A family of equivariant bijections between noncrossing and nonnesting partitions of type A

joint work (in progress) with
B. Dequène, G. Frieden, A. Iraci, H. Thomas, N. Williams

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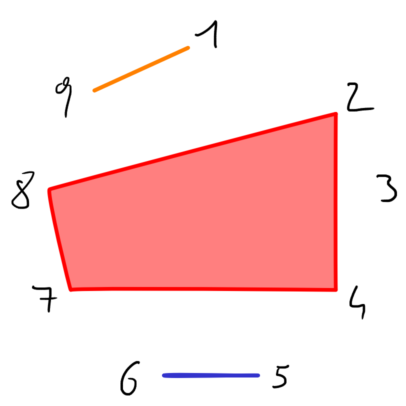
Noncrossing partitions   NC(n)

(there is a Coxeter group definition)

Combinatorially:

place integers 1, ..., n on a circle,
form polygons of integers in the same block,
polygons are not allowed to cross.

{1,9} {2,4,7,8} {3} {5,6}



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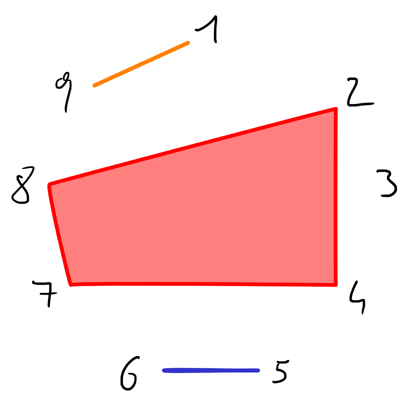
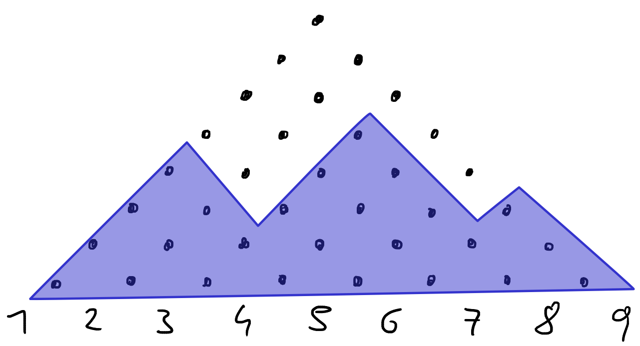


Nonnesting partitions $NN(n)$

- order ideals of the root poset of type A_{n-1}

positive roots: $(i,j) = e_{i,j} = e_j - e_i \quad i < j$

$$
\begin{array}{ccccc}
 & & (1,n) & & \\
 & (1,n-1) & & (2,n) & \\
 \cdots & & & & \cdots \\
(1,2) & (2,3) & \cdots & & (n-1,n)
\end{array}
$$



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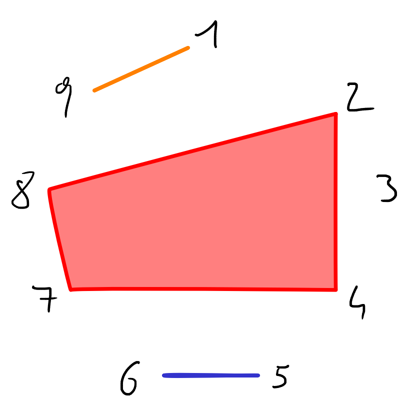

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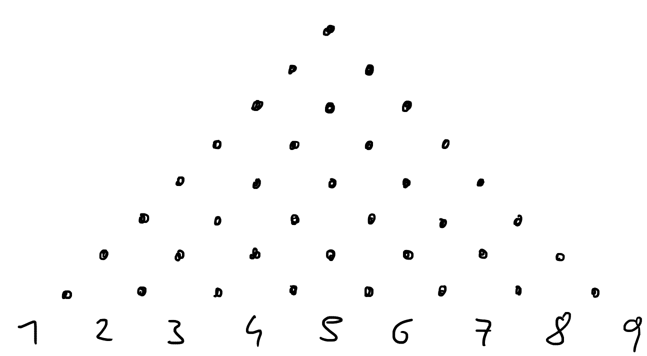

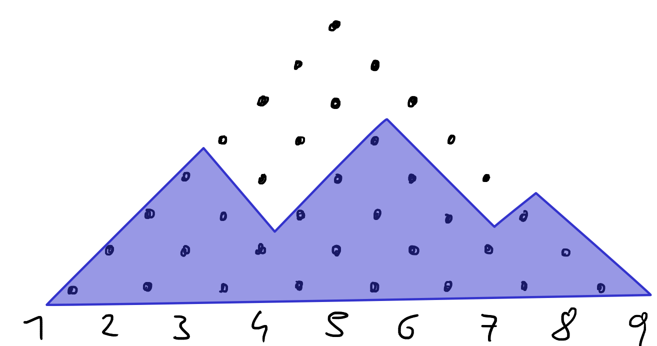

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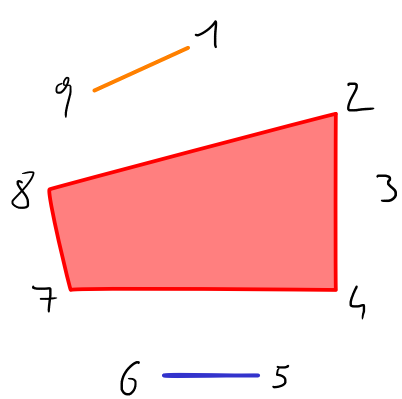

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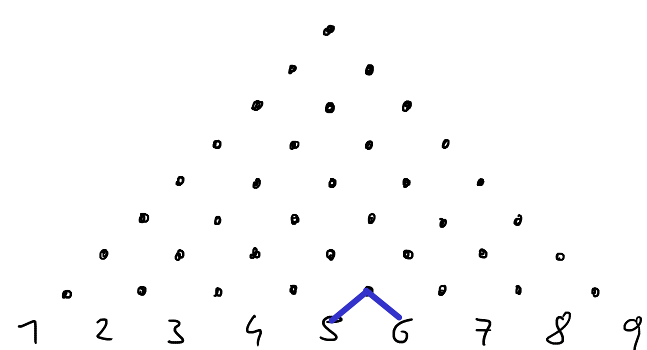

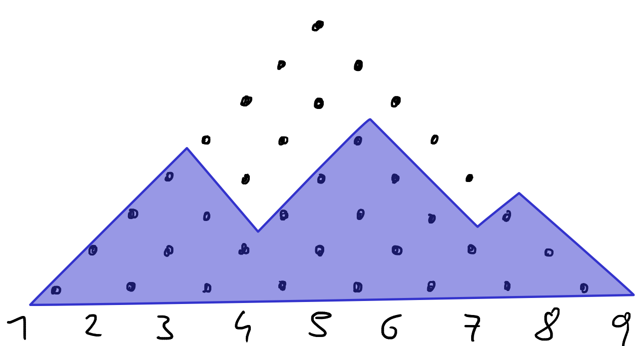

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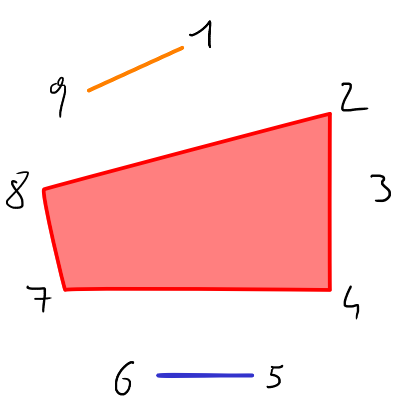

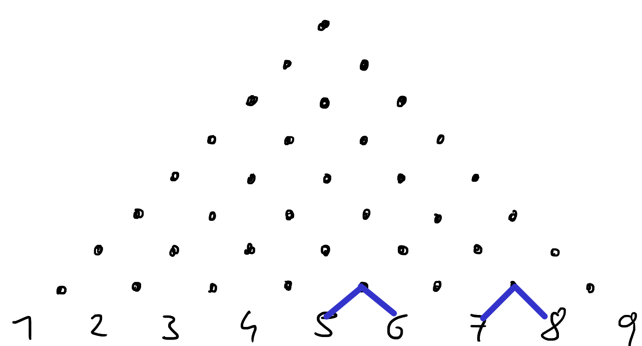

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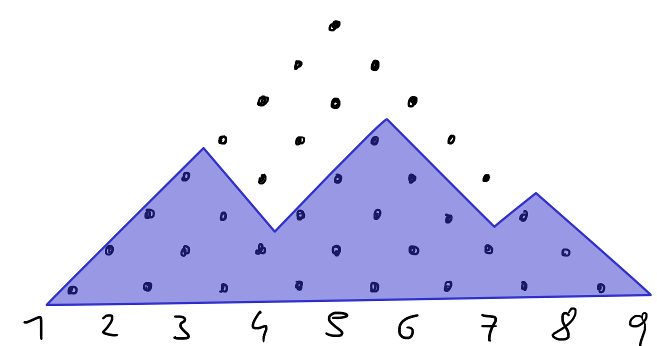

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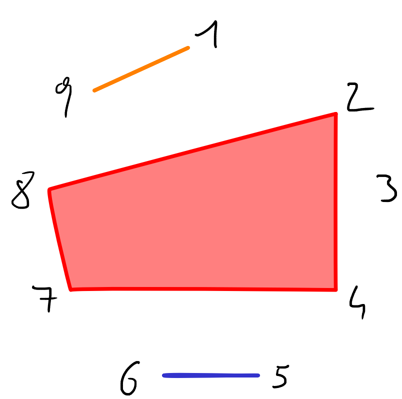

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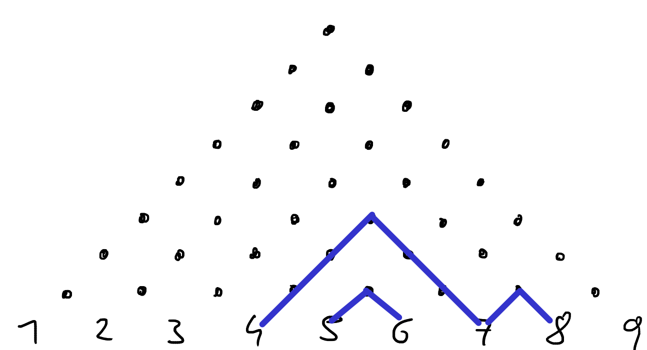

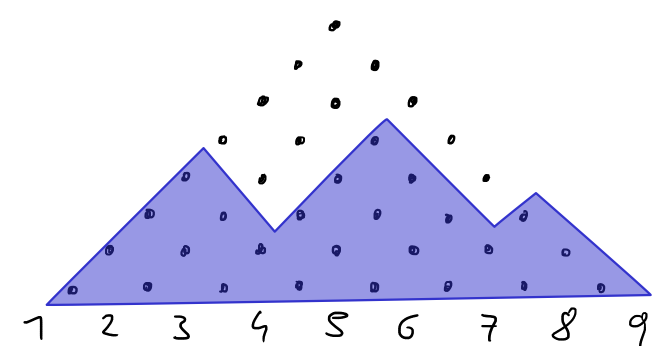

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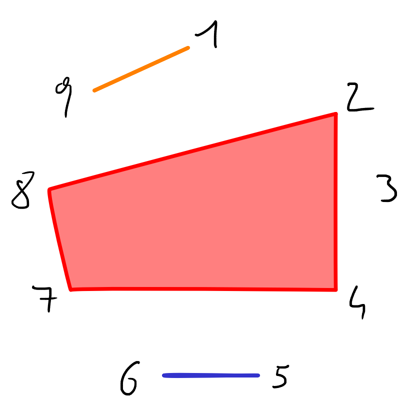

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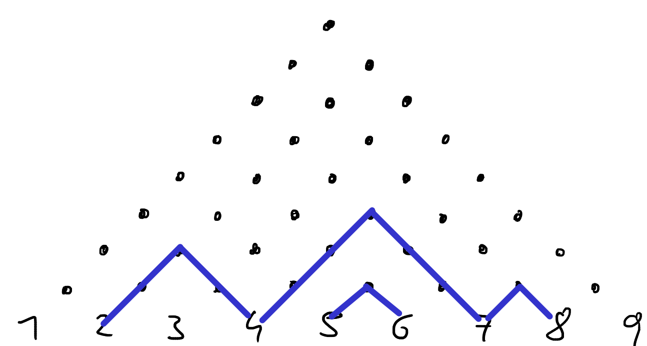

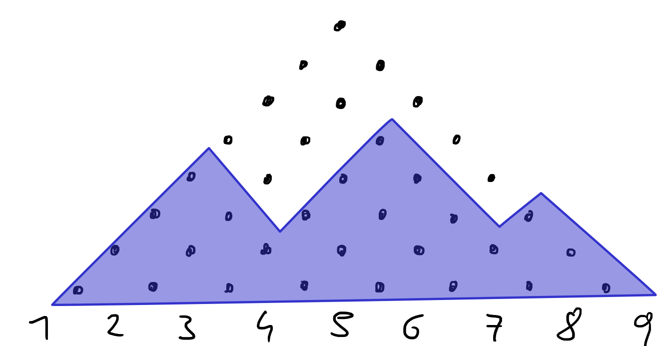

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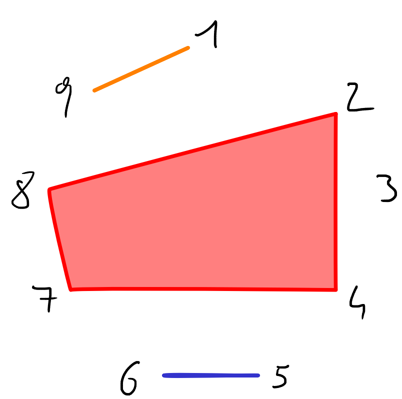

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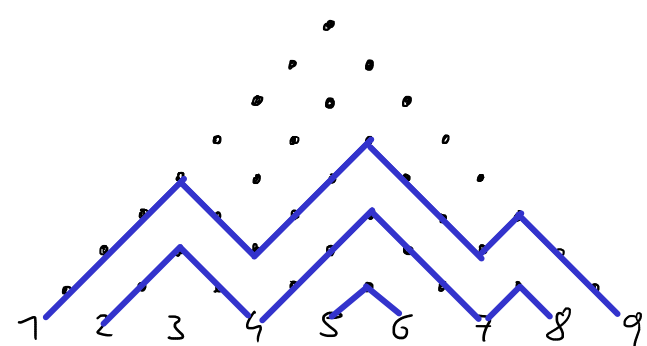
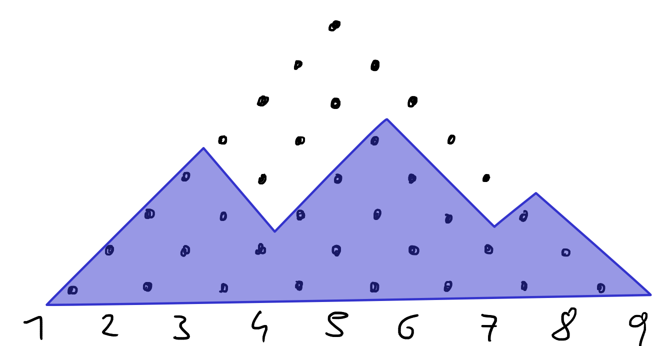
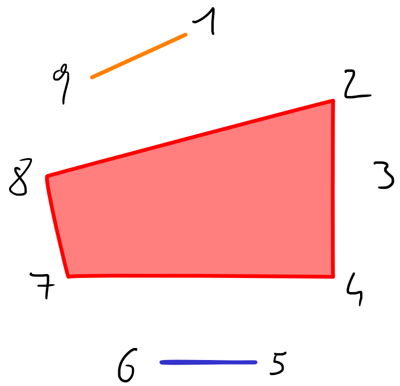




Kreweras Complement

Kr

1) Double numbers in clockwise direction.



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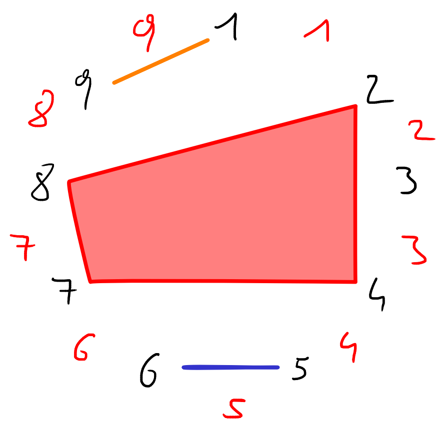

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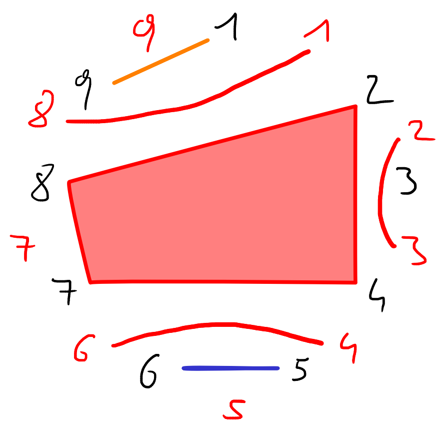

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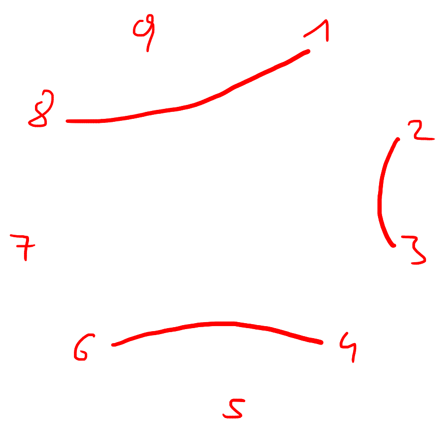
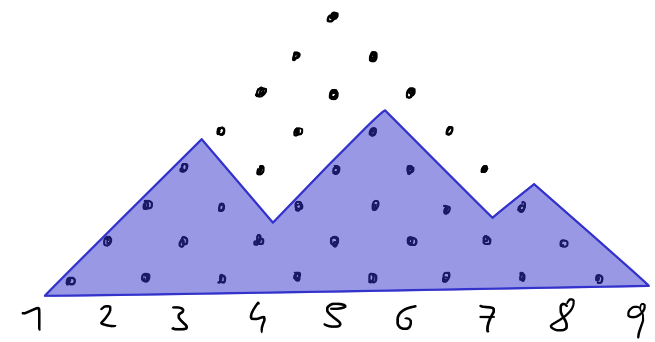


Promotion ∂

- Toggling t_r at root r :
  add / remove r whenever possible
- Toggle along SE-diagonals from left to right



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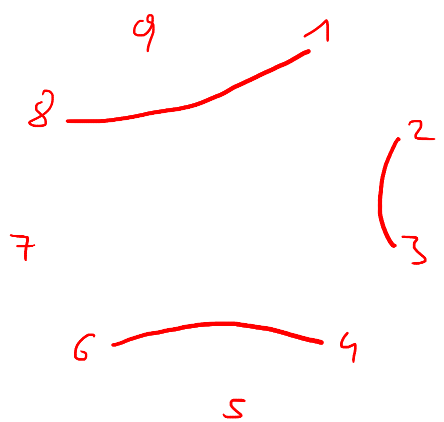

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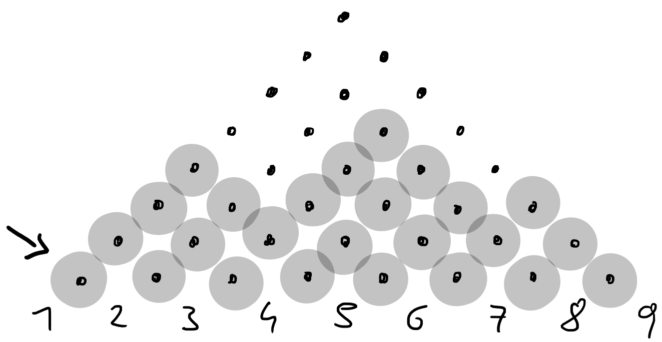

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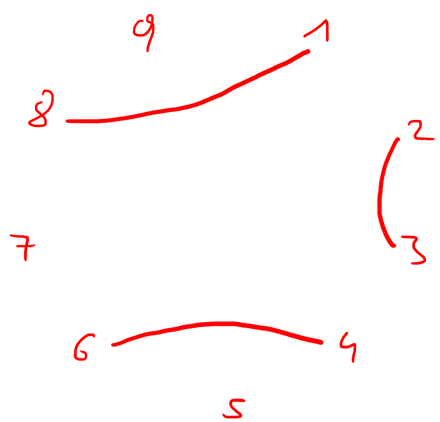

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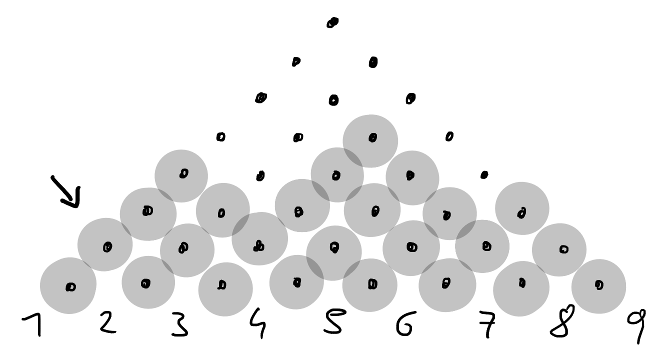

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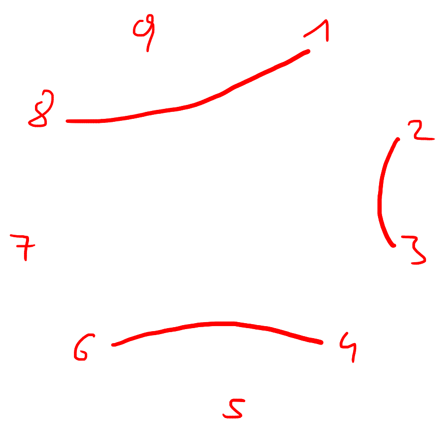

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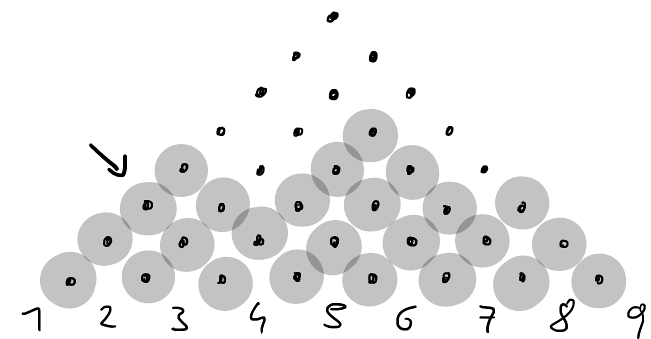

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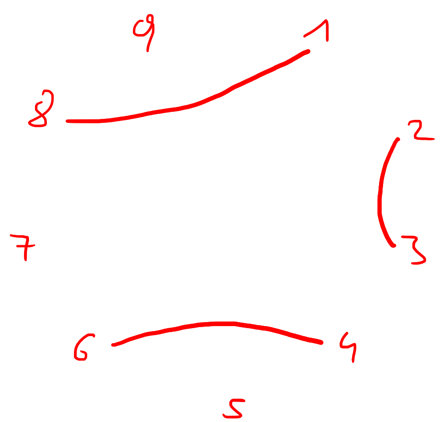

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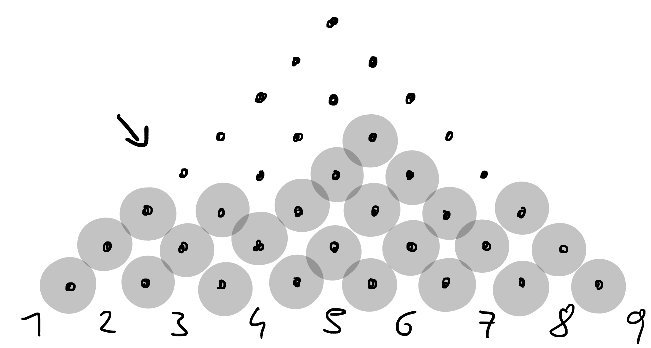

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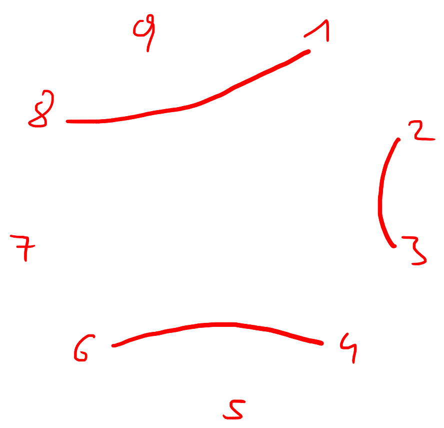

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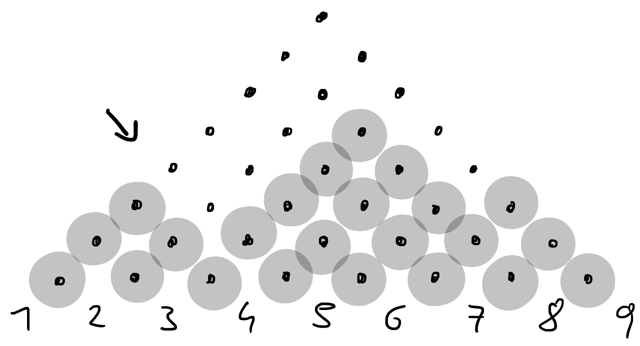

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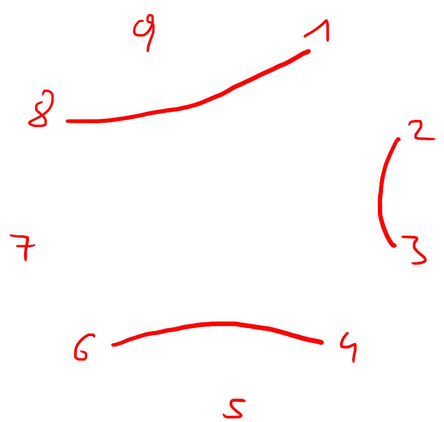

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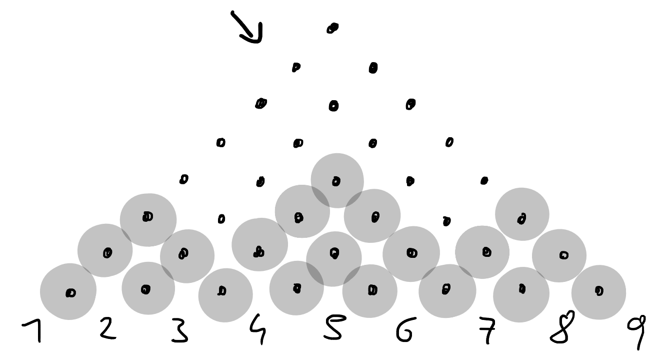

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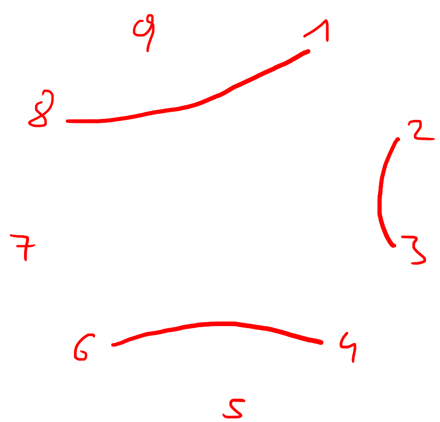

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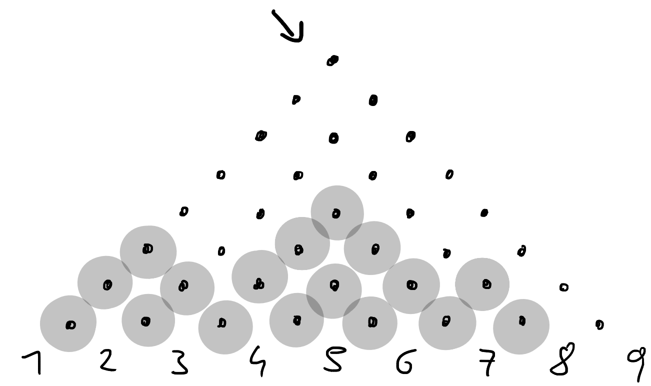
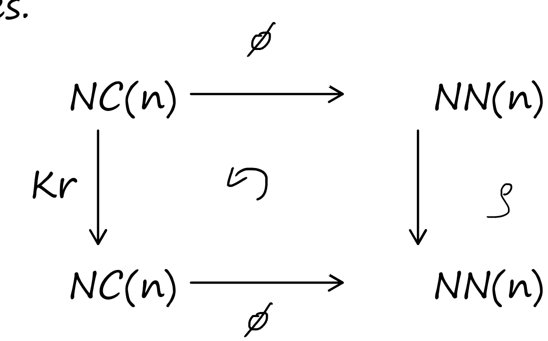


Theorem

Let $\varnothing$ denote the above presented bijection between $NC(n)$ and $NN(n)$.
Then the following diagram commutes.



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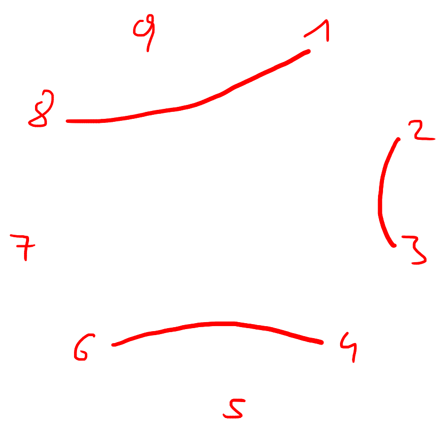

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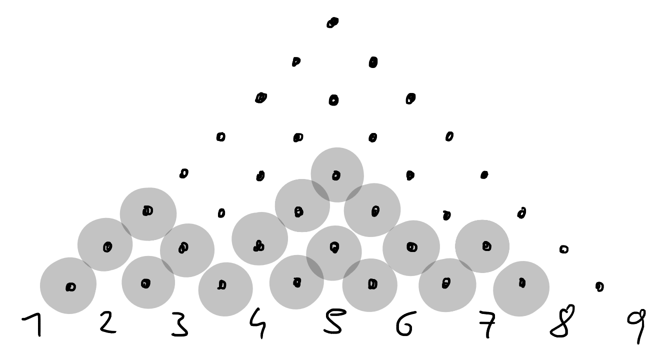

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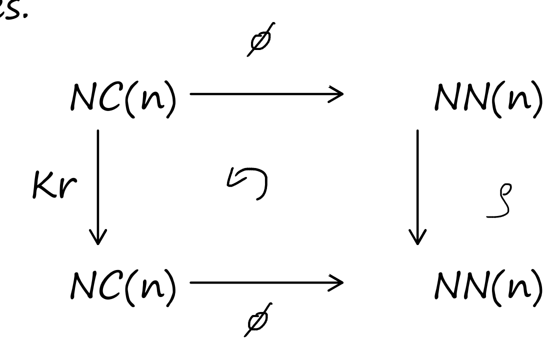

Denote by s_i the transposition $(i,i+1)$.

A coxeter element c is a product of the $s_1, s_2, \ldots, s_n$ in some order.

$$\Longleftrightarrow \quad c = (a_1, a_2, \ldots, a_k, \ldots, a_n) \quad \text{with}$$

$$1 = a_1 < a_2 < \ldots < a_k = n > \ldots > a_n .$$

$$c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1\ 3\ 4\ 7\ 9\ 8\ 6\ 5\ 2)$$

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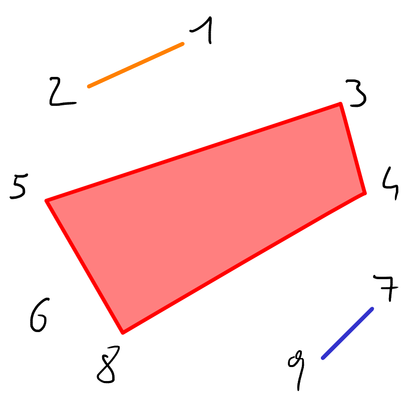
c-Noncrossing partitions $NC(n,c)$

- place integers $a_1, \ldots, a_n$ on a circle,
- form polygons of integers in the same block,
- polygons are not allowed to cross.



Define Kreweras complement Kr_c "as before".

Denote by s_i the transposition $(i,i+1)$.

A coxeter element c is a product of the $s_1, s_2, \dots, s_n$ in some order.

$$\Longleftrightarrow \quad c = (a_1, a_2, \dots, a_k, \dots, a_n) \quad \text{with}$$
$$1 = a_1 < a_2 < \dots < a_k = n > \dots > a_n .$$

$$c = s_2 s_1 s_3 s_6 s_5 s_4 s_8 s_7 = (1\ 3\ 4\ 7\ 9\ 8\ 6\ 5\ 2)$$

c-Noncrossing partitions $NC(n,c)$

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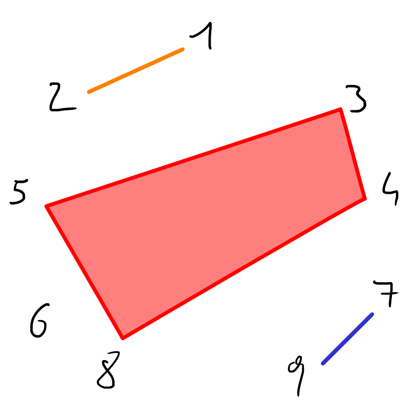

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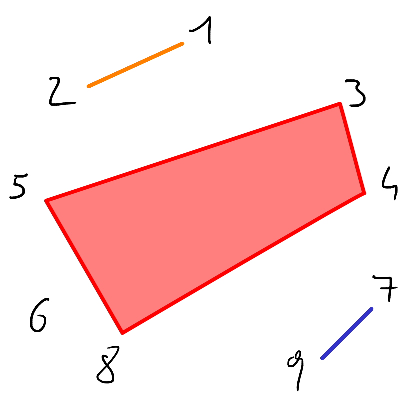

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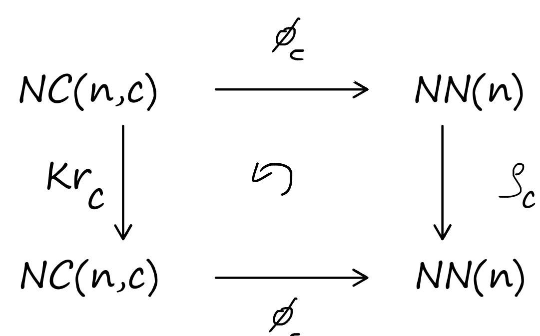
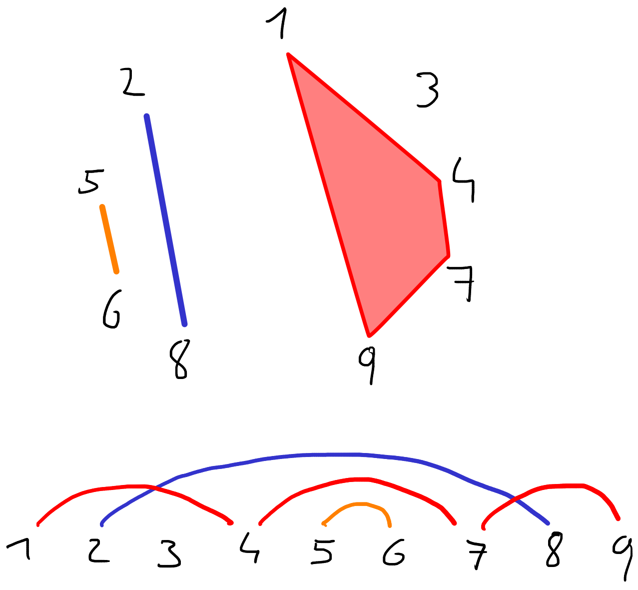
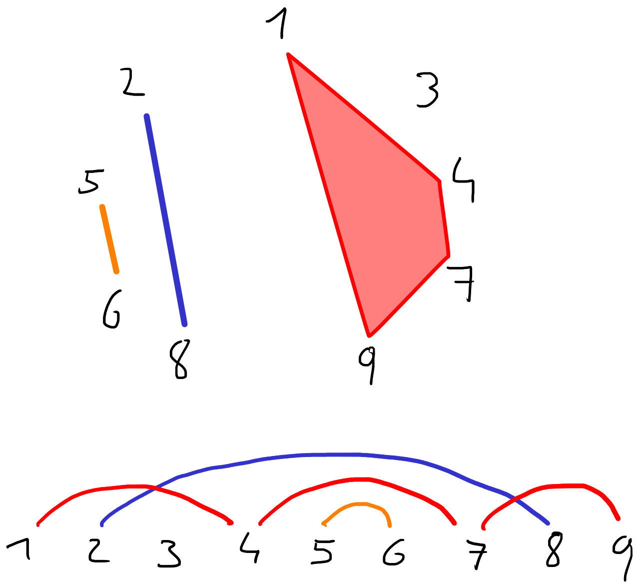
Question:

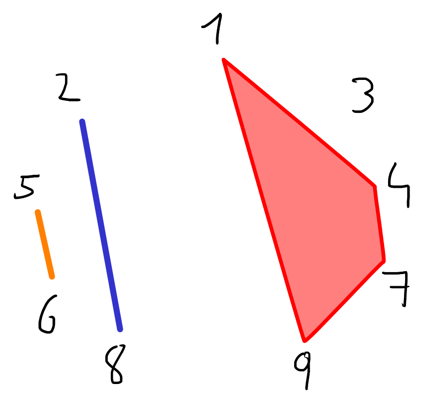
Which map ϕ_c lets the following diagram commute?

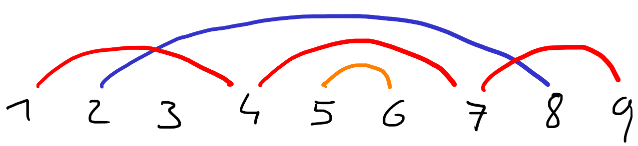



1
2
3
4
5
6
7
8
9
1 2 3 4 5 6 7 8 9



Need to deal with crossings !





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Outgoing set O: set of integers connected to a larger integer.
Incoming set I: smaller integer.

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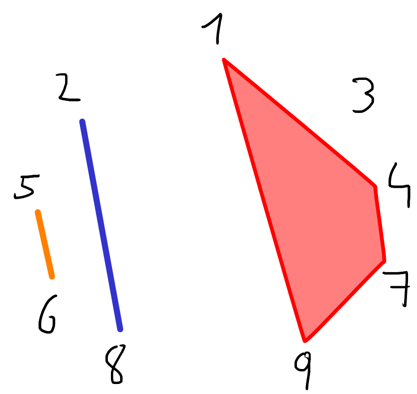

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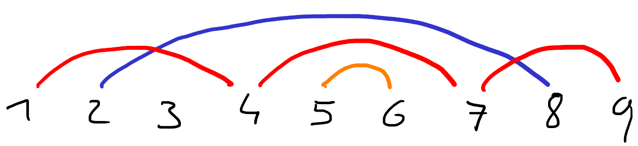

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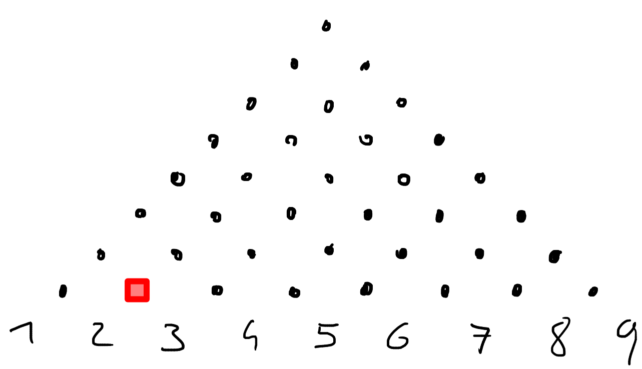

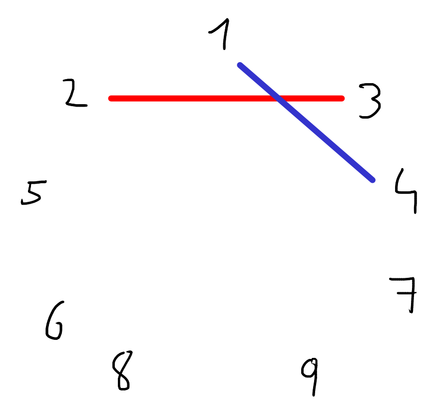

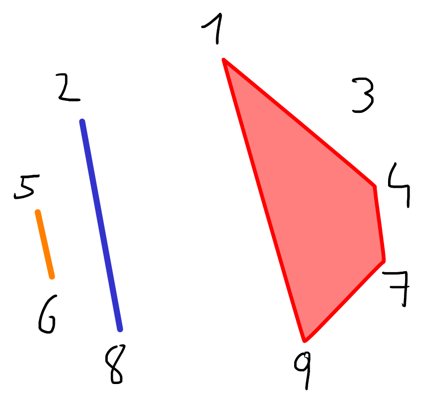

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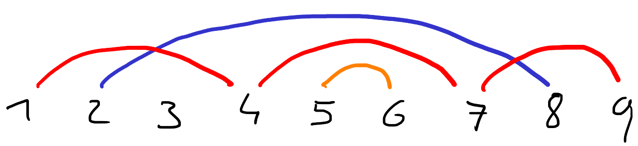

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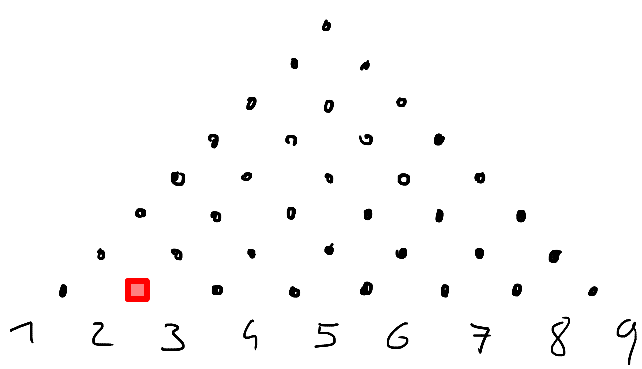

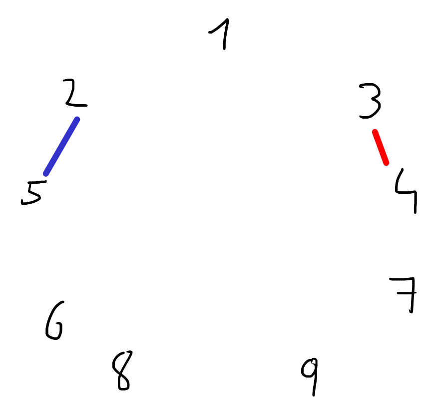

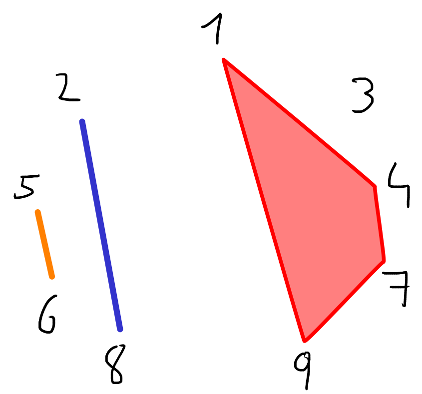

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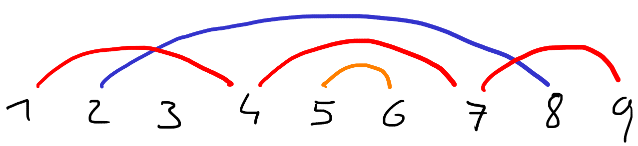

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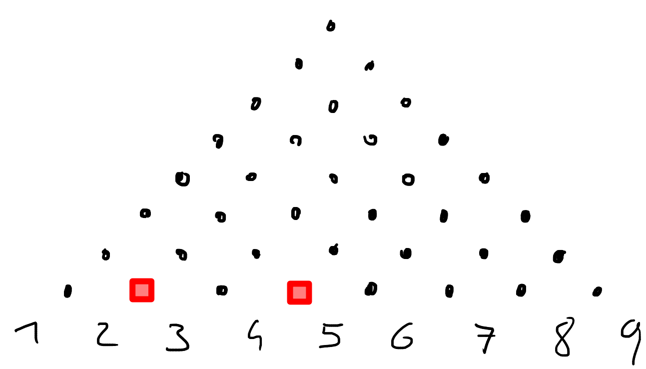

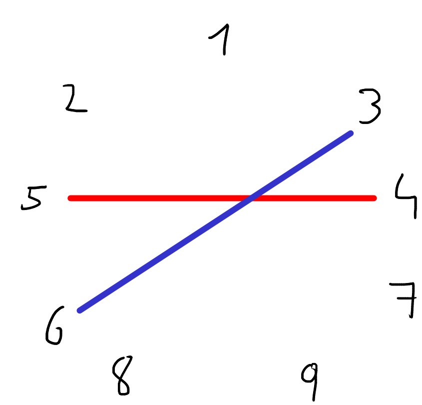

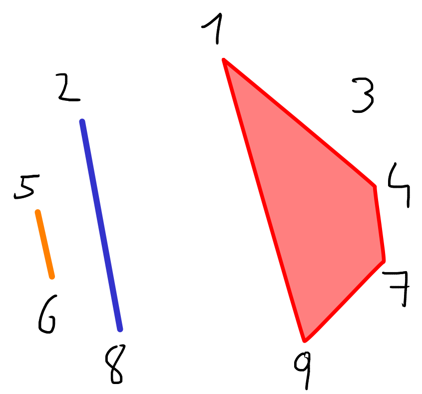

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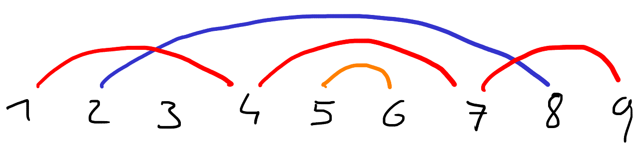

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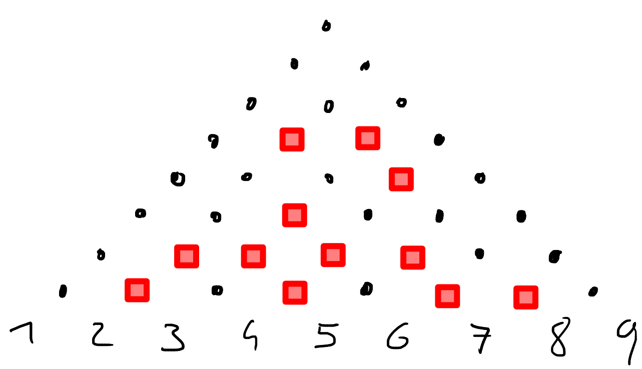
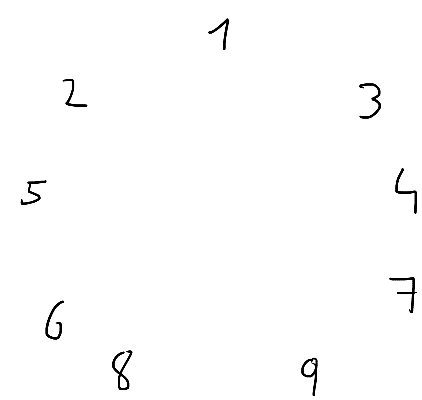
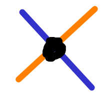
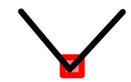




Idea: construct certain families of paths
in the marked root lattice.

Consider families of kissing paths with the follwing forbidden local configurations:

- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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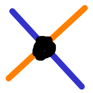
- crossing of two paths in an unmarked root 
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We call such a family of paths minimal, if:

- each unmarked root lies either on a path or above all paths,
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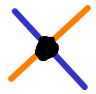
- crossing of two paths in an unmarked root 
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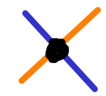
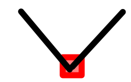
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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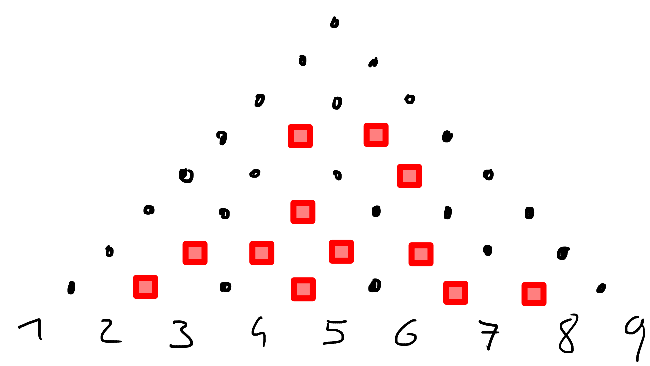

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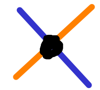
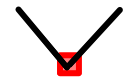
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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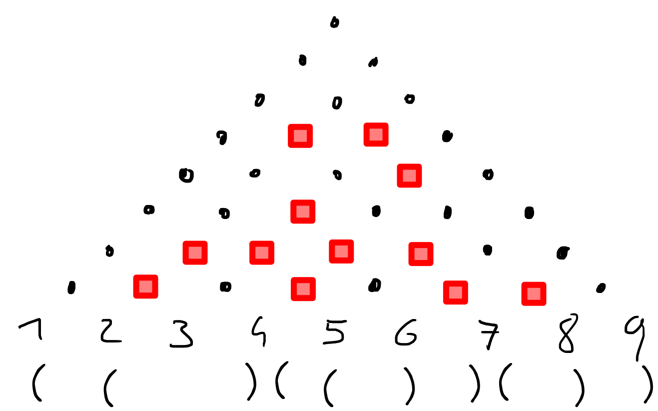

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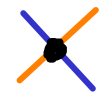
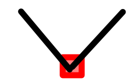
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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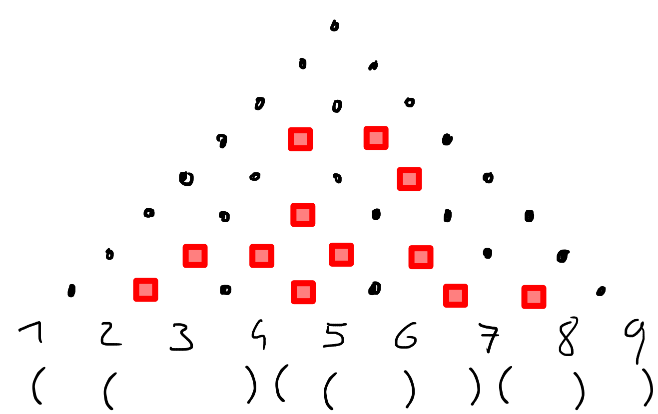

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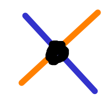
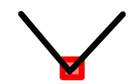
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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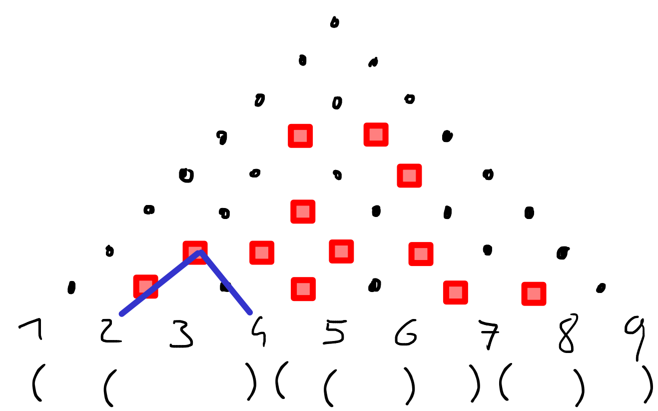

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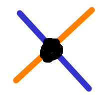
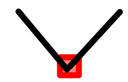
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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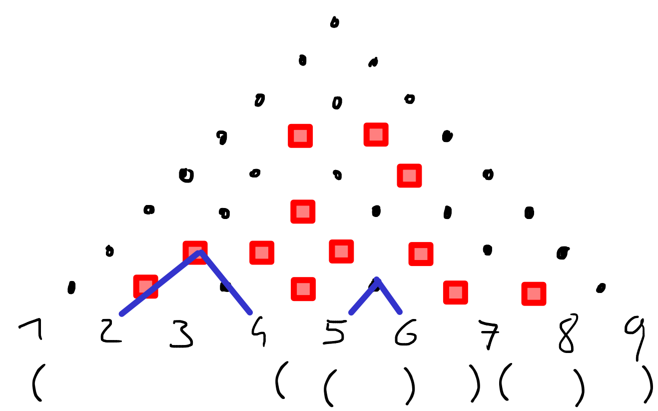

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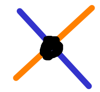
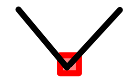
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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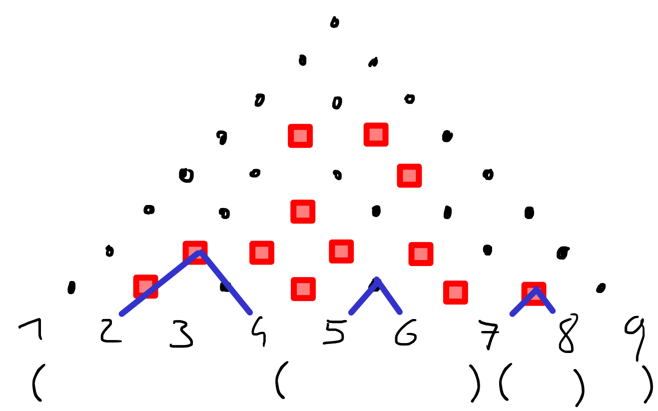

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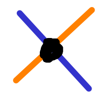
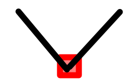
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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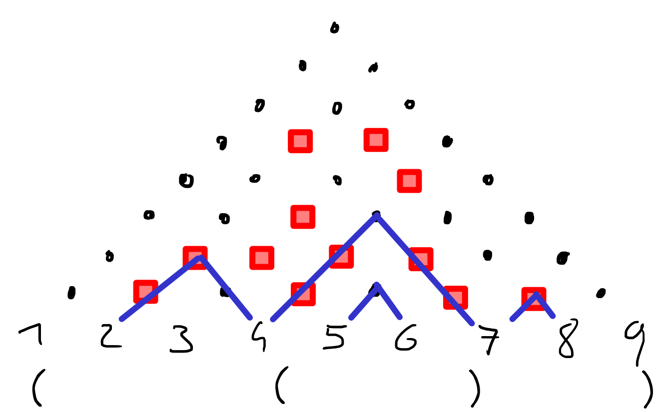

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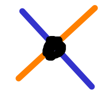
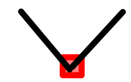
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

We call such a family of paths minimal, if:

- each unmarked root lies either on a path or above all paths,
- if a marked root is a peak of a path, then it is either part of another path, or there is no other path "above" it.

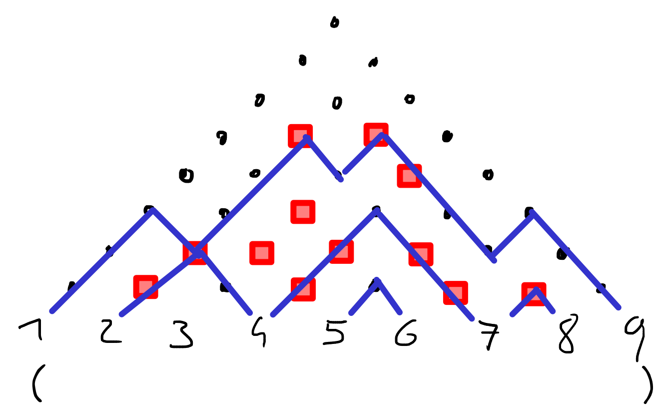
Theorem

The bijection ϕ_c is given by constructing the minimal family of kissing paths with starting points O and endpoints I .

$O=\{1, 2, 4, 5, 7\}$

$I= \{4, 6, 7, 8, 9\}$

- Construct a parenthesis word.
- "Replace" two neighbouring parenthesis ( ) by a path, s.t. the resulting family is minimal.



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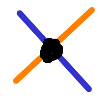
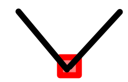
- crossing of two paths in an unmarked root 
- an up and down step of a path at a marked root without crossing another path 

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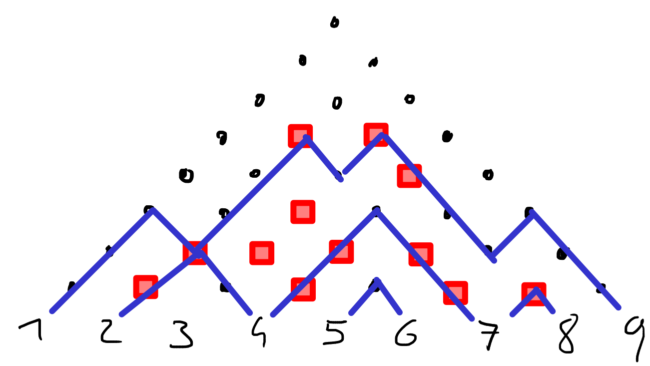

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  Two coxeter elements are related by a sequence of mutations.

  Show that if the theorem holds for c, then it also holds for c' (Cambrian induction).