Cumulants

I. Overview

Philippe Biane

CNRS, LIGM

Séminaire Lotharingien de Combinatoire

Strobl, 04–08/09/2022

Set Partitions

For a finite set S the set-partitions of S, endowed with the refinement order ($\pi_1 \leq \pi_2$ if π_1 is finer than π_2) is a lattice $\mathcal{P}_S$. It has a largest element 1_S (the partition with one part) and a smallest element 0_S, whose parts have cardinal 1.
One has

$$|\mathcal{P}_S| = B_{|S|}$$

where the B_n are the *Bell numbers* with exponential generating function

$$\sum \frac{B_n}{n!} z^n = e^{e^z - 1}$$

Every interval $[\pi_1, \pi_2]$ is canonically isomorphic to a product lattice

$$\prod_i \mathcal{P}_{S_i}$$

where the number of terms in the product is the number of parts of π_2 and S_i is the set of parts of π_1 included in the i^{th} part of π_2. The Möbius function on $\mathcal{P}_S$ is given by

$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{|S_i|-1}(|S_i| - 1)!$$

Non-crossing Partitions

Let $S = \{1, 2, \ldots, n\}$ and π a set-partition of S. A *crossing* of π is a quadruple (i, j, k, l) with

$$i < j < k < l$$

and

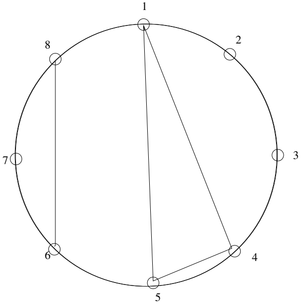
$$i \sim k, \qquad j \sim l$$

but i, j not in the same part of π. A partition is *non-crossing* if it has no crossing.

Example

$$\{1, 4, 5\} \cup \{2\} \cup \{3\} \cup \{6, 8\} \cup \{7\}$$

is non-crossing.



$NC(n)$ is the set of non-crossing partitions of $\{1, 2, \ldots, n\}$. It is a lattice for the refinement order with largest element 1_n and smallest element 0_n.
One has

$$|NC(n)| = C_n \qquad (n^{th} \text{ Catalan number})$$

Every interval $[\pi_1, \pi_2]$ is canonically isomorphic to a product lattice

$$\prod_i NC(k_i)$$

where the number of terms in the product is the number of parts of π_2.
The Möbius function on $NC(n)$ is given by

$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{k_i - 1} C_{k_i - 1}$$

Interval partitions

$I(n)$ is the set of interval partitions of $\{1, 2, \dots, n\}$, whose parts are intervals $[i, j]$. It is a lattice for the refinement order with largest element 1_n and smallest element 0_n.
One has

$$|I(n)| = 2^{n-1}$$

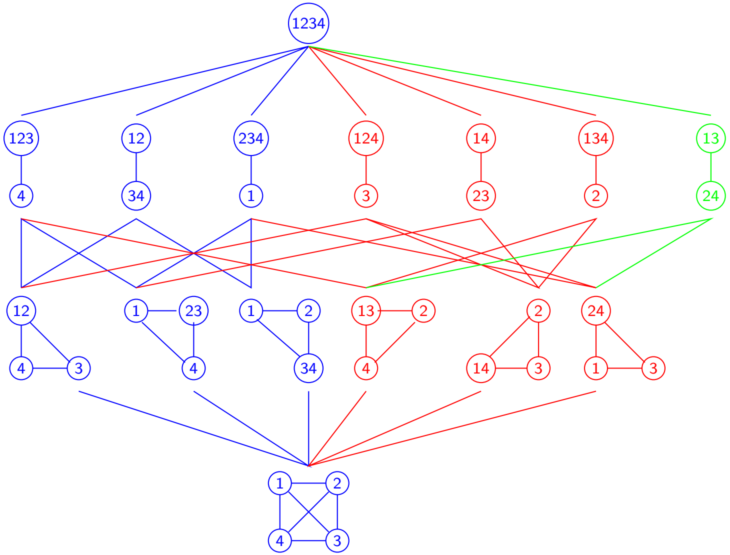
Every interval $[\pi_1, \pi_2]$ is canonically isomorphic to a product lattice

$$\prod_i I(k_i)$$

where the number of terms in the product is the number of parts of π_2.
The Möbius function on $I(n)$ is given by

$$\mu([\pi_1, \pi_2]) = \prod_i (-1)^{k_i - 1}$$

$I(4) \subset NC(4) \subset \mathcal{P}(4)$



One has

$$I(n) \subset NC(n) \subset \mathcal{P}_n$$

For $\pi \in \mathcal{P}_n$ let π^* be the smallest non-crossing partition with $\pi \leq \pi^*$.

Analogously let π^{**} be the smallest interval partition with $\pi \leq \pi^{**}$.
One has

$$\pi \leq \pi^* \leq \pi^{**}$$

Cumulants

Let $\mathcal{A}$ be a k-algebra with $1 \in \mathcal{A}$ and a linear form

$$\varphi : \mathcal{A} \to k, \qquad \varphi(1) = 1$$

In most applications $\mathcal{A}$ is an algebra of random variables (possibly non-commutative like random matrices) over C or R.

The cumulants are n-linear forms K_n (classical) R_n (non-crossing), B_n (Boolean) $n = 1, 2, \dots$ defined implicitly by:

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in \mathcal{P}_n} K_\pi(a_1, \dots, a_n)$$

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in NC(n)} R_\pi(a_1, \dots, a_n)$$

$$\varphi(a_1 a_2 \dots a_n) = \sum_{\pi \in I(n)} B_\pi(a_1, \dots, a_n)$$

here

$$X_\pi(a_1, a_2 \dots, a_n) = \prod_{p \in \pi} X_{|p|}(a_{i_1}, \dots, a_{i_{|p|}})$$

p are the parts of π and $p = \{i_1, i_2, \dots, i_{|p|}\}$

Examples with non-crossing free cumulants

$$\varphi(a_1) = R_1(a_1) \qquad \{1\}$$

$$\begin{array}{lcc} \varphi(a_1 a_2) = & R_2(a_1, a_2) & \{1,2\} \\ & +R_1(a_1)R_1(a_2) & \{1\} \cup \{2\} \end{array}$$

thus

$$\begin{array}{rcl} R_1(a) & = & \varphi(a) \\ R_2(a_1, a_2) & = & \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2) \end{array}$$

$$
\begin{array}{rll}
\varphi(a_1 a_2 a_3) = & R_3(a_1, a_2, a_3) & \{1,2,3\} \\
& +R_1(a_1)R_2(a_2, a_3) & \{1\} \cup \{2,3\} \\
& +R_2(a_1, a_3)R_1(a_2) & \{1,3\} \cup \{2\} \\
& +R_2(a_1, a_2)R_1(a_3) & \{1,2\} \cup \{3\} \\
& +R_1(a_1)R_1(a_2)R_1(a_3) & \{1\} \cup \{2\} \cup \{2\}
\end{array}
$$

$$
\begin{aligned}
R_3(a_1, a_2, a_3) \quad = \quad & \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\varphi(a_3) - \varphi(a_1 a_3)\varphi(a_2) \\
& -\varphi(a_1)\varphi(a_2 a_3) + 2\varphi(a_1)\varphi(a_2)\tau(a_3)
\end{aligned}
$$

Inversion formula

$$
\begin{aligned}
K_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in \mathcal{P}_n} \varphi_\pi(a_1, \ldots, a_n)\mu^{\mathcal{P}}([\pi, 1_n]) \\
R_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \ldots, a_n)\mu^{NC}([\pi, 1_n]) \\
B_n(a_1, a_2, \ldots, a_n) &= \sum_{\pi \in I(n)} \varphi_\pi(a_1, \ldots, a_n)\mu^{B}([\pi, 1_n])
\end{aligned}
$$

The case of one variable

Consider $a \in \mathcal{A}$ and its moments

$$\varphi(a^n); \quad n = 0, 1, 2, \dots$$

with exponential generating series

$$F(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \varphi(a^n)$$

and ordinary generating series

$$M(z) = 1 + \sum_{n=1}^{\infty} z^n \varphi(a^n)$$

The generating series

$$K(z) := \sum_{n=1}^{\infty} \frac{z^n}{n!} K_n(a, a, a \dots, a)$$

$$R(z) := \sum_{n=1}^{\infty} z^n R_n(a, a, a \dots, a)$$

$$B(z) := \sum_{n=1}^{\infty} z^n B_n(a, a, a \dots, a)$$

satisfy the relations:

$$K(z) = \log F(z)$$

$$1 + R(zM(z)) = M(z)$$

$$M(z) = \frac{1}{1 - B(z)}$$

There is a multivariable extension for commuting variables:

$$\log \varphi(e^{\sum_i X_i}) = \sum_{i_1, i_2, \dots} \frac{K_n(X_1^{(i_1)}, X_2^{(i_2)}, \dots)}{i_1! i_2! \dots}$$

Here $X_i^{(j)}$ means that the variable X_i is repeated j times

What are cumulants useful for?

Cumulants are useful for probability theory, they encode independence of random variables:

If $\prod \mathcal{A}_i \subset \mathcal{A}$ are subalgebras they are independent if and only if mixed cumulants vanish:

$$K_n(a_1, \ldots, a_n) = 0$$

if each a_j belongs to one of the $\mathcal{A}_i$ and at least two subalgebras occur.

Independence means that they they commute and

$$\varphi(a_1 a_2 \ldots a_n) = \prod \varphi(a_i)$$

if $a_1 \in \mathcal{A}_{i_1}, a_2 \in \mathcal{A}_{i_2} \ldots$ all i_k distinct

Cumulants are ubiquituous in statistical physics by virtue of the formula for the expansion of the free energy

$$\log \varphi(e^{\sum_i X_i}) = \sum_{i_1, i_2, \ldots} \frac{K_n(X_1^{(i_1)}, X_2^{i_2)}, \ldots)}{i_1! i_2! \ldots}$$

Using non-crossing cumulants one can define the notion of *freness* of subalgebras by the vanishing of mixed non-crossing cumulants. If $\mathcal{A}_i \subset \mathcal{A}$ are subalgebras they are *free* if and only if mixed non-crossing cumulants vanish.
Here free means that for any sequence $a_1, a_2, \ldots, a_n$ such that

- $\varphi(a_i) = 0$;
- $a_i \in \mathcal{A}_{k_i}$ with $k_1 \neq k_2, k_2 \neq k_3, etc$

one has

$$\varphi(a_1 a_2 \ldots a_n) = 0$$

This is the original definition of Voiculescu (1983). The definition of non-crossing cumulants and the connection to freeness is due to Speicher (1990).

The notion of freeness has many applications to operator algebra theory and to random matrix theory. Indeed large independent random matrices give natural models for free random variables.

A notion of Boolean independence is defined similarly using Boolean cumulants but it is less useful (no natural model for Boolean independent variables).

Gaussianity and cumulants

For classical gaussian random variables all cumulants of order ≥ 3 vanish. One can similarly define a notion of gaussianity for free and Boolean cumulants. In the case of non-crossing cumulants the role played by the gaussian law is played by the semi-circle law, which is also the limit law of spectral distribution of large gaussian random matrices.
This leads to the notion of *semi-circular systems* a non-commutative analogue of gaussian family.

Free cumulants also appear (somewhat unexpectedly) in some enumerative problems:

Enumeration of braids (B., Dehornoy, 2014)

The enumeration of Eulerian orientations of planar maps (Bousquet-Mélou, Elvey-Price 2018)

Asymptotic relations between free and classical cumulants 1:
non-crossing cumulants and random matrices:

Take a $N \times N$ hermitian diagonal matrix D with eigenvalues $\lambda_1, \ldots, \lambda_N$ and $M = UDU^*$ with U random unitary matrix with Haar measure.
Let $N \to \infty$ and

$$\frac{1}{N}\sum_i \delta_{\lambda_i} \to \mu(dx)$$

Then the classical cumulants of M_{11} satisfy

$$N^{q-1}K_q(M_{11}) \to (q-1)R_q(\mu)$$

as $N \to \infty$.

Asymptotic relations between free anc classical cumulants 2: QSSEP

The QSSEP (quantum symmetric simple exclusion process) is a model of quantum particles hopping on a finite discrete interval $[1, N]$. It is characterized by a random matrix

$$G_{ij} = Tr(c_i c_j^\dagger \Omega); 1 \leq i, j \leq N$$

giving the correlation between sites ($c_i, c_i \dagger$ are fermionic creation and annihilation, and Ω a steady state fermionic correlation matrix).

Let $N \to \infty$ and $i_1/N \to x_1, i_2/N \to x_2 \ldots, i_n/N \to x_n$ for some $x_1, x_2, \ldots, x_n \in [0, 1]$ then

$$\lim_{N\to\infty} N^{k-1} K_n(G_{i_1 i_2}, G_{i_2 i_3}, \ldots, G_{i_n i_1}) = R_n(\kappa_{x_1}, \ldots, \kappa_{x_n})$$

where $\kappa_x = 1_{[0,x]}$ considered as a random variable on the space $[0, 1]$ with Lebesgue measure.

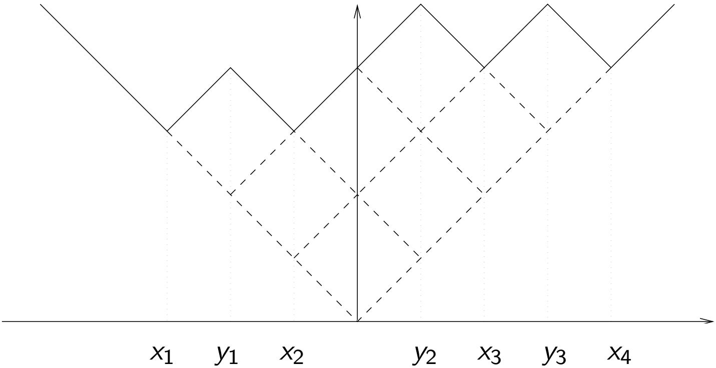
Free cumulants and characters of symmetric groups

The irreducible representations of S_N (symmetric goup) are indexed by integer partitions of N:

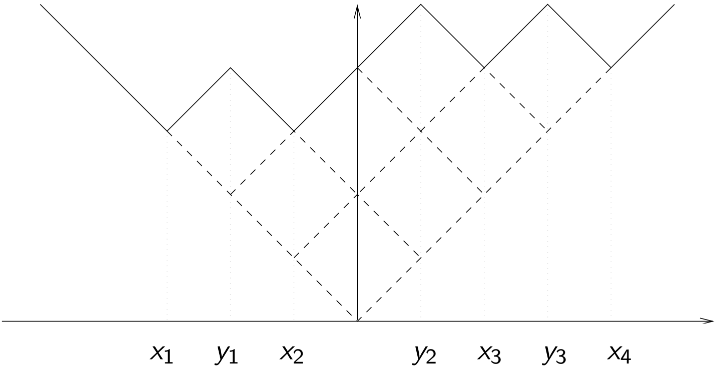
$$N = \lambda_1 + \lambda_2 + \dots$$

It will be convenient to represent partitions in the Russian way.

The partition 4, 3, 1 represented as a piecewise linear function:



TRANSITION MEASURE



S.Kerov: there exists a unique probability measure m_λ such that

$$m_\lambda = \sum_{k=1}^{n} \mu_k \delta_{x_k} \qquad \mu_k = \frac{\prod_{i=1}^{n-1}(x_k - y_i)}{\prod_{i \neq k}(x_k - x_i)}$$

The measure m_λ has moments

$$M_n = \int x^n m_\lambda(dx)$$

and non-crossing cumulants

$$R_n(\lambda)$$

ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with q boxes

Number of rows and columns $= O(\sqrt{q})$.

$\chi_\lambda =$ normalized character of λ.

$$\chi_\lambda(\sigma) \sim q^{-|\sigma|} \prod_{c|\sigma} q^{-1} R_{|c|+2}(\lambda)$$

$|\sigma|=$ length of σ w.r.t generating set of all transpositions, the product is over cycles of σ.

Kerov's formula for characters

Let $K_k = (q)_k \chi_\lambda(c_k)$, c_k=cycle of order k. There exist universal polynomials (independent of q) such that

$$\Sigma_k = P_k(R_{k+1}, \ldots R_2)$$

$$\begin{aligned}
\Sigma_1 &= R_2 \\
\Sigma_2 &= R_3 \\
\Sigma_3 &= R_4 + R_2 \\
\Sigma_4 &= R_5 + 5R_3 \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2 \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3 \\
\Sigma_7 &= R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2
\end{aligned}$$

Theorem (Féray 2009) Kerov's polynomials have nonnegative coefficients.

Doleǵa, Féray, Śniady: found an explicit combinatorial formula for Kerov's polynomials counting certain factorizations in the symmetric groups.

THANK YOU!