# Integrality in the Matching-Jack conjecture and the Farahat-Higman algebra

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# Generating series of permutations and matchings

1. [Representation theory of the symmetric group]

$$\sum_{\theta} t^{|\theta|} \frac{|\theta|!}{\dim(\theta)} s_\theta(\mathbf{p}) s_\theta(\mathbf{q}) s_\theta(\mathbf{r}) = \sum_{n \geq 0} t^n \sum_{\lambda,\mu,\nu \vdash n} \frac{\gamma^{\lambda}_{\mu,\nu}}{z_\lambda} p_\lambda q_\mu r_\nu,$$

$s_\theta$ : the Schur function associated to the partition $\theta$, expressed in the power-sum bases $\mathbf{p} := (p_i)_{i \geq 1}; \mathbf{q} := (q_i)_{i \geq 1}; \mathbf{r} := (r_i)_{i \geq 1}$.
$z_\lambda := \frac{|\lambda|!}{|\mathcal{C}_\lambda|}$.
$\gamma^{\lambda}_{\mu,\nu} := |\{(\sigma_1, \sigma_2) \text{ of type } (\mu,\nu) \text{ such that } \sigma_1 \cdot \sigma_2 = \sigma_\lambda\}|$, where $\sigma_\lambda$ is a fixed permutation of type $\lambda$.

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**Proof:**

- $s_\theta(\mathbf{p}) = \sum_{\lambda\vdash|\theta|} \frac{\chi^\theta(\lambda)}{z_\lambda} p_\lambda$ $\chi^\theta$: characters of the symmetric group.
- $\gamma^{\lambda}_{\mu,\nu} = \sum_{\theta\vdash n} \frac{|\theta|!}{\dim(\theta) z_\mu z_\nu} \chi^\theta(\lambda)\chi^\theta(\mu)\chi^\theta(\nu)$

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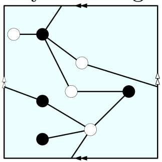
The coefficients $\gamma^{\lambda}_{\mu,\nu}$ also count maps on orientable surfaces



A map on the torus

# Generating series of permutations and matchings

2. Goulden-Jackson '96 [Representation Theory of the Gelfand pair $(\mathfrak{S}_{2n}, \mathfrak{B}_n)$]

$$\sum_{\theta} t^{|\theta|} \frac{\dim(2\theta)}{|2\theta|!} Z_\theta(\mathbf{p}) Z_\theta(\mathbf{q}) Z_\theta(\mathbf{r}) = \sum_{n \geq 0} t^n \sum_{\lambda,\mu,\nu \vdash n} \frac{\widetilde{\gamma}^{\lambda}_{\mu,\nu}}{z_\lambda 2^{\ell(\lambda)}} p_\lambda q_\mu r_\nu,$$

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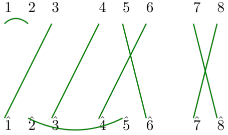


bipartite matchings $\longleftrightarrow$ permutations

A matching of size 8.

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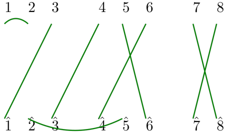


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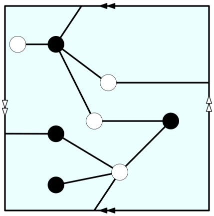
The coefficients $\widetilde{\gamma}^{\lambda}_{\mu,\nu}$ also count maps on general surfaces (orientable or not)



A map on the Klein bottle

# Jack polynomials

We consider the following deformation of the Hall scalar product $\langle .,. \rangle_b$ defined on symmetric functions by
$$\langle p_\lambda, p_\mu \rangle_b = \delta_{\lambda\mu} z_\lambda (1+b)^{\ell(\lambda)}.$$

## Definition

Jack polynomials of parameter $1+b$, denoted $J_\theta^{(b)}$ are defined as follows :

1. Triangularity and normalisation: if $\theta \vdash n$, then
$$J_\theta^{(b)} = \sum_{\mu \vdash n, \mu \leq \theta} u_{\theta\mu} m_\mu,$$
such that $u_{\theta[1^n]} = n!$.
(dominance order $\mu \leq \theta : \mu_1 + \mu_2 + ... + \mu_i \leq \theta_1 + \theta_2 ... + \theta_i \ \forall i$)
2. Orthogonality: if $\theta \neq \xi$ then $\langle J_\theta^{(b)}, J_\xi^{(b)} \rangle_b = 0$.

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- For $b = 0 \longrightarrow$ Schur functions $J_\theta^{(0)} = \frac{|\theta|!}{\dim(\theta)} s_\theta$.
- For $b = 1 \longrightarrow$ Zonal polynomials $J_\theta^{(1)} = Z_\theta$.

# The connection coefficients $c^\lambda_{\mu,\nu}$

$$\sum_{\theta\in\mathbb{Y}} \frac{t^{|\theta|}}{j_\theta^{(1+b)}} J_\theta^{(1+b)}(\mathbf{p}) J_\theta^{(1+b)}(\mathbf{q}) J_\theta^{(1+b)}(\mathbf{r}) = \sum_{n\geq 0} t^n \sum_{\lambda,\mu,\nu\vdash n} \frac{c^\lambda_{\mu,\nu}(b)}{z_\lambda (1+b)^{\ell(\lambda)}} p_\lambda q_\mu r_\nu,$$

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## b=0

$$\begin{aligned} c^{\lambda}_{\mu,\nu}(0) &= |\{(\sigma_1, \sigma_2) \text{ of type } (\mu,\nu) \text{ such that } \sigma_1 \cdot \sigma_2 = \sigma_{\lambda}\}| \\ &= |\{\text{bipartite matchings } \delta \text{ of type } (\mu,\nu) \text{ with respect to } \lambda\}|. \end{aligned}$$

$\sigma_{\lambda}$: fixed permutation of type $\lambda$.

## b=1

$$c^{\lambda}_{\mu,\nu}(1) = |\{\text{matchings } \delta \text{ of type } (\mu,\nu) \text{ with respect to } \lambda\}|.$$

# Matching-Jack conjecture [Goulden and Jackson '96]

## An "algebraic" formulation

*The coefficients $c^{\lambda}_{\mu,\nu}$ are polynomial in the parameter $b$ with non-negative integer coefficients.*

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## A combinatorial formulation

*For every $\lambda \vdash n$ there exists a statistic $\vartheta_\lambda$ on matchings with non-negative integer values, such that:*

- $\vartheta_\lambda(\delta) = 0$ *iff* $\delta$ *is a bipartite matching.*
- *For every* $\mu, \nu \vdash n$

$$c^{\lambda}_{\mu,\nu}(b) = \sum_{\substack{\text{matchings } \delta \text{ of type } (\mu,\nu) \\ \text{with respect to } \lambda}} b^{\vartheta_\lambda(\delta)}.$$

# Partial results and main theorem

Definition of Jack polynomials + basic properties of power-sum functions: the coefficients $c^{\lambda}_{\mu,\nu}$ are rational functions in $b$.

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## Theorem (Dołęga-Féray '15, Duke Math J.)

*The coefficients $c_{\mu,\nu}^{\lambda}$ are polynomial in $b$ with rational coefficients. Moreover,* $\deg(c_{\mu,\nu}^{\lambda}) \leq \mathrm{rk}(\mu) + \mathrm{rk}(\nu) - \mathrm{rk}(\lambda)$.

where $\mathrm{rk}(\lambda) := |\lambda| - \ell(\lambda)$.

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## Main theorem (BD '22)

*The coefficients $c^{\lambda}_{\mu,\nu}$ are polynomial in $b$ with integer coefficients.*

+new proof of the polynomiality

## Starting point of the proof: Matching-Jack conjecture for marginal coefficients $\overline{c}^{\lambda}_{\mu,m}$

Fix $\lambda, \mu \vdash n$ and $m \leq n$. We define $\overline{c}^{\lambda}_{\mu,m} := \sum_{\ell(\nu)=m} c^{\lambda}_{\mu,\nu}$.

### Theorem (BD '21)

*For every $\lambda \vdash n$ there exists a statistic $\vartheta_\lambda$ with non-negative integer values, such that:*

- $\vartheta_\lambda(\delta) = 0$ *iff $\delta$ is a bipartite matching.*
- *For every $\mu \vdash n$ and $m \leq n$*

$$\overline{c}^{\lambda}_{\mu,m}(b) = \sum_{\substack{\text{matchings } \delta \text{ of marginal type } (\mu,m)\\ \text{with respect to } \lambda}} b^{\vartheta_\lambda(\delta)}$$

based on the work of Chapuy and Dołęga '20 on the $b$-conjecture

# Scheme of the proof

Integrality for the marginal coefficients $\overline{c}^{\lambda}_{\mu,m}$

$$\Big\Downarrow \text{ The associativity property}$$

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1. The associativity property: a system of linear equations relating $c^{\lambda}_{\mu,\nu}$ to $\overline{c}^{\lambda}_{\mu,m}$

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Integrality for the coefficients $c^{\lambda}_{\mu,\nu}$

1. The associativity property: a system of linear equations relating $c^{\lambda}_{\mu,\nu}$ to $\overline{c}^{\lambda}_{\mu,m}$
2. The Farahat-Higman algebra: This linear system is invertible in $\mathbb{Z}$.

# The associativity property and a system of linear equations

Jack polynomials orthogonality

$$\Longrightarrow \sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa} c^{\kappa}_{\nu,\rho} = \sum_{\theta \vdash n} c^{\lambda}_{\theta,\rho} c^{\theta}_{\mu,\nu} \qquad \text{for } \lambda, \mu, \nu, \rho \vdash n \geq 1.$$

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Combinatorial interpretation for $b = 0$: Fix $\sigma_\lambda$ of type $\lambda$. Two ways to enumerate the decompositions $\sigma_\lambda = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$ of type $(\mu, \nu, \rho)$:

$$\sigma_\lambda = \sigma_1 \cdot \underbrace{(\sigma_2 \cdot \sigma_3)}_{\text{of type } \kappa} = \underbrace{(\sigma_1 \cdot \sigma_2)}_{\text{of type } \theta} \cdot \sigma_3$$

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Fix $m \leq n$. Taking the sum over $\rho$ of length $m$ we get:

$$\sum_{\kappa \vdash n} c^{\lambda}_{\mu,\kappa} \, \overline{c}^{\kappa}_{\nu,m} = \sum_{\theta \vdash n} \overline{c}^{\lambda}_{\theta,m} \, c^{\theta}_{\mu,\nu}, \qquad \lambda, \mu, \nu \vdash n \text{ and } m \leq n.$$

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**We prove by induction on the rank of $\kappa$ that $c^{\lambda}_{\mu,\kappa}$ has integer coefficients for $\lambda, \mu \vdash n$ :**

- We fix a rank $r$ and two partitions $\lambda$ and $\mu$.
- We choose $(\nu, m)$ in order to select partitions $\kappa$ of rank $\leq$ r.

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## Recall:

$\deg(\overline{c}^{\kappa}_{\nu,m}) \leq n - m + \mathrm{rk}(\nu) - \mathrm{rk}(\kappa)$.

# The associativity property and a system of linear equations

We denote by $\mathcal{T}(n,r)$ the set of such pairs $(\nu,m)$:

$$\mathcal{T}(n,r) := \{(\nu,m) \text{ such that } \mathrm{rk}(\nu)+n-m=r \text{ and } \mathrm{rk}(\nu)<r\}.$$

For $(\nu,m)\in\mathcal{T}(n,r)$:

$$\sum_{\mathrm{rk}(\kappa)=r} c^{\lambda}_{\mu,\kappa}\underbrace{\overline{c}^{\kappa}_{\nu,m}}_{\hookrightarrow \left\{\begin{array}{l}\text{top connection coefficients}\\ \text{independent from } b\end{array}\right.} \text{ is a polynomial in } b \text{ with integer coefficients.}$$

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$$\Longrightarrow \text{A linear system} \begin{cases} c^{\lambda}_{\mu,\kappa} \text{ are the "unknowns".} \\ \overline{c}^{\kappa}_{\nu,m} \text{ are the coefficients of the system.} \end{cases}$$

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**Step 2:** We prove that this linear system is invertible in $\mathbb{Z}$ using a new connection with the the Farahat-Higman algebra.

# The Farahat Higman algebra

For $\nu \vdash n$

$$\mathcal{C}_\nu = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ type(\sigma) = \nu}} \sigma \quad \in Z\left(\mathbb{C}[\mathfrak{S}_n]\right).$$

$\{\mathcal{C}_\nu; \nu \vdash n\}$ form a basis of $Z\left(\mathbb{C}[\mathfrak{S}_n]\right)$.

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## Recall:

$$\mathcal{C}_\nu \cdot \mathcal{C}_\rho = \sum_{\substack{\kappa \vdash n \\ \mathrm{rk}(\kappa) \leq \mathrm{rk}(\nu) + \mathrm{rk}(\rho)}} c^{\kappa}_{\nu,\rho}(0) \mathcal{C}_\kappa$$

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We pass to the graded algebra $\mathcal{Z}_n$, spanned by $\{\mathcal{C}_\nu; \nu \vdash n\}$ and in which the multiplication is given by

$$\mathcal{C}_\nu * \mathcal{C}_\rho = \sum_{\substack{\kappa \vdash n \\ \mathrm{rk}(\kappa) = \mathrm{rk}(\nu) + \mathrm{rk}(\rho)}} c^{\kappa}_{\nu,\rho}(0)\mathcal{C}_\rho.$$

$\mathcal{Z}_n^{(r)}$: the vector space spanned by $\{\mathcal{C}_\nu; \nu \vdash n \text{ and } \mathrm{rk}(\nu) = r\}$.

# The Farahat-Higman algebra

**Fact:** The marginal coefficients $\overline{c}^{\kappa}_{\nu,m}$ encoding the linear system are structure coefficients/change of basis coeffcients in $\mathcal{Z}_n$:

for $(\nu, m) \in \mathcal{T}(n, r)$ and $\kappa$ of rank $r$

$$\overline{c}^{\kappa}_{\nu,\,m} = [\mathcal{C}_\kappa]\mathcal{C}_\nu * \Big( \sum_{\ell(\rho)=m} \mathcal{C}_\rho \Big)$$

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## Theorem (BD '21)

*The family* $\mathcal{C}_\nu * \left( \sum_{\ell(\rho)=m} \mathcal{C}_\rho \right)$ *for* $(\nu, m) \in \mathcal{T}(n, r)$ *is a* $\mathbb{Z}$*-spanning family of* $\mathcal{Z}_n^{(r)}$*. By consequence, the system encoded by* $(\overline{c}^{\kappa}_{\nu,m})$ *is invertible in* $\mathbb{Z}$*.*

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- (Farahat-Higman) Stability by adding parts of size 1:
  $\overline{c}^{\kappa}_{\nu,m} = \overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n,m+n}$, for $n \geq 1$
  $\Longrightarrow$ we pass to the projective limit $\mathcal{Z}_\infty^{(r)} := \varprojlim \mathcal{Z}_n^{(r)}$
  (the graded Farahat-Higman algebra).

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$$\overline{c}^{\kappa}_{\nu,\,m} = [\mathcal{C}_\kappa]\mathcal{C}_\nu * \Big(\sum_{\ell(\rho)=m} \mathcal{C}_\rho\Big)$$

## Theorem (BD '21)

*The family $\mathcal{C}_\nu * \left(\sum_{\ell(\rho)=m} \mathcal{C}_\rho\right)$ for $(\nu, m) \in \mathcal{T}(n, r)$ is a $\mathbb{Z}$-spanning family of $\mathcal{Z}_n^{(r)}$. By consequence, the system encoded by $(\overline{c}^{\kappa}_{\nu,m})$ is invertible in $\mathbb{Z}$.*

- (Farahat-Higman) Stability by adding parts of size 1: $\overline{c}^{\kappa}_{\nu,m} = \overline{c}^{\kappa\cup 1^n}_{\nu\cup 1^n, m+n}$, for $n \geq 1$
  $\Longrightarrow$ we pass to the projective limit $\mathcal{Z}_\infty^{(r)} := \varprojlim \mathcal{Z}_n^{(r)}$ (the graded Farahat-Higman algebra).
- Use two other bases of $\mathcal{Z}_n^{(r)}$ introduced by Farahat-Higman and Matsumoto-Novak.

$$\begin{tikzcd}
(\mathfrak{c}_\rho)_{\rho\vdash r} \arrow[d, "\mathcal{M}^{(r)}"'] \arrow[dr, "\mathcal{L}^{(r)}"] & (\mathfrak{g}_\pi)_{\pi\vdash r} \arrow[l, "(Q^{(r)})^T"'] \arrow[dl, bend right, "\mathcal{N}^{(r)}"'] \\
(\mathfrak{f}_\lambda)_{\lambda\vdash r} & (\mathfrak{m}_\mu)_{\mu\vdash r} \arrow[l, "\mathcal{U}^{(r)}"]
\end{tikzcd}$$

*Thank You!*