e_{i_k} = 1]$.

Example For $k=2$, we have $P_{ij} = -P_{ji}$, hence the Plücker coordinates fit into a skew-symmetric matrix P. Then $Gr(2,n)$ is the projective algebraic variety defined by all the Pfaffians of the 4×4 minors of P.

Schubert varieties

Let $W_{P_x} = \langle \{S_x\}^c \rangle$. The opposite Schubert varieties in G/P_x are those of the form $X_w = \overline{B^- w P_x}/P_x$, $w \in W/W_{P_x}$ where B^- is the opposite Borel subgroup with respect to P.

Remarks

- The codimension of X_w is the length of w.
- The big open cell in G/P_x is given by $P_{top} \neq 0$.

Example (type D_n)

Pick the hyperbolic basis $e_1, \dots, e_n, e_{\bar{1}}, \dots, e_{\bar{n}}$ of $\mathbb{C}^{2n}$ with respect to Q, that is,

$$Q(a_1e_1+\cdots+a_ne_n+a_{\bar{1}}e_{\bar{1}}+\cdots+a_{\bar{n}}e_{\bar{n}},\, b_1e_1+\cdots+b_ne_n+b_{\bar{1}}e_{\bar{1}}+\cdots+b_{\bar{n}}e_{\bar{n}}) = \sum_{i=1}^{n} a_ib_{\bar{i}} + \sum_{i=1}^{n} a_{\bar{i}}b_i.$$

- Then the "big open cell" $\overset{U}{(w=id)}$ in $IG(n,2n)$ is spanned by the rows of matrices of the form $(I_n X)$ where I_n is the $n\times n$ identity matrix and X is a skew-symmetric matrix.

Example The big open cell in $Gr(3,6)$ is given by matrices of the form:

$$\begin{array}{c}\begin{pmatrix} 1 & 0 & 0 & x_{1,4} & x_{1,5} & x_{1,6} \\ 0 & 1 & 0 & x_{2,4} & x_{2,5} & x_{2,6} \\ 0 & 0 & 1 & x_{3,4} & x_{3,5} & x_{3,6} \end{pmatrix}\end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$Q(v,w) = 0$ forall v,w rows

- 1 with itself: $x_{1,4} = 0 = x_{2,5} = x_{3,6}$
- 1 with 2: $x_{2,4} + x_{1,5} = 0$
- 1 with 3 $x_{3,4} + x_{1,6} = 0$
- 2 with 3 $x_{3,5} + x_{2,6} = 0$

Example

Let $$\begin{array}{c} M = \begin{pmatrix} 1 & 0 & 0 & 0 & x_{1,5} & x_{1,6} \\ 0 & 1 & 0 & -x_{1,5} & 0 & x_{2,6} \\ 0 & 0 & 1 & -x_{1,6} & -x_{2,6} & 0 \end{pmatrix} \\ \quad\quad\; 1 \;\; 2 \;\; 3 \;\;\; \bar{1} \;\;\;\; \bar{2} \;\;\;\; \bar{3} \end{array}$$ and X its skew-symmetric part.

Any given subset of $\{\bar{1}, \bar{2}, \bar{3}\}$ of cardinality 2 determines a unique skew-symmetric 2×2 minor of X.

The subsets $\{\bar{1}, \bar{2}\}$, $\{\bar{1}, \bar{3}\}$ and $\{\bar{2}, \bar{3}\}$ correspond to $\{\bar{1}, \bar{2}, 3\}$, $\{\bar{1}, 2, \bar{3}\}$, and $\{\bar{1}, 2, \bar{3}\}$, whose corresponding determinants are the squares of the pfaffians of the appropriate 2×2 skew-symmetric minors.

$G/P_x \hookrightarrow \mathbb{P}(V_{\omega_x})$

- Plücker coordinates are the Pfaffians of X of all sizes. They correspond to subsets of $\{1, \dots, n\}$ of even cardinality.

- If $n = 2m+1$ is odd, the $2m \times 2m$ Pfaffians of X are the defining equations of the intersection of the variety of pure spinors with the big open cell $\mathcal{U}$.

- It is known that these Pfaffians span the generic Gorenstein ideal with resolution of format $(1, n, n, 1)$.

Methods

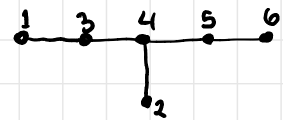
- Set-theoretic description of the defining ideal of a Schubert variety.
- We use descriptions of the minuscule homogeneous spaces by Vavilov, Luzgarev & Pevzner.
- Hands-on inspection assisted by Macaulay 2.

Set-theoretic description of the defining ideal of a Schubert Variety

Let $\sigma \in W/W_{P_x}$ and $X_\sigma \subseteq G/P_x \xhookrightarrow{i} V(\omega_x)$ be a Schubert variety. Let $p_\sigma^* \in V(\omega_x)^*$ be a weight vector dual to $p_\sigma = \sigma(V_{\omega_x}) \in V(\omega_x)$, a "Plücker coordinate". Then:

$$X_\tau = \left\{ x \in i(G/P) \;\middle|\; p_\sigma^*(x) = 0 \quad \forall \; \sigma \nleq \tau \text{ in the Bruhat order} \right\}$$

Example (E_6, ω_1)



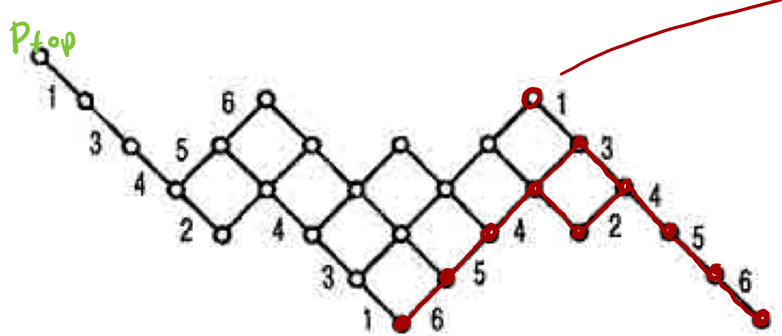


Fig. 20. $(E_6, \overline{\omega}_1)$

The equations for $X_o \cap U$ are given by the vanishing of the ten equations shaded in in red.

$Y_o = X_o \cap U$ is isomorphic to the variety of pure spinors.

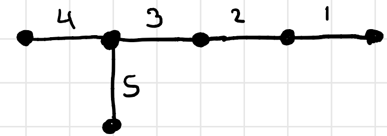
Image: Visual basic representations: an atlas (Plotkin, Semenov, Vavilov)

W/W_{P_1}

node w $\rightsquigarrow$ $I(X_w)$ is gen. by x vert. $x \not\leq w$

Bruhat order

Example (D_6, ω_6)



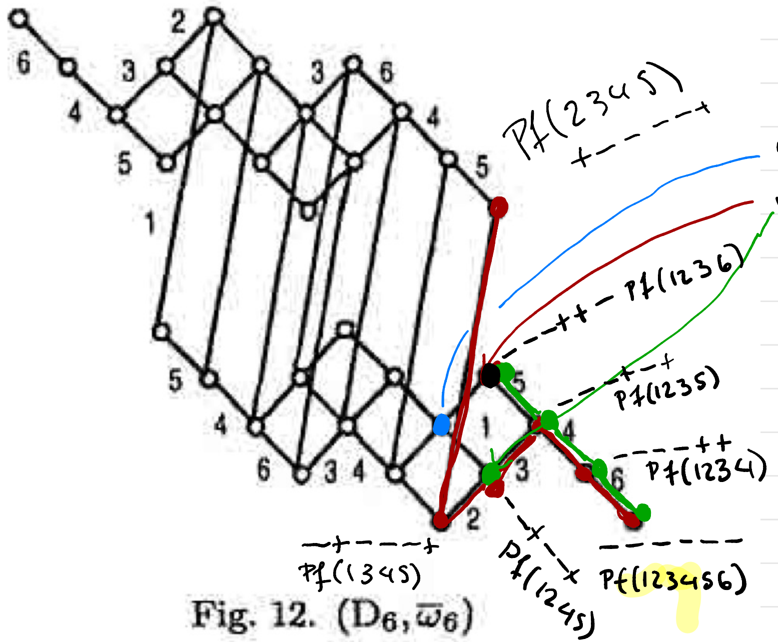


Fig. 12. $(D_6, \overline{\omega}_6)$

Schubert varieties

four equations for the green one

six equations for the red one

$\bullet \rightsquigarrow X_\bullet$ Schubert variety.

generators $\leftrightarrow$ all nodes y

$y \nleq \bullet$

Bruhat order

(D_n, ω_n)

Image: Visual basic representations: an atlas (Plotkin, Semenov, Vavilov)